On the Renormalization Group Transformation for Scalar Hierarchical Models

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Abstract. We give a new proof for the existence of a non–Gaussian hierarchical renormalization group fixed point, using what could be called a beta–function for this problem. We also discuss the asymptotic behavior of this fixed point, and the connection between the hierarchical models of Dyson and Gallavotti.

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1. Introduction and Main Results

We consider the fixed point problem for the nonlinear operator $\mathcal{N}$, defined by the equation

$$\bigl(\mathcal{N}(f)\bigr)(t) = \frac{1}{\sqrt{(1-\beta^2)\pi}} \int_{-\infty}^{\infty} ds\, e^{-\frac{1}{1-\beta^2}s^2} f(s+\beta t)^N\,, \qquad t \in \mathbb{C}\,, \tag{1.1}$$

with $\beta = N^{-5/6}$ and $N = 2$. Our first result is the following.

Theorem 1.1. *There is a function f_{IR} which can be written as a convergent product*

$$f_{IR}(t) = f_{IR}(0) \prod_{k=1}^{\infty} \Bigl(1 + \tfrac{t^2}{r_k}\Bigr)\,, \qquad t \in \mathbb{C}\,, \tag{1.2}$$

with $f_{IR}(0), r_1, r_2, \ldots$ real and positive, and which satisfies $\mathcal{N}(f_{IR}) = f_{IR}$. Furthermore, along the real axis, the limits

$$\ell_+ \equiv \overline{\lim}\; t^{-6/5} \ln f_{IR}(t)\,, \qquad \ell_- \equiv \underline{\lim}\; t^{-6/5} \ln f_{IR}(t)\,, \tag{1.3}$$

exist, and they satisfy $0 < \ell_+ \le 2\ell_- < \infty$.

Interest in this fixed point problem stems from the theory of critical phenomena in statistical mechanics and quantum field theory. The transformation $\mathcal{N}$ is directly related to a renormalization group (RG) transformation on a space of lattice field theories in three dimensions with a certain hierarchical symmetry; more details, including a discussion of the role of N, are given in the remarks below. Although these models are not very physical, they mimic extremely well what appears to be the behavior of more realistic models, both qualitatively as well as quantitatively. Due to their relative simplicity, hierarchical models have a long history in the testing of RG ideas [1–7]; for a more extensive list of references, see [6].

The fact that $\mathcal{N}$ has a non–Gaussian fixed point has been proved before in [6], using computer–assisted analysis. * The proof which we present here follows a different route, close to the traditional beta–function approach in quantum field theory. It still involves a fair number of numerical estimates, but they can be checked without the help of a computer. Our analysis uses three new and interesting facts about the transformation $\mathcal{N}$:

1) Under the iteration of $\mathcal{N}$, the zeros of a polynomial approach the imaginary axis.
2) The derivative of $\mathcal{N}$ is symmetric with respect to some inner product (given below).
3) There is a connection between $\mathcal{N}$ and Dyson's RG transformation.

In [7] it was shown * that the spectrum of the derivative of $\mathcal{N}$ at the fixed point f_{IR} lies in the open unit disk, except for a simple eigenvalue $\delta > 1$ (which is related to the critical index for the free energy) and the trivial eigenvalue N. With the methods used here, it is easy to prove the following additional property. For $0 < b < 1$, denote by $\mathcal{H}_{(b)}$

* For N=8; but the case N=2 could be treated similarly.

the Hilbert space (of entire analytic functions) obtained by completing the vector space of polynomials with respect to the norm associated with the inner product

$$\langle f, g\rangle_{(b)} = \frac{2b}{\sqrt{1-b^2}\,\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds\; e^{-\frac{2b}{1+b}t^2 - \frac{2b}{1-b}s^2} f(t+is)\overline{g(t+is)}\,. \tag{1.4}$$

Theorem 1.2. *The derivative $D\mathcal{N}(f_{IR})$ of $\mathcal{N}$ at the fixed point f_{IR} is a positive trace–class operator on $\mathcal{H}_{(\beta)}$. Its trace is equal to $2|||f_{IR}|||$, where*

$$|||f||| \equiv \frac{1}{\sqrt{(1-\beta^2)\pi}} \int_{-\infty}^{\infty} dt\; e^{-\frac{1-\beta}{1+\beta}t^2} |f(t)|\,. \tag{1.5}$$

The transformation $\mathcal{N}$ considered here differs from (the $N=2$ analogue of) Gallavotti's RG transformation $\mathcal{T}$ [5] in that we use a scaling factor $\beta = N^{-5/6}$ instead of $\alpha = N^{-1/6}$. For general N, the transformation $\mathcal{T}$ is defined by the equation

$$\big(\mathcal{T}(g)\big)(t) = \sqrt{2\nu/\pi} \int_{-\infty}^{\infty} ds\; e^{-2\nu s^2} g(s+\alpha t)^N\,, \qquad t \in \mathbb{R}\,, \tag{1.6}$$

where $\nu = 1/(4\alpha^2 - 1)$, but in what follows we set again $N = 2$. In addition, both $\mathcal{N}$ and $\mathcal{T}$ seem to be quite different from (a version promoted by Baker [2] of) Dyson's original RG transformation $\mathcal{R}$ for hierarchical models [1], which is given by

$$\big(\mathcal{R}(h)\big)(t) = 2\alpha \int_{-\infty}^{\infty} dx\; e^{-2\lambda x^2} h(\alpha t + x) h(\alpha t - x)\,, \qquad t \in \mathbb{R}\,, \tag{1.7}$$

with $\lambda = (2\alpha^2 - 1)\nu$. However, all three of these transformations are related by a "change of coordinates" in function space. To be more precise, let

$$g_{HT}(t) = \sqrt{2}\alpha e^{-\lambda t^2}\,, \qquad t \in \mathbb{R}\,. \tag{1.8}$$

Then a straightforward calculation shows the following.

Formal Identities 1.3.
(F1) $M \circ \mathcal{N} = \mathcal{T} \circ M$, *where* $M =$*"multiplication by g_{HT}".*
(F2) $J \circ \mathcal{R} = \mathcal{T} \circ J$, *where* $J =$*"convolution with g_{HT}".*

Note that $f_{HT} \equiv 1$ is a trivial fixed point for $\mathcal{N}$ (the high temperature fixed point). The corresponding fixed points for $\mathcal{T}$ and $\mathcal{R}$ are g_{HT} and $h_{HT}(x) = \delta(x)$, respectively. Similarly, $g_{UV} \equiv 1$ is a trivial fixed point for $\mathcal{T}$ (the ultraviolet fixed point), and the corresponding fixed points for $\mathcal{R}$ and $\mathcal{N}$ are $h_{UV} \equiv \sqrt{\lambda/(2\pi\alpha^2)}$ and $f_{UV} = 1/g_{HT}$, respectively.

It is clear that from the fixed point f_{IR} of $\mathcal{N}$ (the infrared fixed point), described in Theorem 1.1, we obtain a non–Gaussian fixed point $g_{IR} = g_{HT} f_{IR}$ for $\mathcal{T}$. It is less clear, however, whether the corresponding fixed point for $\mathcal{R}$ exists as a function. But if we extend the definition of $\mathcal{R}$ to measures on $\mathbb{R}$, then the following can be proved.

Denote by $C_0(\mathbb{R})$ the Banach space (with the sup–norm) of all continuous functions on $\mathbb{R}$ that vanish at infinity. On the dual $C_0'(\mathbb{R})$, which consists of all finite Baire measures H on $\mathbb{R}$, we define the transformation $\mathcal{R}$ by the equation

$$\big(\mathcal{R}(H)\big)(\varphi) = \int dH(x) \int dH(y)\, e^{-\frac{\lambda}{2}(x-y)^2} \varphi\big(\tfrac{x+y}{2\alpha}\big), \qquad \varphi \in C_0(\mathbb{R}). \tag{1.9}$$

Theorem 1.4. *The equation $H_{IR} * g_{HT} = g_{IR}$ defines a positive measure H_{IR} in $C_0'(\mathbb{R})$, and this measure is a (nontrivial) fixed point of the transformation $\mathcal{R}$.*

We note that if the function $s \mapsto f_{IR}(is)$ is in $L^1(\mathbb{R})$, then $dH_{IR}(x) = h_{IR}(x)dx$, where h_{IR} is the continuous function given by the equation

$$h_{IR}(t) = e^{\lambda t^2}\, \frac{\lambda}{\pi} \int_{-\infty}^{\infty} ds\; e^{-2i\lambda st} f_{IR}(is)\,, \qquad t \in \mathbb{R}\,, \tag{1.10}$$

and of course, h_{IR} is a non–Gaussian fixed point for $\mathcal{R}$. Unfortunately, we have no proof that $f_{IR}(i.) \in L^1(\mathbb{R})$, but numerical investigations indicate that this is the case. In fact, a computation of $r_1, \ldots, r_{4000}$ (see the second remark below) suggests the following.

Conjecture. *The two limits ℓ_+ and ℓ_- in (1.3) are equal.*

If we assume that indeed $\ell_+ = \ell_-$, then it follows [11] that $|f_{IR}(is)|$ is bounded by $\exp(-\kappa|s|^{6/5})$ as $s \to \infty$, for some $\kappa > 0$, which implies that the function h_{IR}, defined by (1.10), is entire analytic.

Remarks.

- The normalization of the integral in (1.1) and the covariance of the Gaussian in this integral can be changed without changing the essential features of the fixed point equation $\mathcal{N}(f) = f$. If we choose any constants $K, c > 0$ and define

$$\big(\widetilde{\mathcal{N}}(f)\big)(t) = \frac{K}{\sqrt{c}} \int_{-\infty}^{\infty} ds\, e^{-\frac{1}{2c}s^2} f(s+\beta t)^2\,, \qquad t \in \mathbb{C}, \tag{1.11}$$

then there exists a scaling T of the form $Tf = af(b.)$, such that $\widetilde{\mathcal{N}} = T^{-1} \circ \mathcal{N} \circ T$. The same holds for the transformations $\mathcal{T}$ and $\mathcal{R}$.

- The transformations $\mathcal{N}$ and $\mathcal{T}$, as defined by the equations (1.1) and (1.6), depend on three parameters N, α, and β. These parameters are related to the more commonly used quantities d (dimension) and L (linear block size) by the equation

$$N = L^d\,, \qquad \alpha = L^{-(d-2)/2}\,, \qquad \beta = L^{-(d+2)/2}\,. \tag{1.12}$$

and their numerical values were obtained by choosing $d = 3$ (the most interesting case) and $L = 2^{1/3}$. To explain the relations (1.12), we note first that a conjugacy analogous to (F1) holds in general, for any real numbers $d > 0$ and $L > 1$. Thus, given e.g. the first two relations in (1.12), the third one follows if we require that the fixed point problem for $\mathcal{N}$ be equivalent to that of $\mathcal{T}$. Assume now that d is a positive integer. If L is an integer larger than one, then, as shown in [6], there exists a block spin transformation (with hypercubic blocks of linear size L) on a space of d–dimensional hierarchical lattice field theories, which is conjugate to $\mathcal{T}$, and which has a Gaussian ultraviolet fixed point whose covariance decays like the inverse Laplacean in d dimensions. If only $N = L^d$ is an integer larger than one, but not L, then a similar block spin transformation can be found (a product of d block spin transformations, where the k–th factor uses string–shaped blocks of length N that are parallel to the k–th axis of the lattice $\mathbb{Z}^d$, such that the blocks for the product are hypercubes of linear size N), which is conjugate to $\mathcal{T}^d$. In other words, the relations (1.12) are such that the hierarchical models with $N = 2, 3, \ldots$ mimic a translation invariant model with short range interactions. We note that in the translation invariant case, the model (and hence the physics) does not depend on the choice of L. In the hierarchical case, L is actually a model parameter; but numerical results (for $d = 3$) suggest that physically relevant quantities, such as the critical index ν, depend only very weakly on L. Our main reason for choosing $L = 2^{1/3}$ is the conjugacy (F2) which allows us to construct a non–Gaussian fixed point for Dyson's RG transformation.

- Numerically, the zeros r_k of the function $z \mapsto f_{IR}(\sqrt{-z})$, henceforth simply called "zeros", can be computed as follows. If f is given by a product of the form (1.2), then the function $\tilde{f} = \widetilde{\mathcal{N}}(f)$, defined by (1.11), is formally given by a similar product with zeros $\tilde{r}_1, \tilde{r}_2, \ldots$. The idea is to consider c in (1.11) a time parameter which can be increased continuously from $c = 0$, where $\tilde{f} = const\, f(\beta\, .)^2$, to $c = \hat{c} \equiv (1 - \beta^2)/2$, where $\tilde{f} = const\, \mathcal{N}(f)$. Differentiation of (1.11) with respect to c yields an evolution equation for the function $\tilde{f}$, and it is easy to find the corresponding evolution equation (a system of nonlinearly coupled first order differential equations) for the zeros $\tilde{r}_k = \tilde{r}_k(c)$. By integrating these equations, after doubling and multiplying by β^{-2} each zero at time $c = 0$, we obtain a RG transformation $\mathcal{M} : r \mapsto \tilde{r}(\hat{c})$ which maps the zeros $r = (r_1, r_2, \ldots)$ for f onto the zeros for $\mathcal{N}(f)$. Since, as mentioned earlier, the exact value of $\hat{c}$ is irrelevant, we may as well choose $\hat{c} = \hat{c}(r)$ in such a way that $\tilde{r}_1(\hat{c}(r)) = r_1$. In this case, $\mathcal{M}^n(r)$ converges numerically to a fixed point as $n \to \infty$, for any reasonable initial set of zeros r.
- A formal relation similar to (F2) exists between the non–hierarchical analogues of $\mathcal{T}$ and $\mathcal{R}$, that is, the RG transformation of Balaban [8], and the Wilson–Kadanoff transformation used e.g. in [9].
- Many of the results given later in this paper are stronger than what is needed to prove Theorem 1.1 and Theorem 1.2. For more information we refer to the Sections 2–4. In addition, the results of Sections 2 and 4 can easily be generalized to arbitrary integers $N \geq 2$, where N is the power of f that appears in the definition (1.1) of $\mathcal{N}$.

The remaining part of this paper is organized as follows. Section 2 contains some general results concerning the Hilbert space $\mathcal{H}_{(\beta)}$, some Banach space $\mathcal{B}_\rho$, and the action of $\mathcal{N}$ on these spaces; and Theorem 1.2 is proved. In Section 3 we show that $\mathcal{N}$ has a non–Gaussian fixed point in $\mathcal{B}_\rho$, close to an explicitly given polynomial p of degree six.

Here, some bounds are used which can be translated into trivial numerical inequalities; this is done in Section 5. The proof of Theorem 1.1 is completed in Section 4, where we discuss the zeros and the asymptotic behavior of the fixed point f_{IR}. At the end of Section 4 Theorem 1.4 is proved.

2. Basic Properties of $\mathcal{N}$

In this section, we derive some general bounds on the transformation $\mathcal{N}$, and a formula for its "matrix elements" $A_{k,n}^{(m)}$.

In addition to the Hilbert spaces $\mathcal{H}_{(b)}$, defined in the introduction, we will also consider the Hilbert space $\mathcal{H}_{(1)} \equiv L^2(\mathbb{R}, \exp(-t^2)dt)$, with the inner product

$$\langle f, g\rangle_{(1)} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt\ e^{-t^2} f(t)\overline{g(t)} . \tag{2.1}$$

If H_n denotes the n^{th} Hermite polynomial, then the polynomials p_n, defined by the equation $p_n(t) = H_n(\sqrt{2}t)$, are an orthogonal basis for $\mathcal{H}_{(1)}$. The following two identities are well known and easy to prove.

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} p_n(t) = \ G_z(t) \ \ \equiv e^{\sqrt{2}zt - z^2/2} , \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{b^n}{n!} p_n(s) p_n(t) = E_b(s,t) \equiv \frac{1}{\sqrt{1-b^2}}\ e^{-\frac{b^2}{1-b^2}(s^2+t^2) + \frac{2b}{1-b^2}st} . \tag{2.3}$$

Here, z, s, t are arbitrary complex numbers, and $0 < b < 1$.

Lemma 2.1. *Let* $0 < b \leq 1$.

(i) $\langle p_n, p_k\rangle_{(b)} = b^{-n} n!\, \delta_{k,n}$, *for* $k, n = 0, 1, 2, \ldots$.

(ii) *If* $b < 1$ *then the elements of* $\mathcal{H}_{(b)}$ *can be identified with entire analytic functions. Furthermore, for all* $f \in \mathcal{H}_{(b)}$ *and* $t \in \mathbb{C}$,

$$|f(t)| \leq \|f\|_{(b)} \frac{1}{\sqrt[4]{1-b^2}} e^{\frac{b}{1-b^2}|t|^2 - \frac{b^2}{1-b^2}\mathrm{Re}(t^2)} . \tag{2.4}$$

Proof. If $0 < b \leq 1$ then for all $x, y \in \mathbb{C}$ we have

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{y^k}{k!} \langle p_n, p_k\rangle_{(b)} = \langle G_x, G_y\rangle_{(b)} = e^{xy/b} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{y^k}{k!} b^{-n} n!\, \delta_{k,n} . \tag{2.5}$$

The second equality is the result of a straightforward Gaussian integration. The identity (i) now follows by comparing coefficients. Assume now that $0 < b < 1$. From (i) and equation (2.3) we see that the function $E_b(t,.)$ is in $\mathcal{H}_{(b)}$ for all $t \in \mathbb{C}$:

$$\begin{aligned}\|E_b(t,.)\|^2_{(b)} &= \lim_{N\to\infty}\sum_{k,n=0}^{N}\frac{b^{k+n}}{k!\,n!}p_n(t)\overline{p_k(t)}\langle p_n,p_k\rangle_{(b)} = \sum_{k=0}^{\infty}\frac{b^k}{k!}p_k(t)p_k(\bar t)\\ &= E_b(t,\bar t) = \frac{1}{\sqrt{1-b^2}}e^{\frac{2b}{1-b^2}|t|^2-\frac{2b^2}{1-b^2}\mathrm{Re}(t^2)}\,.\end{aligned} \tag{2.6}$$

Furthermore, if $f = p_n$ for some $n \geq 0$, then

$$\langle f, E_b(t,.)\rangle_{(b)} = \sum_{k=0}^{\infty}\frac{b^k}{k!}p_k(\bar t)\langle p_n,p_k\rangle_{(b)} = f(\bar t) \tag{2.7}$$

for all $t \in \mathbb{C}$. This identity extends to arbitrary polynomials f by linearity. Consequently, if $\{f_n\}$ is a Cauchy sequence of polynomials in $\mathcal{H}_{(b)}$, then $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{C}$. Thus, the limit is an entire analytic function, and (2.7) extends to any $f \in \mathcal{H}_{(b)}$. The inequality (ii) is now obtained by substituting the identity (2.6) into the bound

$$|f(t)| = |\langle f, E_b(\bar t,.)\rangle_{(b)}| \leq \|f\|_{(b)}\|E_b(\bar t,.)\|_{(b)}\,. \tag{2.8}$$

■

Next, we consider the derivative of $\mathcal{N}$ at points $f \in \mathcal{H}_{(\beta)}$ for which the norm $|||f|||$, defined in (1.5), is finite.

Proposition 2.2. *Assume that f is a function in $\mathcal{H}_{(\beta)}$ with $|||f||| < \infty$. Then the operator $A(f)$, defined by the equation*

$$\big(A(f)g\big)(t) = \frac{1}{\sqrt{(1-\beta^2)\pi}}\int_{-\infty}^{\infty} ds\; e^{-\frac{1}{1-\beta^2}(s-\beta t)^2}f(s)g(s)\,, \quad g \in \mathcal{H}_{(\beta)}\,,\ t \in \mathbb{C}\,, \tag{2.9}$$

is a bounded linear operator on $\mathcal{H}_{(\beta)}$, whose norm is less than or equal to $|||f|||$. Furthermore, for any $g, h \in \mathcal{H}_{(\beta)}$, the following holds.

$$\langle A(f)g, h\rangle_{(\beta)} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt\; e^{-t^2}f(t)g(t)\overline{h(t)}\,. \tag{2.10}$$

Proof. Let $f_{HT} \equiv 1$. Then for all $z, t \in \mathbb{C}$ we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^n}{n!}\big(A(f_{HT})p_n\big)(t) &= \big(A(f_{HT})G_z\big)(t)\\ &= \frac{1}{\sqrt{(1-\beta^2)\pi}}\int_{-\infty}^{\infty} ds\; e^{-\frac{1}{1-\beta^2}(s-\beta t)^2+\sqrt{2}zs-z^2/2}\\ &= G_{\beta z}(t) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\beta^n p_n(t)\,,\end{aligned} \tag{2.11}$$

which implies that $A(f_{HT})p_n = \beta^n p_n$ for all n. Hence

$$\langle A(f_{HT})p_n, p_k\rangle_{(\beta)} = \beta^n \langle p_n, p_k\rangle_{(\beta)} = \langle p_n, p_k\rangle_{(1)} \tag{2.12}$$

for all n and k. Let now f, g, h be arbitrary polynomials. Then (2.10) is obtained from (2.12) by linear extension:

$$\langle A(f)g, h\rangle_{(\beta)} = \langle A(f_{HT})(fg), h\rangle_{(\beta)} = \langle fg, h\rangle_{(1)}\,. \tag{2.13}$$

By using the bound (2.4), we find that

$$\begin{aligned} |\langle A(f)g, h\rangle_{(\beta)}| &\le \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} dt\ e^{-t^2} |f(t)| \frac{1}{\sqrt{1-\beta^2}} \|g\|_{(\beta)} \|h\|_{(\beta)} e^{\frac{2\beta}{1+\beta}t^2} \\ &= |||f|||\ \ \|g\|_{(\beta)} \|h\|_{(\beta)}\,. \end{aligned} \tag{2.14}$$

This proves the assertion, since polynomials are dense in $\mathcal{H}_{(\beta)}$.

■

The following Corollary, together with Theorem 1.1, implies Theorem 1.2.

Corollary 2.3. *Assume that f is a function in $\mathcal{H}_{(\beta)}$ with $|||f||| < \infty$ and $f(t) > 0$ for all $t \in \mathbb{R}$. Then $D\mathcal{N}(f)$ is a positive trace–class operator on $\mathcal{H}_{(\beta)}$, and $\mathrm{tr}(D\mathcal{N}(f)) = 2|||f|||$.*

Proof. Note first that $D\mathcal{N}(f) = 2A(f)$. It is clear from Proposition 2.2 that, given our assumptions on f, $A(f)$ is self–adjoint and positive. The trace of $A(f)$ can be computed by using equation (2.3):

$$\begin{aligned} \mathrm{tr}\big(A(f)\big) &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \langle A(f)p_n, p_n\rangle_{(\beta)} = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} dt\ e^{-t^2} f(t) p_n(t)^2 \\ &= \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} dt\ e^{-t^2} f(t) E_\beta(t,t) = |||f|||\,. \end{aligned} \tag{2.15}$$

■

From now on we restrict our attention to a subspace of $\mathcal{H}_{(\beta)}$ which is invariant under the action of $\mathcal{N}$.

Definition 2.4. *Denote by $\mathcal{H}^e_{(\beta)}$ the real subspace of $\mathcal{H}_{(\beta)}$ consisting of all even functions in $\mathcal{H}_{(\beta)}$ which take real values when restricted to the real axis. To every such function f we associate an ℓ^2 sequence $(f_0, f_1, \ldots)$ by defining*

$$f_n = \frac{\beta^n}{\sqrt{(2n)!}} \langle f, p_{2n}\rangle_{(\beta)}\,, \qquad n = 0, 1, 2, \ldots\,. \tag{2.16}$$

Lemma 2.5. *Assume that f, g are functions in $\mathcal{H}^e_{(\beta)}$, and that $|||f||| < \infty$. Then $A(f)g$ lies in $\mathcal{H}^e_{(\beta)}$, and*

$$\big(A(f)g\big)_k = \sum_{m,n=0}^{\infty} A^{(m)}_{k,n} f_m g_n \,, \qquad k = 0, 1, 2, \dots \,, \tag{2.17}$$

where $A^{(m)}_{k,n} = 0$ if either $k > m+n$, or $m > n+k$, or $n > k+m$; otherwise

$$\begin{aligned} A^{(m)}_{k,n} &= \beta^{k+m+n} \sqrt{\tbinom{2k}{k+m-n}\tbinom{2m}{m+n-k}\tbinom{2n}{n+k-m}} \\ &\le \beta^{k+m+n} \frac{\sqrt{(2k)!}}{k!} \binom{m+n}{k} . \end{aligned} \tag{2.18}$$

Proof. For $k, m, n = 0, 1, 2, \dots$, define

$$A^{(m)}_{k,n} = \frac{\beta^k}{\sqrt{(2k)!}} \frac{\beta^m}{\sqrt{(2m)!}} \frac{\beta^n}{\sqrt{(2n)!}} \langle p_{2k}, A(p_{2m}) p_{2n} \rangle_{(\beta)} \,. \tag{2.19}$$

Then, with the exception of the last inequality in (2.18), the assertion follows from Proposition 2.2, since

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt\, e^{-t^2} p_{2k}(t) p_{2m}(t) p_{2n}(t) = \frac{(2k)!\,(2m)!\,(2n)!}{(k+m-n)!(m+n-k)!(n+k-m)!} \,. \tag{2.20}$$

Here, $|m-n| \le k \le m+n$; otherwise the integral is zero. To prove the last inequality in (2.18), it suffices to show that $A^{(m)}_{k,n}$, if nonzero, is bounded by

$$X^{(m)}_{k,n} \equiv \beta^{k+m+n} \sqrt{\tbinom{2k}{k+m-n}} \tbinom{m+n}{k} \,. \tag{2.21}$$

But

$$\begin{aligned} \big[A^{(m)}_{k,n}\big]^2 &= \big[X^{(m)}_{k,n}\big]^2 \frac{(2m)!\,(2n)!}{(m+n)!(m+n)!} \; \frac{k!\,k!}{(k+m-n)!(k+n-m)!} \\ &= \big[X^{(m)}_{k,n}\big]^2 \prod_{j=1}^{|m-n|} \frac{(m+n)+j}{(m+n)-|m-n|+j} \; \frac{k-|m-n|+j}{k+j} \\ &= \big[X^{(m)}_{k,n}\big]^2 \prod_{j=1}^{|m-n|} \frac{(m+n+j)(k+j) - |m-n|(m+n+j)}{(m+n+j)(k+j) - |m-n|(k+j)} \,, \end{aligned} \tag{2.22}$$

and each factor in the last product is clearly ≤ 1, whenever $|m-n| \le k \le m+n$.

■

Remark. Since the coefficients $A_{k,n}^{(m)}$ are symmetric under permutations of the indices, the fixed point equation for $\mathcal{N}$ is formally the equation for the stationary points of the functional,

$$f \mapsto \sum_{k,m,n=0}^{\infty} A_{k,n}^{(m)} f_k f_m f_n \,, \tag{2.23}$$

on a sphere $\sum_k |f_k|^2 = const.$

Since it is difficult in general to find good estimates for the norm of a bounded operator on ℓ^2, we will continue our analysis of $\mathcal{N}$ on weighted ℓ^1 spaces.

Definition 2.6. *For any given $\rho \geq 1$, denote by $\mathcal{B}_\rho$ the vector space of all functions in $\mathcal{H}^e_{(\beta)}$, for which the sum*

$$\|f\|_\rho \equiv \sum_{n=0}^{\infty} |f_n| \rho^n \tag{2.24}$$

is finite. Equipped with the norm $\|.\|_\rho$, $\mathcal{B}_\rho$ is a Banach space.

Lemma 2.7. *Assume that $r \geq 1$ and $\rho \equiv \beta + 2\beta^2 r \geq 1$. If f is a function in $\mathcal{B}_\rho$ then $A(f)$ is a bounded linear operator from $\mathcal{B}_\rho$ to $\mathcal{B}_r$, and*

$$\|A(f)g\|_r \leq \|f\|_\rho \, \|g\|_\rho \,, \qquad \forall g \in \mathcal{B}_\rho \,. \tag{2.25}$$

Proof. By using the bound (2.18) and the fact that $\binom{2k}{k} \leq 4^k$, for all $k \geq 0$, we obtain

$$\begin{aligned}
\|A(f)g\|_r &= \sum_{k=0}^{\infty} \left| \, \big(A(f)g\big)_k \, \right| r^k \\
&\leq \sum_{m,n=0}^{\infty} \sum_{k=0}^{m+n} \beta^{k+m+n} \sqrt{\tbinom{2k}{k}} \, \tbinom{m+n}{k} |f_m| \; |g_n| \, r^k \\
&\leq \sum_{m,n=0}^{\infty} |f_m| \; |g_n| \sum_{k=0}^{m+n} \binom{m+n}{k} \beta^{m+n-k} (2\beta^2 r)^k \\
&= \sum_{m,n=0}^{\infty} |f_m| \; |g_n| (\beta + 2\beta^2 r)^{m+n} = \|f\|_\rho \, \|g\|_\rho \,.
\end{aligned} \tag{2.26}$$

■

3. Existence of a Non–Gaussian Fixed Point

In this section we prove the existence of a fixed point for the map $\mathcal{N} : f \mapsto A(f)f$ in the space $\mathcal{B}_\rho$, for some

$$\rho > \frac{\beta}{1-2\beta^2}\,. \tag{3.1}$$

We assume the validity of five estimates (N1–5), which will be checked (i.e. broken down into trivial numerical inequalities) in Section 5. Our basic strategy is the following.

Denote by P the projection in $\mathcal{B}_\rho$, defined by the equation

$$(Pf)_n = \begin{cases} f_n\,, & \text{if } n \le d, \\ 0\,, & \text{if } n > d, \end{cases} \tag{3.2}$$

with $d = 6$. We seek a fixed point of $\mathcal{N}$ in a region $G \oplus H$, consisting of functions $f \in \mathcal{B}_\rho$ with a relatively well specified polynomial part $Pf \in G$, and a small higher order part $(\mathrm{I}-P)f \in H$. For any given $g \in G$, we define a map T_g by the equation

$$T_g(h) = (\mathrm{I}-P)\mathcal{N}(g+h)\,, \qquad h \in H. \tag{3.3}$$

It will be shown that this map is a contraction on H. Hence T_g has a unique fixed point $h = h^*(g)$ in H. Using this fixed point, which can be obtained by simply iterating T_g, we define a new map B,

$$B(g) = P\mathcal{N}\bigl(g + h^*(g)\bigr)\,, \qquad g \in G. \tag{3.4}$$

If we can prove that this map B (the beta function) has a fixed point $g^* \in G$, then the function $f^* = g^* + h^*(g^*)$ is a fixed point of $\mathcal{N}$ in $G \oplus H$:

$$\begin{aligned} \mathcal{N}(f^*) &= (\mathrm{I}-P)\mathcal{N}\bigl(g^* + h^*(g^*)\bigr) + P\mathcal{N}\bigl(g^* + h^*(g^*)\bigr) \\ &= T_{g^*}\bigl(h^*(g^*)\bigr) + B(g^*) = h^*(g^*) + g^* = f^*. \end{aligned} \tag{3.5}$$

Remark. For a purely numerical computation of f^*, the procedure outlined above already works for $d = 4$. Numerically, the three largest eigenvalues of $D\mathcal{N}(f^*)$ are approximately 2, 1.427, and 0.859. Thus, even a projection P of rank two (which makes B a map on $\mathbb{R}^2$) should do the job, if chosen appropriately. Our reason for taking $d = 6$ is that this choice seems to minimize the amount of work needed to estimate both B and T_g.

To carry out the abovementioned steps, we decompose a function $f \in \mathcal{B}_\rho$ into a polynomial part $g = Pf$, and a remainder h. By decomposing the same way each of the terms in the sum $\mathcal{N}(f) = A(g)g + 2A(g)h + A(h)h$, we end up with six terms, and each of them will now be estimated separately.

Lemma 3.1. *Let $\kappa = 0.46$. Then for any two functions $h, \tilde h$ in $(\mathrm{I}-P)\mathcal{B}_\rho$ we have*
(i) $\bigl|\bigl(A(h)\tilde h\bigr)_k\bigr| \le \rho^{-14} A^{(7)}_{k,7}\, \|h\|_\rho \|\tilde h\|_\rho\,, \quad k = 0, 1, \ldots, 6.$
(ii) $\|(\mathrm{I}-P)A(h)\tilde h\|_\rho \le \kappa \|h\|_\rho \|\tilde h\|_\rho\,.$

Proof. Without loss of generality we may assume that h and $\tilde h$ are of norm one. To prove (i), we use that for $k \le 6$ and $i \ge 14$,

$$\rho^{-(i+1)}\beta^{k+(i+1)}\binom{i+1}{k} = \rho^{-i}\beta^{k+i}\binom{i}{k}\frac{i+1}{i+1-k}\,\frac{\beta}{\rho} \le \rho^{-i}\beta^{k+i}\binom{i}{k}. \tag{3.6}$$

The last inequality in (3.6) follows since

$$\rho > \frac{2\beta}{2-4\beta^2} = \frac{2\beta}{2-2^{1/3}} > 2\beta\,. \tag{3.7}$$

By combining the bounds (2.18) and (3.6), we find that

$$\begin{aligned} \left|\,\bigl(A(h)\tilde{h}\bigr)_k\,\right| &\le \sup_{m,n\ge 7} \rho^{-(m+n)} A^{(m)}_{k,n} \le \sup_{i\ge 14}\, \rho^{-i}\beta^{k+i}\sqrt{\tbinom{2k}{k}}\tbinom{i}{k} \\ &\le \rho^{-14}\beta^{k+14}\sqrt{\tbinom{2k}{k}}\tbinom{14}{k} = \rho^{-14}A^{(7)}_{k,7}\,, \end{aligned} \tag{3.8}$$

and (i) follows. To prove (ii), we use again the bound (2.18):

$$\begin{aligned} \|(\mathrm{I}-P)A(h)\tilde{h}\|_\rho &\le \sup_{m,n\ge 7} \sum_{k=7}^{m+n} \rho^{k-m-n} A^{(m)}_{k,n} \\ &\le \sup_{i\ge 14}\, (\beta/\rho)^i \sum_{k=7}^{i} (\beta\rho)^k \sqrt{\tbinom{2k}{k}}\tbinom{i}{k} \\ &\le \sqrt{4^{-7}\tbinom{14}{7}}\, \sup_{i\ge 14}\, (\beta/\rho)^i \sum_{k=0}^{i} (\beta\rho)^k 2^k \tbinom{i}{k} \\ &= \sqrt{4^{-7}\tbinom{14}{7}}\, \sup_{i\ge 14}\, (\beta/\rho)^i (1+2\beta\rho)^i < \sqrt{4^{-7}\tbinom{14}{7}}\,. \end{aligned} \tag{3.9}$$

The last square root in (3.9) equals $\sqrt{429/2048}$, which is smaller than 0.46, as can easily be checked by verifying e.g. that $2048 \times 0.46 \times 0.46 > 429$.

■

In order to estimate mixed terms of the form $A(g)h$, we will need bounds on the following sums $S_{m,n}$.

$$S_{m,n} \equiv \sum_{k\ge 7} \rho^{k-7} A^{(m)}_{k,n}\,. \tag{3.10}$$

Lemma 3.2. *Define* $\sigma_m \equiv \max\{S_{m,7}\,, \rho^{-1}S_{m,8}\}$. *If* g *is a function in* $P\mathcal{B}_\rho$, *and* h *a function in* $(\mathrm{I}-P)\mathcal{B}_\rho$, *then*

(i) $\left|\,\bigl(A(g)h\bigr)_k\,\right| \le \|h\|_\rho \rho^{-7} \sum_{m=0}^{6} A^{(m)}_{k,7}|g_m|\,, \quad k = 0, 1, \ldots, 6.$

(ii) $\|(\mathrm{I}-P)A(g)h\|_\rho \le \|h\|_\rho \sum_{m=0}^{6} \sigma_m |g_m|\,.$

Proof. Clearly,

$$\left|\,\bigl(A(g)h\bigr)_k\,\right| \le \|h\|_\rho \sum_{m=0}^{6} |g_m| \max_{7\le n\le k+m} \rho^{-n} A^{(m)}_{k,n}\,. \tag{3.11}$$

The following bound shows that the maximum in (3.11) is taken for $n = 7$. If $j \equiv |k-m| < n$ and $\ell \equiv \max\{k, m\} < n$, then

$$\begin{aligned}\frac{\rho^{-(n+1)} A_{k,n+1}^{(m)}}{\rho^{-n} A_{k,n}^{(m)}} &= \frac{\beta}{\rho} \frac{\sqrt{(2n+2)(2n+1)}(k+m-n)}{(n+1+m-k)(n+1-m+k)} \leq \frac{2\beta}{\rho} \frac{(n+1)(2\ell-n-j)}{(n+1+j)(n+1-j)} \\ &= \frac{2\beta}{\rho} \frac{(n+1)(2\ell-n)-(n+1)j}{(n+1)^2-j^2} \leq \frac{2\beta}{\rho} < 2-2^{1/3} < 1.\end{aligned} \tag{3.12}$$

This proves (i). In order to verify (ii), we note first that since

$$\begin{aligned}A_{k+1,n+1}^{(m)} &= \beta^2 \frac{\sqrt{(2k+2)(2n+2)}\sqrt{(2k+1)(2n+1)}}{(k+n+2-m)(k+n+1-m)} A_{k,n}^{(m)} \\ &\leq \beta^2 \frac{(k+n+2)(k+n+1)}{(k+n+2-m)(k+n+1-m)} A_{k,n}^{(m)},\end{aligned} \tag{3.13}$$

we have

$$\begin{aligned}S_{m,n+1} &= \sum_{k\geq 7} \rho^{k-7} A_{k,n+1}^{(m)} = A_{7,n+1}^{(m)} + \rho \sum_{k\geq 7} \rho^{k-7} A_{k+1,n+1}^{(m)} \\ &\leq A_{7,n+1}^{(m)} + \rho\beta^2 \frac{(7+n+2)(7+n+1)}{(7+n+2-m)(7+n+1-m)} \sum_{k\geq 7} \rho^{k-7} A_{k,n}^{(m)},\end{aligned} \tag{3.14}$$

and thus

$$S_{m,n+1} \leq A_{7,n+1}^{(m)} + \rho\beta^2 \frac{(9+n)(8+n)}{(3+n)(2+n)} S_{m,n}, \qquad m \leq 6. \tag{3.15}$$

If $m < 7 < n$, then by combining (3.15) with the bound

$$\rho^{7-(n+1)} A_{7,n+1}^{(m)} = \rho^{-2(n-6)} \rho^{(n+1)-7} A_{n+1,7}^{(m)} \leq \rho^{-4} \sum_{k\geq 7} \rho^{k-7} A_{k,7}^{(m)} = \rho^{-4} S_{m,7}, \tag{3.16}$$

we find that

$$\begin{aligned}\rho^{7-(n+1)} S_{m,n+1} &\leq \rho^{-4} S_{m,7} + \rho^{7-n} \beta^2 \frac{17 \times 16}{11 \times 10} S_{m,n} \\ &\leq \left[\rho^{-4} + \frac{136}{55}\beta^2\right] \max\{S_{m,7}, \rho^{7-n} S_{m,n}\}.\end{aligned} \tag{3.17}$$

The assertion (ii) now follows since

$$\|(\mathrm{I}-P)A(g)h\|_\rho \leq \|h\|_\rho \sup_{n\geq 7} \sum_{m=0}^{6} \rho^{7-n} S_{m,n} |g_m|, \tag{3.18}$$

and since the term $[\ldots]$ in (3.17) is bounded by

$$[\ldots] < \left(\frac{\beta}{1-2\beta^2}\right)^{-4} + \frac{136}{55}\beta^2 = 24 - 30 \times 2^{-1/3} + \frac{34}{55} 2^{1/3} < 1, \tag{3.19}$$

as can easily be verified by using the inequalities (I1) below.

■

The following inequalities will be used later on, in order to estimate powers of $2^{1/6}$ and $\rho^{\pm 1}$.

Inequalities 3.3. *If $\rho - \beta/(1-2\beta^2)$ is a sufficiently small positive number, then*

(I1)	$0.793700 < 2^{-1/3} < 0.793701\,,$	$1.259920 < 2^{1/3} < 1.259922\,,$
(I2)	$0.890897 < 2^{-1/6} < 0.890900\,,$	$1.122461 < 2^{1/6} < 1.122463\,,$
(I3)	$0.659331 < \rho^{-1} < 0.659339\,,$	$1.516670 < \rho < 1.516689\,.$

Proof. One way to prove (I1) is by comparing the third power of 635 with twice the third power of 504, which shows immediately that

$$\bigl(\tfrac{635}{504}\bigr)^3 < 2 < \bigl(1+10^{-6}\bigr)\bigl(\tfrac{635}{504}\bigr)^3,$$

and then computing 504/635 and 635/504 to seven decimals. The other bounds are easy to verify, if one proceeds in the order they are given. For example, the upper bound on $2^{1/6}$ follows from (I1) since $(1.122463)^2 > 1.259922$. Note that it suffices to verify (I3) for $\rho = \beta/(1-2\beta^2)$; in this case the identity $\rho^{-1} = 2 \times 2^{-1/6} - 2^{1/6}$ can be used.

■

We are now ready to discuss the transformation T_g given by equation (3.3). From now on, ρ is assumed to be a fixed number larger than $\beta/(1-2\beta^2)$, that satisfies the bounds (I3). The sets G and H are given by

$$\begin{aligned} G &= \{g \in \mathcal{B}_\rho : |g_n - p_n| \le \delta_n \quad \forall n\}\,, \\ H &= \{h \in \mathcal{B}_\rho : \|h\|_\rho \le \tfrac{1}{10}\,,\ Ph = 0,\ h_n \ge 0 \quad \forall n\}\,, \end{aligned} \tag{3.20}$$

where p and δ are the following two functions (to simplify notation, we identify a function $f \in \mathcal{B}_\rho$ with the sequence of its coefficients f_n).

$$\begin{aligned} p &= (\,0.909\,,\ 0.492\,,\ 0.251\,,\ 0.112\,,\ 0.045\,,\ 0.016\,,\ 0.005\,,\ 0\,,\ 0\,,\ \ldots)\,, \\ \delta &= (\,0.005\,,\ 0.010\,,\ 0.004\,,\ 0.008\,,\ 0.007\,,\ 0.005\,,\ 0.003\,,\ 0\,,\ 0\,,\ \ldots)\,. \end{aligned} \tag{3.21}$$

Proposition 3.4. *Assume that*

(N1) $$\sigma \equiv \sum_{m=0}^{6} \sigma_m (p_m + \delta_m) \le 0.157\,,$$

(N2) $$\tau \equiv \rho^7 \sum_{m,n=0}^{6} S_{m,n}(p_m + \delta_m)(p_n + \delta_n) \le 0.064\,.$$

Then the map T_g has a unique fixed point $h^*(g)$ in H, for every $g \in G$, and the map $g \mapsto h^*(g)$ is continuous on G.

Proof. Let $g, \tilde{g}, \hat{g} \in G$ and $h, \tilde{h}, \hat{h} \in H$, and assume that (N1) and (N2) hold. Then by Lemma 3.1 and Lemma 3.2 we have the bounds

$$\begin{aligned} \|T_g(h)\|_\rho &\leq \|(\mathrm{I}-P)A(g)g\|_\rho + 2\|(\mathrm{I}-P)A(g)h\|_\rho + \|(\mathrm{I}-P)A(h)h\|_\rho \\ &\leq \tau + 2\sigma\tfrac{1}{10} + \kappa\tfrac{1}{100} \leq \tfrac{1}{10}\,, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \|T_{\tilde{g}}(\tilde{h}) - T_{\hat{g}}(\hat{h})\|_\rho &\leq \|(\mathrm{I}-P)A(\tilde{g}+\hat{g}+2\hat{h})(\tilde{g}-\hat{g})\|_\rho \\ &\quad + \|(\mathrm{I}-P)A(2\tilde{g}+\tilde{h}+\hat{h})(\tilde{h}-\hat{h})\|_\rho \\ &\leq K\|\tilde{g}-\hat{g}\|_\rho + 2\big(\sigma + \kappa\tfrac{1}{10}\big)\|\tilde{h}-\hat{h}\|_\rho \\ &\leq K\|\tilde{g}-\hat{g}\|_\rho + \tfrac{1}{2}\|\tilde{h}-\hat{h}\|_\rho\,, \end{aligned} \tag{3.23}$$

for some constant K. Since $T_g(h)$ has no negative coefficients, it follows from (3.22) that T_g maps H to H. Furthermore, by setting $\tilde{g} = \hat{g} = g$ in (3.23), we see that T_g is a contraction on H. Thus, T_g has a unique fixed point $h^*(g)$ in H. This fixed point can be obtained by iterating T_g, starting e.g. with $h = 0$. By using (3.23), we find that

$$\begin{aligned} \|h^*(\tilde{g}) - h^*(\hat{g})\|_\rho &= \lim_{n\to\infty} \|T_{\tilde{g}}^n(0) - T_{\hat{g}}^n(0)\|_\rho \\ &\leq \sum_{n=0}^{\infty} 2^{-n} K\|\tilde{g}-\hat{g}\|_\rho = 2K\|\tilde{g}-\hat{g}\|_\rho\,. \end{aligned} \tag{3.24}$$

This shows that the map $g \mapsto h^*(g)$ is continuous on G.

■

Note that the assertion of Proposition 3.4 implies that the map B, given by equation (3.4), is well defined on G, and continuous.

In order to simplify the necessary numerical computations, we now approximate each of the coefficients $A_{k,n}^{(m)}$, for $0 \leq k, m, n \leq 6$, by the largest number $Q_{k,n}^{(m)}$ with the property that $1000 Q_{k,n}^{(m)}$ is a non–negative integer, and that

(N3) $$\big| A_{k,n}^{(m)} - Q_{k,n}^{(m)} \big| \leq \tfrac{1}{2000}\,, \qquad 0 \leq k, m, n \leq 6\,.$$

Definition 3.5. *If g is a vector in $P\mathcal{B}_\rho$, denote by $Q(g)$ be the symmetric 7×7 matrix with entries*

$$Q(g)_{k,n} = \sum_{m=0}^{6} Q_{k,n}^{(m)} g_m\,, \qquad 0 \leq k, n \leq 6\,. \tag{3.25}$$

Define vectors u, v, w in $P\mathcal{B}_\rho \approx \mathbb{R}^7$ by the following equations.

$$
\begin{aligned}
u_k &= \tfrac{1}{2000} \sum_{m=0}^{6} (p_m + \delta_m) \sum_{n=|k-m|}^{\min\{k+m,6\}} (p_n + \delta_n)\,, \\
v_k &= \tfrac{1}{10}\, \rho^{-7} \sum_{m=0}^{6} A_{k,7}^{(m)} (p_m + \delta_m)\,, \\
w_k &= \tfrac{1}{100}\, \rho^{-14} A_{k,7}^{(7)}\,, \qquad\qquad 0 \le k \le 6\,.
\end{aligned}
\tag{3.26}
$$

To simplify notation, if g is a vector in $P\mathcal{B}_\rho$, denote by $|g|$ the vector whose components are $|g_k|$. Similarly, if M is a matrix, then $|M|$ denotes the matrix whose entries are $|M_{k,n}|$. Finally, if f and g are vectors in $P\mathcal{B}_\rho$, we shall write $g \le f$ iff $g_k \le f_k$ for all k.

Proposition 3.6. *Assume that* (N1–3) *hold, and let $\gamma_k = 3 \times 10^{-5}$, for $0 \le k \le 6$. If there is a symmetric 7×7 matrix M, such that*

(N4) $$\left| \big(\mathrm{I} - M(2Q(p) - \mathrm{I})\big)_{k,n} \right| \le \tfrac{1}{2000}\,, \qquad 0 \le k, n \le 6\,,$$

(N5) $$|M|\big(|Q(p)p - p| + Q(\delta)\delta + u + 2v + w\big) + \gamma \le \delta\,,$$

then B has a fixed point in G.

Proof. First, we note that M is nonsingular. This follows since $(\mathrm{I} - (2Q(p) - \mathrm{I})M)g = g$ whenever $Mg = 0$, which by (N4) implies that $g = 0$. Consequently, we may consider the fixed point problem for the map $g \mapsto \tilde{g}$,

$$\tilde{g} = g - M(B(g) - g)\,, \tag{3.27}$$

instead of B. Assume now that $g \in G$. Then

$$
\begin{aligned}
\tilde{g} - p &= (g - p) - M\big[PA(g + h^*(g))(g + h^*(g)) - g\big] \\
&= \big(\mathrm{I} - M(2Q(p) - \mathrm{I})\big)(g - p) - M\big[(Q(p)p - p) + Q(g - p)(g - p) \\
&\quad + (PA - Q)(g)g + 2PA(g)h^*(g) + PA\big(h^*(g)\big)h^*(g)\big]\,.
\end{aligned}
\tag{3.28}
$$

From (N4) and (3.21), together with the fact that $|g - p| \le \delta$, it follows that

$$\left| \big(\mathrm{I} - M(2Q(p) - \mathrm{I})\big)(g - p) \right|_k \le \tfrac{1}{2000} \sum_{n=0}^{6} \delta_n \le \gamma_k\,, \qquad 0 \le k \le 6. \tag{3.29}$$

By the definition of Q, and by Lemma 3.2 and Lemma 3.1, we also have

$$|(PA - Q)(g)g| \le u\,, \qquad |PA(g)h^*(g)| \le v\,, \qquad |PA(h^*(g))h^*(g)| \le w\,. \tag{3.30}$$

This shows that $|\tilde{g} - p|$ is bounded by the left hand side of (N5) which, by assumption, is bounded by δ. Thus, $\tilde{g} \in G$. The assertion now follows by Brower's fixed point theorem.

■

As shown earlier in (3.5), the conclusions of the last two propositions imply the following theorem. The proof of this theorem is completed in Section 5, by verifying that the numerical bounds (N1-5) are satisfied.

Theorem 3.7. *$\mathcal{N}$ has a fixed point in $G \oplus H$.*

4. Proof of Theorems 1.1 and 1.4.

In this section we show that any nonzero fixed point f^* of $\mathcal{N}$, that lies in $\mathcal{B}_\rho$ for some $\rho > \beta/(1-2\beta^2)$, and is different from $f_{HT} \equiv 1$, has the properties described in Theorem 1.1 and Theorem 1.4. An example of such a fixed point (and probably the only example) is described in Section 3. We note that the Gaussian fixed point f_{UV} lies in $\mathcal{B}_\rho$ if and only if $\rho < \beta/(1-2\beta^2)$.

Lemma 4.1. *Assume that $f^* \in \mathcal{B}_\rho$ for some $\rho > \beta/(1-2\beta^2)$, and that f^* is a fixed point for $\mathcal{N}$. Then there are constants K, ℓ such that*

$$|f^*(t)| \le K e^{\ell |t|^{6/5}}, \qquad \forall t \in \mathbb{C}. \tag{4.1}$$

Proof. We may assume that $f^* \not\equiv 0$. Let $r = \beta/(1-2\beta^2)$, and define $\phi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ by the equation $\phi(s) = \|f^*\|_{r+s}$. Then, as a consequence of Lemma 2.7, we have

$$\phi(s) \le \phi(2\beta^2 s)^2 \le \ldots \le \phi\bigl((2\beta^2)^n s\bigr)^{2^n} \tag{4.2}$$

for all n. This shows that in fact $f^* \in \mathcal{B}_\rho$ for all $\rho > r$. Let $I = \bigl[1, \frac{1}{2\beta^2}\bigr]$, and define

$$c \equiv \max_{x \in I} x^{-3/2} \ln \phi(x). \tag{4.3}$$

For any $s \ge 1$, there exists $n \ge 0$ and $x \in I$, such that $s = (2\beta^2)^{-n} x$. Since $2\beta^2 = 2^{-2/3}$, we have thus

$$\phi(s) \le \phi(x)^{2^n} = \phi(x)^{(s/x)^{3/2}} \le e^{c s^{3/2}}. \tag{4.4}$$

Let now t be any complex number of modulus at least $(r+1)^{5/4}$, and define $\rho = |t|^{4/5}$. From the bound (4.4) and Lemma 2.1 it follows that

$$\begin{aligned} \|f^*\|^2_{(\beta/\rho)} &= \sum_{n=0}^{\infty} \frac{(\beta/\rho)^{2n}}{(2n)!} |\langle f^*, p_{2n} \rangle_{(\beta/\rho)}|^2 \\ &= \sum_{n=0}^{\infty} \frac{(\beta/\rho)^{2n}}{(2n)!} \rho^{4n} |\langle f^*, p_{2n} \rangle_{(\beta)}|^2 = \sum_{n=0}^{\infty} \rho^{2n} |f^*_n|^2 \\ &\le \|f^*\|^2_\rho \le e^{2c(\rho - r)^{3/2}}, \end{aligned} \tag{4.5}$$

and thus, using again Lemma 2.1,

$$|f^*(t)| \le e^{c(\rho-r)^{3/2}} \frac{1}{\sqrt[4]{1-(\beta/\rho)^2}} e^{\frac{\beta}{\rho-\beta}|t|^2} \le 2 e^{(c+2\beta)|t|^{6/5}}. \tag{4.6}$$

∎

As mentioned in the introduction, $\mathcal{N}$ has the following property.

Lemma 4.2. *If g is a polynomial whose zeros lie in a strip $|\mathrm{Re}(z)| \leq a$, then $\mathcal{N}(g)$ is a polynomial whose zeros lie in the strip $|\mathrm{Re}(z)| \leq \beta a$.*

Proof. Let d be the degree of g. We use the following formula for $\mathcal{N}(g)$.

$$\big(\mathcal{N}(g)\big)(\beta^{-1}.) = e^{\frac{1}{4}(1-\beta^2)D^2} g^2 = \lim_{n\to\infty} g_n\,, \tag{4.7}$$

where

$$g_n \equiv \Big(1 + i\frac{\sqrt{1-\beta^2}}{2\sqrt{n}}D\Big)^n \Big(1 - i\frac{\sqrt{1-\beta^2}}{2\sqrt{n}}D\Big)^n g^2\,, \tag{4.8}$$

and where D denotes differentiation. Equation (4.7) can be regarded as an identity in the vector space $\mathcal{V}$ of polynomials of degree $\leq 2d$, equipped with some norm, where D acts as a bounded linear operator. Since convergence in $\mathcal{V}$ implies uniform convergence on compact subsets of $\mathbb{C}$, the assertion will follow if we show that all zeros of $g_1, g_2, \ldots$ lie in the strip $|\mathrm{Re}(z)| \leq a$. But this is a consequence of the following theorem by Takagi [10]: If p is a polynomial of degree n, Z the set of its zeros, and s a complex number, then the zeros of the polynomial $t \mapsto p(t) - sp'(t)$ lie in the convex hull of the set $Z \cup (Z + ns)$.

■

That this property of $\mathcal{N}$ yields some information about the fixed points of $\mathcal{N}$ is evident from the following fact.

Lemma 4.3. *Assume that $f^* \in \mathcal{H}^e_{(\beta)}$ is a fixed point of $\mathcal{N}$, and that $|||f^*||| < \infty$. Then there exists a polynomial g such that $\mathcal{N}(g) \to f^*$ in $\mathcal{H}^e_{(\beta)}$, as $n \to \infty$.*

Proof. We may assume that $f^* \not\equiv 0$, which actually implies that $f^*(t) > 0$ for all $t \in \mathbb{R}$. Let N be the map on $\mathcal{H}^e_{(\beta)}$, defined by $N(f) = \mathcal{N}(f^* + f) - f^*$. This map is clearly C^1, and by assumption $N(0) = 0$. By Corollary 2.3 the derivative $DN_0 = 2A(f^*)$ of N at zero is positive and compact. Denote by $\mathcal{H}^s$ and $\mathcal{H}^u$ the spectral subspaces for DN_0 associated with the spectrum lying in $[0, 1)$ and $[1, \infty)$, respectively. By the stable manifold theorem [12], N has a local stable manifold $\mathcal{W}^s \ni 0$, which is the graph of a C^1 map $\phi : U \cap \mathcal{H}^s \to \mathcal{H}^u$, where U is some open neighborhood of zero in $\mathcal{H}^e_{(\beta)}$.

$\mathcal{W}^s$ can also be characterized as the inverse image of zero of the map $\psi : U \to \mathcal{H}^u$, defined by $\psi(f) = \phi(f^s) - f^u$, where f^s and f^u are the components of f in $\mathcal{H}^s$ and $\mathcal{H}^u$, respectively. This map ψ is clearly C^1, and the rank of $D\psi_0$ is equal to $n \equiv \dim\mathcal{H}^u$, i.e. if we choose a basis (linear coordinates) in $\mathcal{H}^u$ and denote by $\psi_k(f)$ the k^{th} coordinate of $\psi(f)$, then the gradients $h_k \equiv \nabla\psi_k(0)$, $1 \leq k \leq n$, are linearly independent. Thus, for $h_0 = f^*$, the map

$$F_{h_0,\cdots,h_n} : (\lambda_1, \ldots, \lambda_n) \mapsto \psi\big(h_0 + \sum_{k=1}^{n} \lambda_k h_k - f^*\big) \tag{4.9}$$

is a C^1 diffeomorphism from some open neighborhood $V \ni 0$ in $\mathbb{R}^n$ to some open neighborhood of zero in $\mathcal{H}^u$. By the implicit function theorem, the equation $F_{g_0,\ldots,g_n}(\lambda) = 0$ has a solution $\lambda \in V$ for any $(g_0, \ldots, g_n)$ in $\bigoplus_{k=0}^n \mathcal{H}^e_{(\beta)}$ sufficiently close to $(h_0, \ldots, h_n)$. But since polynomials are dense in $\mathcal{H}^e_{(\beta)}$, this implies that $\psi(g - f^*) = 0$ for some polynomial g. Since $g - f^* \in \mathcal{W}^s$, we have $N^n(g - f^*) \to 0$ as $n \to \infty$, and the assertion follows.

■

Corollary 4.4. *Assume that $f^* \in \mathcal{H}^e_{(\beta)}$ is a fixed point of $\mathcal{N}$, and that $|||f^*||| < \infty$. Then all zeros of f^* lie on the imaginary axis.*

Proof. By Lemma 4.3 there is a sequence of polynomials $g, \mathcal{N}(g), \mathcal{N}^2(g), \ldots$ converging to f^*, and by Lemma 4.2 the zeros of $\mathcal{N}^n(g)$ lie in the strip $|\mathrm{Re}(z)| \leq \beta^n a$ for some $a < \infty$. Thus, since convergence in $\mathcal{H}^e_{(\beta)}$ implies uniform convergence on compact subsets of $\mathbb{C}$, f^* cannot have any zeros off the imaginary axis (using e.g. the standard formula for the number of zeros inside a circle).

■

Proof of Theorem 1.1. We take f_{IR} to be the fixed point of $\mathcal{N}$ described in Section 3, or any other non–constant fixed point f^* that lies in one of the spaces $\mathcal{B}_\rho$ with $\rho > \beta/(1-2\beta^2)$. Then, by Lemma 4.1, the logarithm of $f^*(t)$ is bounded from above by a constant times $|t|^{6/5}$, i.e. the limit ℓ_+ in (1.3) is finite. For the same reason, the norm $|||f^*|||$ is finite, which, by Corollary 4.4, implies that all zeros of f^* lie on the imaginary axis. Thus, $z \mapsto f^*(\sqrt{z})$ is an entire function of order $\leq 3/5 < 1$, whose zeros are all real and negative. It now follows from Hadamard's factorization theorem that f^* is given by a canonical product (1.2). This implies in particular that f^* is convex, and thus

$$
\begin{aligned}
f^*(\beta^{-1}t) &= \frac{1}{\sqrt{(1-\beta^2)\pi}} \int_{-\infty}^{\infty} ds\, e^{-\frac{1}{1-\beta^2}s^2}\, \tfrac{1}{2}\left[f^*(t+s)^2 + f^*(t-s)^2\right] \\
&\geq \frac{1}{\sqrt{(1-\beta^2)\pi}} \int_{-\infty}^{\infty} ds\, e^{-\frac{1}{1-\beta^2}s^2}\, f^*(t)^2 = f^*(t)^2\,,
\end{aligned}
\tag{4.10}
$$

for all $t \in \mathbb{R}$. The inequality (4.10) shows that the sequence $n \mapsto (\beta^{-n}t)^{-6/5} \ln f^*(\beta^{-n}t)$ is non–decreasing, and hence convergent, for any given $t \in \mathbb{R}$. In particular, if $f(t) > 1$ (such a t exists since f^* is not constant), then the limit is positive, proving that $\ell_+ > 0$. The inequality (4.10) also shows that the function $t \mapsto t^{-6/5} \ln f^*(t)$ varies by no more than a factor of two on any interval $[t, \beta^{-1}t]$. Hence $\ell_- \geq \ell_+/2$. This completes the proof of Theorem 1.1.

■

Proof of Theorem 1.4. Let f_{IR} be the fixed point of $\mathcal{N}$ described in Theorem 1.1, or any other fixed point with the properties: f_{IR} is entire analytic, and real–valued when restricted to the real axis, and $t \mapsto f_{IR}(t)\exp(-\varepsilon|t|^2)$ is a bounded function on $\mathbb{C}$, for every $\varepsilon > 0$. Given any positive integer k, let $d = 2^{k-1}$. By using that $f_{IR} = \mathcal{N}^k(f_{IR})$, we obtain a representation for f_{IR} of the form

$$f_{IR}(t) = \exp\bigl(-2^{k-1}\beta^{2k}\tfrac{1}{1-\beta^2}t^2\bigr)f_k(t)\,, \tag{4.11}$$

with

$$f_k(t) = \int d\mu_k(s)\, e^{tL_k(s)} \prod_{j=1}^{d} f_{IR}(s_j)^2 = \int_{-\infty}^{\infty} dx\, e^{tx} \int d\mu_k(s)\, \delta\bigl(x - L_k(s)\bigr) \prod_{j=1}^{d} f_{IR}(s_j)^2, \tag{4.12}$$

where μ_k is some Gaussian measure on $\mathbb{R}^d$, and where L_k is some real–valued linear functional on $\mathbb{R}^d$. This shows that f_{IR} is the limit (uniformly on compact subsets of $\mathbb{C}$) of functions f_k with the property that $s \mapsto \bigl(D^{2n}f_k\bigr)(is)$ is the Fourier transform of a positive finite measure on $\mathbb{R}$, for all $n \geq 0$. Thus, by Bochner's theorem [13], the function $s \mapsto f_{IR}(is)$ is the Fourier transform of a positive measure whose moments are all finite (by "measure" we always mean Baire measure). In particular, $f_{IR}(i.)$ may be represented as follows.

$$f_{IR}(is) = \int dH_{IR}(x)\, e^{-\lambda x^2} e^{2i\lambda sx}, \qquad s \in \mathbb{R}, \tag{4.13}$$

for some positive measure H_{IR} on $\mathbb{R}$. Since the n^{th} moment of $H_{IR}(g_{HT}\,.)$ is equal to $\sqrt{2}\alpha(2\lambda)^{-n}$ times the n^{th} derivative of f_{IR} at zero, the identity (4.13) can be extended (using the dominated convergence theorem) to arbitrary $s \in \mathbb{C}$ by means of a power series in s. Furthermore, by replacing the second exponential in (4.13) by the power series for $\exp(\lambda x^2)$, we see (using the monotone convergence theorem) that H_{IR} is a finite measure. This shows that $H_{IR} * g_{HT}$ is equal to $g_{IR} \equiv g_{HT} f_{IR}$, and that the equation (1.9) defines a finite positive measure $\mathcal{R}(H_{IR})$ on $\mathbb{R}$.

In order to prove that $\mathcal{R}(H_{IR}) = H_{IR}$, we need a more precise version of the relation (F2) between the transformations $\mathcal{T}$ and $\mathcal{R}$. By using the identity

$$\begin{aligned} &2\nu s^2 + \lambda(s + \alpha t - x)^2 + \lambda(s + \alpha t - y)^2 \\ &\qquad = \tfrac{\lambda}{2}(x-y)^2 + \lambda\bigl(t - \tfrac{x+y}{2\alpha}\bigr)^2 + 4\alpha^2\nu\bigl[s + \tfrac{\lambda}{4\alpha^2\nu}(2\alpha t - x - y)\bigr]^2, \end{aligned}$$

together with Fubini's theorem, we find that for any finite measure H,

$$\begin{aligned} &\bigl(\mathcal{T}(H * g_{HT})\bigr)(t) \\ &\quad = 2\alpha^2\sqrt{\tfrac{2\nu}{\pi}} \int_{-\infty}^{\infty} ds\, e^{-2\nu s^2} \int dH(x)\, e^{-\lambda(s+\alpha t-x)^2} \int dH(y)\, e^{-\lambda(s+\alpha t-y)^2} \\ &\quad = 2\alpha^2\sqrt{\tfrac{2\nu}{\pi}} \int dH(x) \int dH(y)\, e^{-\frac{\lambda}{2}(x-y)^2} e^{-\lambda(t-\frac{x+y}{2\alpha})^2} \int_{-\infty}^{\infty} ds\, e^{-4\alpha^2\nu[s+\ldots]^2} \\ &\quad = \int dH(x) \int dH(y)\, e^{-\frac{\lambda}{2}(x-y)^2} g_{HT}\bigl(t - \tfrac{x+y}{2\alpha}\bigr) \\ &\quad = \bigl(\mathcal{R}(H) * g_{HT}\bigr)(t)\,, \qquad t \in \mathbb{R}. \end{aligned} \tag{4.14}$$

Thus, since g_{IR} is a fixed point of $\mathcal{T}$, it follows that $\mathcal{R}(H_{IR}) * g_{HT}$ is equal to $H_{IR} * g_{HT} = g_{IR}$. Let now H be any positive measure satisfying $H * g_{HT} = g_{IR}$. Then by analytic continuation we obtain the analogue of the identity (4.13), where H_{IR} is replaced by H. Consequently, the Fourier transform of $H(g_{HT}\,.)$ agrees with that of $H_{IR}(g_{HT}\,.)$. This implies that $H(g_{HT}\,.) = H_{IR}(g_{HT}\,.)$, and therefore $H = H_{IR}$.

∎

5. Numerical Estimates

In this section we prove the numerical bounds (N1–5) used in Section 3. To check this proof, the only skill needed is the ability to correctly add and multiply simple floating point numbers; and no particular computing tool is required. We chose to use a computer (instructed not to round) for most of the calculations; but by sacrificing two or three weekends, we could have done the same with just pencil and paper.

The symmetric matrix M which will be used to verify (N4) and (N5) is the following.

$$M = \begin{pmatrix} 1.053 & 0.594 & -0.768 & -0.994 & -0.688 & -0.354 & -0.147 \\ \cdot & -1.967 & 2.086 & 2.652 & 1.837 & 0.946 & 0.393 \\ \cdot & \cdot & -0.399 & 0.166 & 0.079 & 0.040 & 0.017 \\ \cdot & \cdot & \cdot & -2.370 & -1.201 & -0.649 & -0.271 \\ \cdot & \cdot & \cdot & \cdot & -2.329 & -0.824 & -0.366 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1.651 & -0.350 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1.258 \end{pmatrix}$$

The table below gives the components of $g \equiv p + \delta$, where p and δ are the vectors (polynomials) defined earlier in (3.21), and bounds on powers of ρ. These quantities will be used frequently later on.

$p_0 = 0.909$,	$\delta_0 = 0.005$,	$g_0 = 0.914$,	$\rho^1 \leq 1.517$,	$\rho^{-1} \leq 0.660$
$p_1 = 0.492$,	$\delta_1 = 0.010$,	$g_1 = 0.502$,	$\rho^2 \leq 2.301$,	$\rho^{-2} \leq 0.435$
$p_2 = 0.251$,	$\delta_2 = 0.004$,	$g_2 = 0.255$,	$\rho^3 \leq 3.489$,	$\rho^{-3} \leq 0.287$
$p_3 = 0.112$,	$\delta_3 = 0.008$,	$g_3 = 0.120$,	$\rho^4 \leq 5.292$,	$\rho^{-4} \leq 0.189$
$p_4 = 0.045$,	$\delta_4 = 0.007$,	$g_4 = 0.052$,	$\rho^5 \leq 8.026$,	$\rho^{-5} \leq 0.125$
$p_5 = 0.016$,	$\delta_5 = 0.005$,	$g_5 = 0.021$,	$\rho^6 \leq 12.18$,	$\rho^{-6} \leq 0.083$
$p_6 = 0.005$,	$\delta_6 = 0.003$,	$g_6 = 0.008$,	$\rho^7 \leq 18.47$,	$\rho^{-7} \leq 0.055$

Table 1. The vectors p, δ, g, and bounds on $\rho^{-7}, \ldots, \rho^{7}$.

The above bounds of the form $\rho^k \leq r_k$ were obtained iteratively by setting $r_k = r_{k-i} r_i$, rounded up to 5 digits after the decimal point, where i is the integer part of $k/2$. The iteration was started with the upper bounds $r_{\pm 1}$ in the inequalities (I3) of Section 3, and with r_0=1. At the end, the numbers r_k were rounded up to the number of decimals shown in Table 1.

5.1. Proof of (N1)

Most of the work in proving that $\sigma \leq 0.157$ and $\tau \leq 0.064$ goes into estimating 62 of the coefficients $A_{k,n}^{(m)}$. The following bounds will turn out to be sufficient.

$$
\begin{array}{llll}
A_{7,7}^{(0)} & = 2^{-1/6} \times \frac{1000}{4096} \times \frac{\sqrt{2}}{1000} & \leq 0.9 \times 0.3 \times 0.002 & \leq 0.001 \\
A_{6,7}^{(1)} & = 2^{-1/6} \times \frac{1000}{4096} \times \frac{\sqrt{182}}{1000} & \leq 0.891 \times 0.245 \times 0.0135 & \leq 0.003 \\
A_{7,7}^{(1)} & = \frac{700}{2048} \times \frac{\sqrt{1}}{100} & \leq 0.4 \times 0.01 & \leq 0.004 \\
A_{7,8}^{(1)} & = 2^{-1/3} \times \frac{1000}{4096} \times \frac{\sqrt{30}}{1000} & \leq 0.8 \times 0.25 \times 0.01 & \leq 0.002 \\
A_{5,7}^{(2)} & = 2^{-1/6} \times \frac{1000}{4096} \times \frac{\sqrt{2002}}{1000} & \leq 0.90 \times 0.245 \times 0.045 & \leq 0.010 \\
A_{6,7}^{(2)} & = \frac{1000}{1024} \times \frac{\sqrt{546}}{1000} & \leq 0.98 \times 0.0234 & \leq 0.023 \\
A_{6,8}^{(2)} & = 2^{-1/3} \times \frac{1000}{4096} \times \frac{\sqrt{455}}{1000} & \leq 0.80 \times 0.25 \times 0.022 & \leq 0.005 \\
A_{7,7}^{(2)} & = 2^{-1/3} \times \frac{910}{8192} \times \frac{\sqrt{6}}{10} & \leq 0.80 \times 0.112 \times 0.245 & \leq 0.022 \\
A_{7,8}^{(2)} & = 2^{-1/6} \times \frac{700}{2048} \times \frac{\sqrt{10}}{100} & \leq 0.891 \times 0.35 \times 0.032 & \leq 0.010 \\
A_{7,9}^{(2)} & = \frac{3000}{16384} \times \frac{\sqrt{85}}{1000} & \leq 0.2 \times 0.01 & \leq 0.002 \\
A_{4,7}^{(3)} & = 2^{-1/6} \times \frac{1000}{4096} \times \frac{\sqrt{6006}}{1000} & \leq 0.891 \times 0.2442 \times 0.078 & \leq 0.017 \\
A_{5,7}^{(3)} & = \frac{1000}{2048} \times \frac{\sqrt{15015}}{1000} & \leq 0.489 \times 0.1226 & \leq 0.060 \\
A_{5,8}^{(3)} & = 2^{-1/3} \times \frac{1000}{4096} \times \frac{\sqrt{2002}}{1000} & \leq 0.80 \times 0.245 \times 0.045 & \leq 0.009 \\
A_{6,7}^{(3)} & = 2^{-1/3} \times \frac{3300}{8192} \times \frac{\sqrt{910}}{100} & \leq 0.794 \times 0.403 \times 0.302 & \leq 0.097 \\
A_{6,8}^{(3)} & = 2^{-1/6} \times \frac{300}{2048} \times \frac{\sqrt{546}}{100} & \leq 0.891 \times 0.147 \times 0.234 & \leq 0.031 \\
A_{6,9}^{(3)} & = \frac{10000}{16384} \times \frac{\sqrt{4641}}{10000} & \leq 0.7 \times 0.007 & \leq 0.005 \\
A_{7,7}^{(3)} & = 2^{-1/6} \times \frac{910}{2048} \times \frac{\sqrt{5}}{10} & \leq 0.891 \times 0.445 \times 0.224 & \leq 0.089 \\
A_{7,8}^{(3)} & = \frac{4550}{16384} \times \frac{\sqrt{3}}{10} & \leq 0.278 \times 0.174 & \leq 0.049 \\
A_{7,9}^{(3)} & = 2^{-1/3} \times \frac{2100}{8192} \times \frac{\sqrt{51}}{100} & \leq 0.80 \times 0.257 \times 0.072 & \leq 0.015 \\
A_{7,10}^{(3)} & = 2^{-1/6} \times \frac{10000}{32768} \times \frac{\sqrt{4845}}{10000} & \leq 0.9 \times 0.31 \times 0.007 & \leq 0.002 \\
A_{4,7}^{(4)} & = \frac{100}{256} \times \frac{\sqrt{429}}{100} & \leq 0.3907 \times 0.2072 & \leq 0.081 \\
A_{4,8}^{(4)} & = 2^{-1/3} \times \frac{3000}{8192} \times \frac{\sqrt{1430}}{1000} & \leq 0.7938 \times 0.3663 \times 0.03782 & \leq 0.011 \\
A_{5,7}^{(4)} & = 2^{-1/3} \times \frac{2100}{4096} \times \frac{\sqrt{2145}}{100} & \leq 0.794 \times 0.513 \times 0.464 & \leq 0.189 \\
A_{5,8}^{(4)} & = 2^{-1/6} \times \frac{500}{1024} \times \frac{\sqrt{143}}{100} & \leq 0.90 \times 0.49 \times 0.120 & \leq 0.053 \\
A_{5,9}^{(4)} & = \frac{30000}{32768} \times \frac{\sqrt{4862}}{10000} & \leq 1.0 \times 0.007 & \leq 0.007 \\
A_{6,7}^{(4)} & = 2^{-1/6} \times \frac{770}{2048} \times \frac{\sqrt{65}}{10} & \leq 0.891 \times 0.376 \times 0.807 & \leq 0.271 \\
A_{6,8}^{(4)} & = \frac{770}{4096} \times \frac{\sqrt{39}}{10} & \leq 0.1880 \times 0.625 & \leq 0.118 \\
A_{6,9}^{(4)} & = 2^{-1/3} \times \frac{300}{2048} \times \frac{\sqrt{663}}{100} & \leq 0.7938 \times 0.1465 \times 0.2575 & \leq 0.030 \\
A_{6,10}^{(4)} & = 2^{-1/6} \times \frac{10000}{65536} \times \frac{\sqrt{62985}}{10000} & \leq 0.90 \times 0.16 \times 0.026 & \leq 0.004 \\
A_{7,7}^{(4)} & = \frac{10010}{32768} \times \frac{\sqrt{70}}{10} & \leq 0.3055 \times 0.837 & \leq 0.256 \\
A_{7,8}^{(4)} & = 2^{-1/3} \times \frac{910}{2048} \times \frac{\sqrt{21}}{10} & \leq 0.794 \times 0.4444 \times 0.459 & \leq 0.162 \\
A_{7,9}^{(4)} & = 2^{-1/6} \times \frac{9100}{32768} \times \frac{\sqrt{714}}{100} & \leq 0.891 \times 0.278 \times 0.27 & \leq 0.067 \\
A_{7,10}^{(4)} & = \frac{10000}{16384} \times \frac{\sqrt{67830}}{10000} & \leq 0.611 \times 0.0261 & \leq 0.016 \\
A_{7,11}^{(4)} & = 2^{-1/3} \times \frac{30000}{262144} \times \frac{\sqrt{35530}}{10000} & \leq 0.8 \times 0.12 \times 0.02 & \leq 0.002 \\
A_{5,7}^{(5)} & = 2^{-1/6} \times \frac{150}{1024} \times \frac{\sqrt{858}}{10} & \leq 0.8909 \times 0.1465 \times 2.93 & \leq 0.383
\end{array}
$$

$A^{(5)}_{5,8} = \frac{13500}{32768} \times \frac{\sqrt{1430}}{100}$	$\leq 0.4120 \times 0.3782$	≤ 0.156
$A^{(5)}_{5,9} = 2^{-1/3} \times \frac{5000}{16384} \times \frac{\sqrt{24310}}{1000}$	$\leq 0.794 \times 0.306 \times 0.156$	≤ 0.038
$A^{(5)}_{5,10} = 2^{-1/6} \times \frac{10000}{65536} \times \frac{\sqrt{92378}}{10000}$	$\leq 0.90 \times 0.16 \times 0.031$	≤ 0.005
$A^{(5)}_{6,7} = \frac{3465}{32768} \times \sqrt{26}$	$\leq 0.1058 \times 5.10$	≤ 0.540
$A^{(5)}_{6,8} = 2^{-1/3} \times \frac{550}{2048} \times \frac{\sqrt{195}}{10}$	$\leq 0.7938 \times 0.2686 \times 1.397$	≤ 0.298
$A^{(5)}_{6,9} = 2^{-1/6} \times \frac{9900}{65536} \times \frac{\sqrt{6630}}{100}$	$\leq 0.891 \times 0.1511 \times 0.815$	≤ 0.110
$A^{(5)}_{6,10} = \frac{5000}{32768} \times \frac{\sqrt{25194}}{1000}$	$\leq 0.153 \times 0.16$	≤ 0.025
$A^{(5)}_{6,11} = 2^{-1/3} \times \frac{100000}{262144} \times \frac{\sqrt{646646}}{100000}$	$\leq 0.80 \times 0.39 \times 0.009$	≤ 0.003
$A^{(5)}_{7,7} = 2^{-1/3} \times \frac{3003}{16384} \times \sqrt{14}$	$\leq 0.7938 \times 0.1833 \times 3.742$	≤ 0.545
$A^{(5)}_{7,8} = 2^{-1/6} \times \frac{10010}{32768} \times \frac{\sqrt{210}}{10}$	$\leq 0.8909 \times 0.3055 \times 1.45$	≤ 0.395
$A^{(5)}_{7,9} = \frac{3900}{8192} \times \frac{\sqrt{1785}}{100}$	$\leq 0.477 \times 0.423$	≤ 0.202
$A^{(5)}_{7,10} = 2^{-1/3} \times \frac{195000}{262144} \times \frac{\sqrt{13566}}{1000}$	$\leq 0.794 \times 0.744 \times 0.1165$	≤ 0.069
$A^{(5)}_{7,11} = 2^{-1/6} \times \frac{35000}{131072} \times \frac{\sqrt{3553}}{1000}$	$\leq 0.9 \times 0.27 \times 0.06$	≤ 0.015
$A^{(5)}_{7,12} = \frac{100000}{524288} \times \frac{\sqrt{490314}}{100000}$	$\leq 0.2 \times 0.01$	≤ 0.002
$A^{(6)}_{6,7} = 2^{-1/3} \times \frac{990}{2048} \times \frac{\sqrt{429}}{10}$	$\leq 0.7938 \times 0.4834 \times 2.0713$	≤ 0.795
$A^{(6)}_{6,8} = 2^{-1/6} \times \frac{14850}{65536} \times \frac{\sqrt{715}}{10}$	$\leq 0.8909 \times 0.2266 \times 2.674$	≤ 0.540
$A^{(6)}_{6,9} = \frac{5500}{32768} \times \frac{\sqrt{24310}}{100}$	$\leq 0.1679 \times 1.56$	≤ 0.262
$A^{(6)}_{6,10} = 2^{-1/3} \times \frac{33000}{65536} \times \frac{\sqrt{46189}}{1000}$	$\leq 0.7938 \times 0.5036 \times 0.2150$	≤ 0.086
$A^{(6)}_{6,11} = 2^{-1/6} \times \frac{30000}{65536} \times \frac{\sqrt{176358}}{10000}$	$\leq 0.9 \times 0.46 \times 0.042$	≤ 0.018
$A^{(6)}_{6,12} = \frac{100000}{524288} \times \frac{\sqrt{676039}}{100000}$	$\leq 0.2 \times 0.01$	≤ 0.002
$A^{(6)}_{7,7} = 2^{-1/6} \times \frac{30030}{65536} \times \frac{\sqrt{462}}{10}$	$\leq 0.8909 \times 0.4583 \times 2.150$	≤ 0.878
$A^{(6)}_{7,8} = \frac{4290}{16384} \times \frac{\sqrt{770}}{10}$	$\leq 0.2619 \times 2.775$	≤ 0.727
$A^{(6)}_{7,9} = 2^{-1/3} \times \frac{128700}{262144} \times \frac{\sqrt{13090}}{100}$	$\leq 0.7938 \times 0.4910 \times 1.1442$	≤ 0.446
$A^{(6)}_{7,10} = 2^{-1/6} \times \frac{65000}{65536} \times \frac{\sqrt{49742}}{1000}$	$\leq 0.891 \times 0.992 \times 0.224$	≤ 0.198
$A^{(6)}_{7,11} = \frac{100100}{524288} \times \frac{\sqrt{969}}{100}$	$\leq 0.191 \times 0.312$	≤ 0.060
$A^{(6)}_{7,12} = 2^{-1/3} \times \frac{21000}{131072} \times \frac{\sqrt{7429}}{1000}$	$\leq 0.794 \times 0.1603 \times 0.0862$	≤ 0.011
$A^{(6)}_{7,13} = 2^{-1/6} \times \frac{500000}{2097152} \times \frac{\sqrt{193154}}{100000}$	$\leq 0.90 \times 0.239 \times 0.0044$	≤ 0.001

Table 2. Bounds on $A^{(m)}_{k,n}$ for $0 \leq m \leq 6$ and $m \leq k \leq 7 \leq n \leq 13$.

The equalities in this table are obtained by simplifying (in an obvious way) the expression (2.18) for each of the coefficients $A^{(m)}_{k,n}$. Then, in order to verify e.g. that $A^{(5)}_{7,11} \leq 0.015$, it suffices to check that $0.27 \times 131072 \geq 35000$ and $(1000 \times 0.06)^2 \geq 3553$, and that $0.9 \times 0.27 \times 0.06 \leq 0.015$. Here, the bound $2^{-1/6} \leq 0.9$ follows from inequality (I2) in Section 3.

By using the inequalities from the last two tables, we get immediately the following bounds on the sums $S_{m,n}$, defined in (3.10). Note that the coefficients $A^{(m)}_{k,n}$ are symmetric under permutations of the indices k, m, n, and that many of these coefficients are zero; see Lemma 2.5.

$S_{m,n} = A^{(m)}_{7,n} + \rho A^{(m)}_{8,n} + \rho^2 A^{(m)}_{9,n} + \rho^3 A^{(m)}_{10,n} + \rho^4 A^{(m)}_{11,n} + \rho^5 A^{(m)}_{12,n} + \rho^6 A^{(m)}_{13,n}$	
$S_{0,7} \le 0.001$	≤ 0.001
$S_{1,6} \le 0.003$	≤ 0.003
$S_{1,7} \le 0.004+0.00304$	≤ 0.008
$S_{2,5} \le 0.010$	≤ 0.010
$S_{2,6} \le 0.023+0.00759$	≤ 0.031
$S_{2,7} \le 0.022+0.01517+0.00461$	≤ 0.042
$S_{3,4} \le 0.017$	≤ 0.017
$S_{3,5} \le 0.060+0.01366$	≤ 0.074
$S_{3,6} \le 0.097+0.04703+0.01151$	≤ 0.156
$S_{3,7} \le 0.089+0.07434+0.03452+0.00698$	≤ 0.205
$S_{4,4} \le 0.081+0.01669$	≤ 0.098
$S_{4,5} \le 0.189+0.08041+0.01611$	≤ 0.286
$S_{4,6} \le 0.271+0.17901+0.06903+0.01396$	≤ 0.533
$S_{4,7} \le 0.256+0.24576+0.15417+0.05583+0.01059$	≤ 0.723
$S_{5,5} \le 0.383+0.23666+0.08744+0.01745$	≤ 0.725
$S_{5,6} \le 0.540+0.45207+0.25311+0.08723+0.01588$	≤ 1.349
$S_{5,7} \le 0.545+0.59922+0.46481+0.24075+0.07938+0.01606$	≤ 1.946
$S_{6,6} \le 0.795+0.81918+0.60287+0.30006+0.09526+0.01606$	≤ 2.629
$S_{6,7} \le 0.878+1.10286+1.02625+0.69083+0.31752+0.08829+0.01218$	≤ 4.116

Table 3. Bounds on the sums $S_{m,n}$, for $0 \le m \le n \le 7$, which are not zero.

In order to bound $\rho^{-1}S_{m,8}$, and then $\sigma_m = \max\{S_{m,7}, \rho^{-1}S_{m,8}\}$, we use the inequality (3.15) with $n = 7$, together with the bounds from Table 2 and Table 3.

$\rho^{-1}S_{m,8} \le \rho^{-1} \times A^{(m)}_{8,7} + \frac{8}{3}\beta^2 \times S_{m,7}$	
$\rho^{-1}S_{0,8} \le 0.66 \times 0.000+0.84 \times 0.001 \le 0.001$,	$\sigma_0 \le 0.001$
$\rho^{-1}S_{1,8} \le 0.66 \times 0.002+0.84 \times 0.008 \le 0.009$,	$\sigma_1 \le 0.009$
$\rho^{-1}S_{2,8} \le 0.66 \times 0.010+0.84 \times 0.042 \le 0.042$,	$\sigma_2 \le 0.042$
$\rho^{-1}S_{3,8} \le 0.66 \times 0.049+0.84 \times 0.205 \le 0.205$,	$\sigma_3 \le 0.205$
$\rho^{-1}S_{4,8} \le 0.66 \times 0.162+0.84 \times 0.723 \le 0.715$,	$\sigma_4 \le 0.723$
$\rho^{-1}S_{5,8} \le 0.66 \times 0.395+0.84 \times 1.946 \le 1.896$,	$\sigma_5 \le 1.946$
$\rho^{-1}S_{6,8} \le 0.66 \times 0.727+0.84 \times 4.116 \le 3.938$,	$\sigma_6 \le 4.116$

Table 4. Bounds on $\sigma_0, \ldots, \sigma_6$.

From these bounds, using the values of $g_0, \ldots, g_6$ given in Table 1, it follows that

$$\begin{aligned}\sigma &= \sigma_0 g_0 + \sigma_1 g_1 + \sigma_2 g_2 + \sigma_3 g_3 + \sigma_4 g_4 + \sigma_5 g_5 + \sigma_6 g_6 \\ &\le 0.000914+0.004518+0.010710+0.024600+0.037596+0.040866+0.032928 \\ &\le 0.153\,,\end{aligned}$$

which completes the proof of (N1).

5.2. Proof of (N2)

By definition, τ is obtained by adding up all the products $\rho^7 g_m S_{m,n} g_n$, for $0 \le m, n \le 6$. By using the values from Table 1 and Table 3, the sum over all $n \le 6$ of $S_{m,n} g_n$ is bounded as follows.

$(Sg)_m \le$	$S_{m,1}g_1$	$+S_{m,2}g_2$	$+S_{m,3}g_3$	$+S_{m,4}g_4$	$+S_{m,5}g_5$	$+S_{m,6}g_6$	
$(Sg)_1 \le$						0.000024	≤ 0.00003
$(Sg)_2 \le$					0.000210	+0.000248	≤ 0.00046
$(Sg)_3 \le$				0.000884	+0.001554	+0.001248	≤ 0.00369
$(Sg)_4 \le$			0.002040	+0.005096	+0.006006	+0.004264	≤ 0.01741
$(Sg)_5 \le$		0.002550	+0.008880	+0.014872	+0.015225	+0.010792	≤ 0.05232
$(Sg)_6 \le$	0.001506	+0.007905	+0.018720	+0.027716	+0.028329	+0.021032	≤ 0.10521

Table 5. Bounds on $(Sg)_m$, for $1 \le m \le 6$; $(Sg)_0 = 0$.

Given these inequalities, and from Table 1 the values of g_n and the bound on ρ^7, we find

$$\begin{aligned}\tau \;&=\; \rho^7 \left[\, g_1(Sg)_1 + g_2(Sg)_2 + g_3(Sg)_3 + g_4(Sg)_4 + g_5(Sg)_5 + g_6(Sg)_6 \,\right]\\ &\le 18.47 \left[0.000016{+}0.000118{+}0.000443{+}0.000906{+}0.001099{+}0.000842 \right]\\ &= 18.47 \times 0.003424\\ &\le 0.064\,,\end{aligned}$$

which completes the proof of (N2).

5.3. Proof of (N3)

In the next table we define the numbers $Q^{(m)}_{k,n}$ and verify (N3), by giving bounds of the form $A^{(m)}_{k,n} \in Q^{(m)}_{k,n} \pm \epsilon$, where ϵ denotes the interval $[0, 1/2000]$. Here, $a \in q + \epsilon$ means that $a - q \in \epsilon$, and $a \in q - \epsilon$ means that $q - a \in \epsilon$. The values $Q^{(m)}_{k,n}$ which are not given below are either zero (if $m + k < n$), or determined by symmetry.

$A^{(m)}_{k,n}$			$Q^{(m)}_{k,n}$
$A^{(0)}_{0,0}$	$= 1 \times \sqrt{1}$	$\in\ 1 \times 1$	$\in\ 1.000 - \epsilon$
$A^{(0)}_{1,1}$	$= 2^{-1/6} \times \frac{1}{4} \times \sqrt{2}$	$\in\ 0.890[89,90] \times 0.25 \times 1.414[2,3]$	$\in\ 0.315 - \epsilon$
$A^{(0)}_{2,2}$	$= 2^{-1/3} \times \frac{1}{8} \times \sqrt{1}$	$\in\ 0.79[3,4] \times 0.125 \times 1$	$\in\ 0.099 + \epsilon$
$A^{(0)}_{3,3}$	$= \frac{10}{32} \times \frac{\sqrt{1}}{10}$	$\in\ 0.31[2,3] \times 0.1$	$\in\ 0.031 + \epsilon$
$A^{(0)}_{4,4}$	$= 2^{-1/6} \times \frac{100}{128} \times \frac{\sqrt{2}}{100}$	$\in\ 0.89[0,1] \times 0.7[8,9] \times 0.014[1,2]$	$\in\ 0.010 - \epsilon$
$A^{(0)}_{5,5}$	$= 2^{-1/3} \times \frac{100}{256} \times \frac{\sqrt{1}}{100}$	$\in\ 0.[79,80] \times 0.[39,40] \times 0.01$	$\in\ 0.003 + \epsilon$
$A^{(0)}_{6,6}$	$= \frac{1000}{1024} \times \frac{\sqrt{1}}{1000}$	$\in\ [0.9,1.0] \times 0.001$	$\in\ 0.001 - \epsilon$
$A^{(1)}_{1,1}$	$= \frac{1}{2} \times \sqrt{1}$	$\in\ 0.5 \times 1$	$\in\ 0.500 - \epsilon$
$A^{(1)}_{1,2}$	$= 2^{-1/3} \times \frac{1}{8} \times \sqrt{6}$	$\in\ 0.793[7,8] \times 0.125 \times 2.449[4,5]$	$\in\ 0.243 + \epsilon$
$A^{(1)}_{2,2}$	$= 2^{-1/6} \times \frac{1}{4} \times \sqrt{2}$	$\in\ 0.890[89,90] \times 0.25 \times 1.414[2,3]$	$\in\ 0.315 - \epsilon$
$A^{(1)}_{2,3}$	$= \frac{10}{32} \times \frac{\sqrt{15}}{10}$	$\in\ 0.3125 \times 0.387[2,3]$	$\in\ 0.121 + \epsilon$
$A^{(1)}_{3,3}$	$= 2^{-1/3} \times \frac{3}{16} \times \sqrt{1}$	$\in\ 0.79[3,4] \times 0.1875 \times 1$	$\in\ 0.149 - \epsilon$
$A^{(1)}_{3,4}$	$= 2^{-1/6} \times \frac{10}{64} \times \frac{\sqrt{14}}{10}$	$\in\ 0.890[8,9] \times 0.156[2,3] \times 0.37[4,5]$	$\in\ 0.052 + \epsilon$
$A^{(1)}_{4,4}$	$= \frac{10}{16} \times \frac{\sqrt{1}}{10}$	$\in\ 0.625 \times 0.1$	$\in\ 0.063 - \epsilon$
$A^{(1)}_{4,5}$	$= 2^{-1/3} \times \frac{30}{256} \times \frac{\sqrt{5}}{10}$	$\in\ 0.79[3,4] \times 0.11[7,8] \times 0.22[3,4]$	$\in\ 0.021 - \epsilon$

$A^{(1)}_{5,5}$	$= 2^{-1/6} \times \frac{50}{256} \times \frac{\sqrt{2}}{10}$	$\in 0.89[0,1] \times 0.195[3,4] \times 0.14[1,2]$	$\in 0.025 - \epsilon$
$A^{(1)}_{5,6}$	$= \frac{1000}{1024} \times \frac{\sqrt{66}}{1000}$	$\in 0.97[6,7] \times 0.0081[2,3]$	$\in 0.008 - \epsilon$
$A^{(1)}_{6,6}$	$= 2^{-1/3} \times \frac{30}{256} \times \frac{\sqrt{1}}{10}$	$\in 0.[79,80] \times 0.11[7,8] \times 0.1$	$\in 0.009 + \epsilon$
$A^{(2)}_{2,2}$	$= \frac{3}{16} \times \sqrt{6}$	$\in 0.1875 \times 2.4[49,50]$	$\in 0.459 + \epsilon$
$A^{(2)}_{2,3}$	$= 2^{-1/3} \times \frac{1}{8} \times \sqrt{10}$	$\in 0.793[7,8] \times 0.125 \times 3.16[2,3]$	$\in 0.314 - \epsilon$
$A^{(2)}_{2,4}$	$= 2^{-1/6} \times \frac{10}{64} \times \frac{\sqrt{35}}{10}$	$\in 0.89[0,1] \times 0.156[2,3] \times 0.59[1,2]$	$\in 0.082 + \epsilon$
$A^{(2)}_{3,3}$	$= 2^{-1/6} \times \frac{15}{64} \times \sqrt{3}$	$\in 0.890[89,90] \times 0.234[3,4] \times 1.73[2,3]$	$\in 0.362 - \epsilon$
$A^{(2)}_{3,4}$	$= \frac{10}{32} \times \frac{\sqrt{42}}{10}$	$\in 0.3125 \times 0.64[8,9]$	$\in 0.203 - \epsilon$
$A^{(2)}_{3,5}$	$= 2^{-1/3} \times \frac{100}{256} \times \frac{\sqrt{210}}{100}$	$\in 0.793[7,8] \times 0.390[6,7] \times 0.14[4,5]$	$\in 0.045 - \epsilon$
$A^{(2)}_{4,4}$	$= 2^{-1/3} \times \frac{7}{64} \times \sqrt{6}$	$\in 0.793[7,8] \times 0.1093[7,8] \times 2.4[49,50]$	$\in 0.213 - \epsilon$
$A^{(2)}_{4,5}$	$= 2^{-1/6} \times \frac{10}{32} \times \frac{\sqrt{15}}{10}$	$\in 0.890[8,9] \times 0.3125 \times 0.387[2,3]$	$\in 0.108 - \epsilon$
$A^{(2)}_{4,6}$	$= \frac{300}{1024} \times \frac{\sqrt{55}}{100}$	$\in 0.29[2,3] \times 0.07[4,5]$	$\in 0.022 - \epsilon$
$A^{(2)}_{5,5}$	$= \frac{450}{1024} \times \frac{\sqrt{6}}{10}$	$\in 0.4[39,40] \times 0.24[49,50]$	$\in 0.108 - \epsilon$
$A^{(2)}_{5,6}$	$= 2^{-1/3} \times \frac{50}{256} \times \frac{\sqrt{11}}{10}$	$\in 0.793[7,8] \times 0.195[3,4] \times 0.33[1,2]$	$\in 0.051 + \epsilon$
$A^{(2)}_{6,6}$	$= 2^{-1/6} \times \frac{330}{1024} \times \frac{\sqrt{3}}{10}$	$\in 0.890[8,9] \times 0.322[2,3] \times 0.17[3,4]$	$\in 0.050 - \epsilon$
$A^{(3)}_{3,3}$	$= \frac{5}{32} \times \sqrt{10}$	$\in 0.15625 \times 3.16[2,3]$	$\in 0.494 + \epsilon$
$A^{(3)}_{3,4}$	$= 2^{-1/3} \times \frac{150}{256} \times \frac{\sqrt{70}}{10}$	$\in 0.793[7,8] \times 0.58[59,60] \times 0.836[6,7]$	$\in 0.389 + \epsilon$
$A^{(3)}_{3,5}$	$= 2^{-1/6} \times \frac{90}{128} \times \frac{\sqrt{7}}{10}$	$\in 0.890[8,9] \times 0.70[3,4] \times 0.264[5,6]$	$\in 0.166 - \epsilon$
$A^{(3)}_{3,6}$	$= \frac{100}{512} \times \frac{\sqrt{231}}{100}$	$\in 0.195[3,4] \times 0.15[19,20]$	$\in 0.030 - \epsilon$
$A^{(3)}_{4,4}$	$= 2^{-1/6} \times \frac{7}{32} \times \sqrt{5}$	$\in 0.890[89,90] \times 0.21875 \times 2.23[6,7]$	$\in 0.436 - \epsilon$
$A^{(3)}_{4,5}$	$= \frac{105}{512} \times \sqrt{2}$	$\in 0.2050[7,8] \times 1.414[2,3]$	$\in 0.290 + \epsilon$
$A^{(3)}_{4,6}$	$= 2^{-1/3} \times \frac{30}{128} \times \frac{\sqrt{33}}{10}$	$\in 0.793[7,8] \times 0.234[3,4] \times 0.57[4,5]$	$\in 0.107 - \epsilon$
$A^{(3)}_{5,5}$	$= 2^{-1/3} \times \frac{15}{128} \times \sqrt{10}$	$\in 0.7937[0,1] \times 0.1171[8,9] \times 3.16[2,3]$	$\in 0.294 + \epsilon$
$A^{(3)}_{5,6}$	$= 2^{-1/6} \times \frac{450}{4096} \times \frac{\sqrt{330}}{10}$	$\in 0.890[8,9] \times 0.109[8,9] \times 1.81[6,7]$	$\in 0.178 - \epsilon$
$A^{(3)}_{6,6}$	$= \frac{550}{1024} \times \frac{\sqrt{10}}{10}$	$\in 0.537[1,2] \times 0.316[2,3]$	$\in 0.170 - \epsilon$
$A^{(4)}_{4,4}$	$= \frac{350}{512} \times \frac{\sqrt{70}}{10}$	$\in 0.683[5,6] \times 0.836[6,7]$	$\in 0.572 - \epsilon$
$A^{(4)}_{4,5}$	$= 2^{-1/3} \times \frac{21}{128} \times \sqrt{14}$	$\in 0.7937[0,1] \times 0.1640[6,7] \times 3.74[1,2]$	$\in 0.487 + \epsilon$
$A^{(4)}_{4,6}$	$= 2^{-1/6} \times \frac{70}{512} \times \frac{\sqrt{462}}{10}$	$\in 0.890[8,9] \times 0.1367[1,2] \times 2.1[49,50]$	$\in 0.262 - \epsilon$
$A^{(4)}_{5,5}$	$= 2^{-1/6} \times \frac{105}{1024} \times \sqrt{35}$	$\in 0.890[89,90] \times 0.1025[39,40] \times 5.916[0,1]$	$\in 0.540 + \epsilon$
$A^{(4)}_{5,6}$	$= \frac{30}{256} \times \frac{\sqrt{1155}}{10}$	$\in 0.1171[8,9] \times 3.39[8,9]$	$\in 0.398 + \epsilon$
$A^{(4)}_{6,6}$	$= 2^{-1/3} \times \frac{4950}{8192} \times \frac{\sqrt{70}}{10}$	$\in 0.793[7,8] \times 0.604[2,3] \times 0.836[6,7]$	$\in 0.401 + \epsilon$
$A^{(5)}_{5,5}$	$= \frac{189}{1024} \times \sqrt{14}$	$\in 0.18457[0,1] \times 3.741[6,7]$	$\in 0.691 - \epsilon$
$A^{(5)}_{5,6}$	$= 2^{-1/3} \times \frac{1050}{2048} \times \frac{\sqrt{231}}{10}$	$\in 0.7937[0,1] \times 0.512[69,70] \times 1.5198[6,7]$	$\in 0.618 + \epsilon$
$A^{(5)}_{6,6}$	$= 2^{-1/6} \times \frac{297}{1024} \times \sqrt{7}$	$\in 0.890[89,90] \times 0.2900[3,4] \times 2.645[7,8]$	$\in 0.684 - \epsilon$
$A^{(6)}_{6,6}$	$= \frac{2310}{4096} \times \frac{\sqrt{231}}{10}$	$\in 0.56[39,40] \times 1.519[8,9]$	$\in 0.857 + \epsilon$

Table 6. Definition of $Q^{(m)}_{k,n}$ and bound on $A^{(m)}_{k,n}$, for $0 \le m \le k \le n \le 6$.

In the third column of this table we have used an abbreviated notation for intervals by writing e.g. $0.29[2,3]$ instead of $[0.292, 0.293]$; and the product of two intervals $[x,y]$ and $[s,t]$ in $\mathbb{R}_+$ is defined to be the interval $[xs, yt]$. Now, in order to verify e.g. that $A^{(2)}_{4,6} \in 0.022 - \epsilon$, it suffices to check that $1024 \times 0.292 \le 300 \le 1024 \times 0.293$ and $(100 \times 0.074)^2 \le 55 \le (100 \times 0.075)^2$, and that $0.292 \times 0.074 \ge 0.0215$ and $0.293 \times 0.075 \le 0.022$.

5.4. Proof of (N4)

We first compute the matrix $Q(p)$, as defined in (3.25), using the values of p_m and $Q^{(m)}_{k,n}$ from Table 1 and Table 6, respectively.

$Q(p)_{k,n}=$	$Q^{(0)}_{k,n}p_0$	$+\ Q^{(1)}_{k,n}p_1$	$+\ Q^{(2)}_{k,n}p_2$	$+\ Q^{(3)}_{k,n}p_3$	$+\ Q^{(4)}_{k,n}p_4$	$+\ Q^{(5)}_{k,n}p_5$	$+\ Q^{(6)}_{k,n}p_6$	
$Q(p)_{0,0} =$	0.909000							= 0.909000
$Q(p)_{1,0} =$		0.154980						= 0.154980
$Q(p)_{1,1} =$	0.286335	+0.246000	+0.060993					= 0.593328
$Q(p)_{2,0} =$			0.024849					= 0.024849
$Q(p)_{2,1} =$		0.119556	+0.079065	+0.013552				= 0.212173
$Q(p)_{2,2} =$	0.089991	+0.154980	+0.115209	+0.035168	+0.003690			= 0.399038
$Q(p)_{3,0} =$				0.003472				= 0.003472
$Q(p)_{3,1} =$			0.030371	+0.016688	+0.002340			= 0.049399
$Q(p)_{3,2} =$		0.059532	+0.078814	+0.040544	+0.009135	+0.000720		= 0.188745
$Q(p)_{3,3} =$	0.028179	+0.073308	+0.090862	+0.055328	+0.017505	+0.002656	+0.000150	= 0.267988
$Q(p)_{4,0} =$					0.000450			= 0.000450
$Q(p)_{4,1} =$				0.005824	+0.002835	+0.000336		= 0.008995
$Q(p)_{4,2} =$			0.020582	+0.022736	+0.009585	+0.001728	+0.000110	= 0.054741
$Q(p)_{4,3} =$		0.025584	+0.050953	+0.043568	+0.019620	+0.004640	+0.000535	= 0.144900
$Q(p)_{4,4} =$	0.009090	+0.030996	+0.053463	+0.048832	+0.025740	+0.007792	+0.001310	= 0.177223
$Q(p)_{5,0} =$						0.000048		= 0.000048
$Q(p)_{5,1} =$					0.000945	+0.000400	+0.000040	= 0.001385
$Q(p)_{5,2} =$				0.005040	+0.004860	+0.001728	+0.000255	= 0.011883
$Q(p)_{5,3} =$			0.011295	+0.018592	+0.013050	+0.004704	+0.000890	= 0.048531
$Q(p)_{5,4} =$		0.010332	+0.027108	+0.032480	+0.021915	+0.008640	+0.001990	= 0.102465
$Q(p)_{5,5} =$	0.002727	+0.012300	+0.027108	+0.032928	+0.024300	+0.011056	+0.003090	= 0.113509
$Q(p)_{6,0} =$							0.000005	= 0.000005
$Q(p)_{6,1} =$						0.000128	+0.000045	= 0.000173
$Q(p)_{6,2} =$					0.000990	+0.000816	+0.000250	= 0.002056
$Q(p)_{6,3} =$				0.003360	+0.004815	+0.002848	+0.000850	= 0.011873
$Q(p)_{6,4} =$			0.005522	+0.011984	+0.011790	+0.006368	+0.002005	= 0.037669
$Q(p)_{6,5} =$		0.003936	+0.012801	+0.019936	+0.017910	+0.009888	+0.003420	= 0.067891
$Q(p)_{6,6} =$	0.000909	+0.004428	+0.012550	+0.019040	+0.018045	+0.010944	+0.004285	= 0.070201

Table 7. The symmetric matrix $Q(p)$.

Let T be the symmetric matrix $T = 2Q(p) - \mathrm{I}$. Using the values $Q(p)_{k,n}$ given above, we obtain the following for T.

$$T = \begin{pmatrix} 0.818000 & 0.309960 & 0.049698 & 0.006944 & 0.000900 & 0.000096 & 0.000010 \\ \cdot & 0.186656 & 0.424346 & 0.098798 & 0.017990 & 0.002770 & 0.000346 \\ \cdot & \cdot & -0.201924 & 0.377490 & 0.109482 & 0.023766 & 0.004112 \\ \cdot & \cdot & \cdot & -0.464024 & 0.289800 & 0.097062 & 0.023746 \\ \cdot & \cdot & \cdot & \cdot & -0.645554 & 0.204930 & 0.075338 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -0.772982 & 0.135782 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -0.859598 \end{pmatrix}$$

The verification of (N4) is now reduced to the straightforward (but tedious) task of computing the product MT, where M is the matrix given at the beginning of this section.

(k,n)	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(0,4)$	$(0,5)$	$(0,6)$
$M_{k,0}T_{0,n}$	0.861354000	0.326387880	0.052331994	0.007312032	0.000947700	0.000101088	0.000010530
$M_{k,1}T_{1,n}$	0.184116240	0.110873664	0.252061524	0.058686012	0.010686060	0.001645380	0.000205524
$M_{k,2}T_{2,n}$	−0.038168064	−0.325897728	0.155077632	−0.289912320	−0.084082176	−0.018252288	−0.003158016
$M_{k,3}T_{3,n}$	−0.006902336	−0.098205212	−0.375225060	0.461239856	−0.288061200	−0.096479628	−0.023603524
$M_{k,4}T_{4,n}$	−0.000619200	−0.012377120	−0.075323616	−0.199382400	0.444141152	−0.140991840	−0.051832544
$M_{k,5}T_{5,n}$	−0.000033984	−0.000980580	−0.008413164	−0.034359948	−0.072545220	0.273635628	−0.048066828
$M_{k,6}T_{6,n}$	−0.000001470	−0.000050862	−0.000604464	−0.003490662	−0.011074686	−0.019959954	0.126360906
$(MT)_{k,n}$	0.999745186	−0.000249958	−0.000095154	0.000092570	0.000011630	−0.000301614	−0.000083952

(k,n)	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
$M_{k,0}T_{0,n}$	0.485892000	0.184116240	0.029520612	0.004124736	0.000534600	0.000057024	0.000005940
$M_{k,1}T_{1,n}$	−0.609691320	−0.367152352	−0.834688582	−0.194335666	−0.035386330	−0.005448590	−0.000680582
$M_{k,2}T_{2,n}$	0.103670028	0.885185756	−0.421213464	0.787444140	0.228379452	0.049575876	0.008577632
$M_{k,3}T_{3,n}$	0.018415488	0.262012296	1.001103480	−1.230591648	0.768549600	0.257408424	0.062974392
$M_{k,4}T_{4,n}$	0.001653300	0.033047630	0.201118434	0.532362600	−1.185882698	0.376456410	0.138395906
$M_{k,5}T_{5,n}$	0.000090816	0.002620420	0.022482636	0.091820652	0.193863780	−0.731240972	0.128449772
$M_{k,6}T_{6,n}$	0.000003930	0.000135978	0.001616016	0.009332178	0.029607834	0.053362326	−0.337822014
$(MT)_{k,n}$	0.000034242	0.999965968	−0.000060868	0.000156992	−0.000333762	0.000170498	−0.000098954

(k,n)	$(2,0)$	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$M_{k,0}T_{0,n}$	−0.628224000	−0.238049280	−0.038168064	−0.005332992	−0.000691200	−0.000073728	−0.000007680
$M_{k,1}T_{1,n}$	0.646576560	0.389364416	0.885185756	0.206092628	0.037527140	0.005778220	0.000721756
$M_{k,2}T_{2,n}$	−0.019829502	−0.169314054	0.080567676	−0.150618510	−0.043683318	−0.009482634	−0.001640688
$M_{k,3}T_{3,n}$	0.001152704	0.016400468	0.062663340	−0.077027984	0.048106800	0.016112292	0.003941836
$M_{k,4}T_{4,n}$	0.000071100	0.001421210	0.008649078	0.022894200	−0.050998766	0.016189470	0.005951702
$M_{k,5}T_{5,n}$	0.000003840	0.000110800	0.000950640	0.003882480	0.008197200	−0.030919280	0.005431280
$M_{k,6}T_{6,n}$	0.000000170	0.000005882	0.000069904	0.000403682	0.001280746	0.002308294	−0.014613166
$(MT)_{k,n}$	−0.000249128	−0.000060558	0.999918330	0.000293504	−0.000261398	−0.000087366	−0.000214960

(k,n)	$(3,0)$	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
$M_{k,0}T_{0,n}$	−0.813092000	−0.308100240	−0.049399812	−0.006902336	−0.000894600	−0.000095424	−0.000009940
$M_{k,1}T_{1,n}$	0.822013920	0.495011712	1.125365592	0.262012296	0.047709480	0.007346040	0.000917592
$M_{k,2}T_{2,n}$	0.008249868	0.070441436	−0.033519384	0.062663340	0.018174012	0.003945156	0.000682592
$M_{k,3}T_{3,n}$	−0.016457280	−0.234151260	−0.894651300	1.099736880	−0.686826000	−0.230036940	−0.056278020
$M_{k,4}T_{4,n}$	−0.001080900	−0.021605990	−0.131487882	−0.348049800	0.775310354	−0.246120930	−0.090480938
$M_{k,5}T_{5,n}$	−0.000062304	−0.001797730	−0.015424134	−0.062993238	−0.132999570	0.501665318	−0.088122518
$M_{k,6}T_{6,n}$	−0.000002710	−0.000093766	−0.001114352	−0.006435166	−0.020416598	−0.036796922	0.232951058
$(MT)_{k,n}$	−0.000431406	−0.000295838	−0.000231272	1.000031976	0.000057078	−0.000093702	−0.000340174

(k,n)	$(4,0)$	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
$M_{k,0}T_{0,n}$	−0.562784000	−0.213252480	−0.034192224	−0.004777472	−0.000619200	−0.000066048	−0.000006880
$M_{k,1}T_{1,n}$	0.569396520	0.342887072	0.779523602	0.181491926	0.033047630	0.005088490	0.000635602
$M_{k,2}T_{2,n}$	0.003926142	0.033523334	−0.015951996	0.029821710	0.008649078	0.001877514	0.000324848
$M_{k,3}T_{3,n}$	−0.008339744	−0.118656398	−0.453365490	0.557292824	−0.348049800	−0.116571462	−0.028518946
$M_{k,4}T_{4,n}$	−0.002096100	−0.041898710	−0.254983578	−0.674944200	1.503495266	−0.477281970	−0.175462202
$M_{k,5}T_{5,n}$	−0.000079104	−0.002282480	−0.019583184	−0.079979088	−0.168862320	0.636937168	−0.111884368
$M_{k,6}T_{6,n}$	−0.000003660	−0.000126636	−0.001504992	−0.008691036	−0.027573708	−0.049696212	0.314612868
$(MT)_{k,n}$	0.000020054	0.000193702	−0.000057862	0.000214664	1.000086946	0.000287480	−0.000299078

(k,n)	$(5,0)$	$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
$M_{k,0}T_{0,n}$	−0.289572000	−0.109725840	−0.017593092	−0.002458176	−0.000318600	−0.000033984	−0.000003540
$M_{k,1}T_{1,n}$	0.293222160	0.176576576	0.401431316	0.093462908	0.017018540	0.002620420	0.000327316
$M_{k,2}T_{2,n}$	0.001987920	0.016973840	−0.008076960	0.015099600	0.004379280	0.000950640	0.000164480
$M_{k,3}T_{3,n}$	−0.004506656	−0.064119902	−0.244991010	0.301151576	−0.188080200	−0.062993238	−0.015411154
$M_{k,4}T_{4,n}$	−0.000741600	−0.014823760	−0.090213168	−0.238795200	0.531936496	−0.168862320	−0.062078512
$M_{k,5}T_{5,n}$	−0.000158496	−0.004573270	−0.039237666	−0.160249362	−0.338339430	1.276193282	−0.224176082
$M_{k,6}T_{6,n}$	−0.000003500	−0.000121100	−0.001439200	−0.008311100	−0.026368300	−0.047523700	0.300859300
$(MT)_{k,n}$	0.000227828	0.000186544	−0.000119780	−0.000099754	0.000227786	1.000351100	−0.000318192

(k,n)	$(6,0)$	$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$
$M_{k,0}T_{0,n}$	−0.120246000	−0.045564120	−0.007305606	−0.001020768	−0.000132300	−0.000014112	−0.000001470
$M_{k,1}T_{1,n}$	0.121814280	0.073355808	0.166767978	0.038827614	0.007070070	0.001088610	0.000135978
$M_{k,2}T_{2,n}$	0.000844866	0.007213882	−0.003432708	0.006417330	0.001861194	0.000404022	0.000069904
$M_{k,3}T_{3,n}$	−0.001881824	−0.026774258	−0.102299790	0.125750504	−0.078535800	−0.026303802	−0.006435166
$M_{k,4}T_{4,n}$	−0.000329400	−0.006584340	−0.040070412	−0.106066800	0.236272764	−0.075004380	−0.027573708
$M_{k,5}T_{5,n}$	−0.000033600	−0.000969500	−0.008318100	−0.033971700	−0.071725500	0.270543700	−0.047523700
$M_{k,6}T_{6,n}$	−0.000012580	−0.000435268	−0.005172896	−0.029872468	−0.094775204	−0.170813756	1.081374284
$(MT)_{k,n}$	0.000155742	0.000242204	0.000168466	0.000063712	0.000035224	−0.000099718	1.000046122

Table 8. The matrix product MT.

By looking at the values of $(MT)_{k,n}$ in Table 8, it is easy to see that $|\delta_{k,n} - (MT)_{k,n}| \leq 0.0005$, for $0 \leq k, n \leq 6$. This completes the proof of (N4).

5.5. Proof of (N5)

We begin with the matrix $Q(\delta)$, as defined in (3.25), using the values of δ_m and $Q^{(m)}_{k,n}$ from Table 1 and Table 6, respectively.

$Q(\delta)_{k,n}$	$= Q^{(0)}_{k,n}\delta_0$	$+Q^{(1)}_{k,n}\delta_1$	$+Q^{(2)}_{k,n}\delta_2$	$+Q^{(3)}_{k,n}\delta_3$	$+Q^{(4)}_{k,n}\delta_4$	$+Q^{(5)}_{k,n}\delta_5$	$+Q^{(6)}_{k,n}\delta_6$	
$Q(\delta)_{0,0}$	= 0.005000							= 0.005000
$Q(\delta)_{1,0}$	=	0.003150						= 0.003150
$Q(\delta)_{1,1}$	= 0.001575	+0.005000	+0.000972					= 0.007547
$Q(\delta)_{2,0}$	=		0.000396					= 0.000396
$Q(\delta)_{2,1}$	=	0.002430	+0.001260	+0.000968				= 0.004658
$Q(\delta)_{2,2}$	= 0.000495	+0.003150	+0.001836	+0.002512	+0.000574			= 0.008567
$Q(\delta)_{3,0}$	=			0.000248				= 0.000248
$Q(\delta)_{3,1}$	=		0.000484	+0.001192	+0.000364			= 0.002040
$Q(\delta)_{3,2}$	=	0.001210	+0.001256	+0.002896	+0.001421	+0.000225		= 0.007008
$Q(\delta)_{3,3}$	= 0.000155	+0.001490	+0.001448	+0.003952	+0.002723	+0.000830	+0.000090	= 0.010688
$Q(\delta)_{4,0}$	=				0.000070			= 0.000070
$Q(\delta)_{4,1}$	=			0.000416	+0.000441	+0.000105		= 0.000962
$Q(\delta)_{4,2}$	=		0.000328	+0.001624	+0.001491	+0.000540	+0.000066	= 0.004049
$Q(\delta)_{4,3}$	=	0.000520	+0.000812	+0.003112	+0.003052	+0.001450	+0.000321	= 0.009267
$Q(\delta)_{4,4}$	= 0.000050	+0.000630	+0.000852	+0.003488	+0.004004	+0.002435	+0.000786	= 0.012245
$Q(\delta)_{5,0}$	=					0.000015		= 0.000015
$Q(\delta)_{5,1}$	=				0.000147	+0.000125	+0.000024	= 0.000296
$Q(\delta)_{5,2}$	=			0.000360	+0.000756	+0.000540	+0.000153	= 0.001809
$Q(\delta)_{5,3}$	=		0.000180	+0.001328	+0.002030	+0.001470	+0.000534	= 0.005542
$Q(\delta)_{5,4}$	=	0.000210	+0.000432	+0.002320	+0.003409	+0.002700	+0.001194	= 0.010265
$Q(\delta)_{5,5}$	= 0.000015	+0.000250	+0.000432	+0.002352	+0.003780	+0.003455	+0.001854	= 0.012138
$Q(\delta)_{6,0}$	=						0.000003	= 0.000003
$Q(\delta)_{6,1}$	=					0.000040	+0.000027	= 0.000067
$Q(\delta)_{6,2}$	=				0.000154	+0.000255	+0.000150	= 0.000559
$Q(\delta)_{6,3}$	=			0.000240	+0.000749	+0.000890	+0.000510	= 0.002389
$Q(\delta)_{6,4}$	=		0.000088	+0.000856	+0.001834	+0.001990	+0.001203	= 0.005971
$Q(\delta)_{6,5}$	=	0.000080	+0.000204	+0.001424	+0.002786	+0.003090	+0.002052	= 0.009636
$Q(\delta)_{6,6}$	= 0.000005	+0.000090	+0.000200	+0.001360	+0.002807	+0.003420	+0.002571	= 0.010453

Table 9. The symmetric matrix $Q(\delta)$.

Then, from the data in Tables 1,7,9, we can determine the two vectors $Q(p)p$ and $Q(\delta)\delta$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$Q(p)_{k,0}\,p_0$	0.826281000	0.140876820	0.022587741	0.003156048	0.000409050	0.000043632	0.000004545
$Q(p)_{k,1}\,p_1$	0.076250160	0.291917376	0.104389116	0.024304308	0.004425540	0.000681420	0.000085116
$Q(p)_{k,2}\,p_2$	0.006237099	0.053255423	0.100158538	0.047374995	0.013739991	0.002982633	0.000516056
$Q(p)_{k,3}\,p_3$	0.000388864	0.005532688	0.021139440	0.030014656	0.016228800	0.005435472	0.001329776
$Q(p)_{k,4}\,p_4$	0.000020250	0.000404775	0.002463345	0.006520500	0.007975035	0.004610925	0.001695105
$Q(p)_{k,5}\,p_5$	0.000000768	0.000022160	0.000190128	0.000776496	0.001639440	0.001816144	0.001086256
$Q(p)_{k,6}\,p_6$	0.000000025	0.000000865	0.000010280	0.000059365	0.000188345	0.000339455	0.000351005
$(Q(p)p)_k$	0.909178166	0.492010107	0.250938588	0.112206368	0.044606201	0.015909681	0.005067859
	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$Q(\delta)_{k,0}\,\delta_0$	0.000025000	0.000015750	0.000001980	0.000001240	0.000000350	0.000000075	0.000000015
$Q(\delta)_{k,1}\,\delta_1$	0.000031500	0.000075470	0.000046580	0.000020400	0.000009620	0.000002960	0.000000670
$Q(\delta)_{k,2}\,\delta_2$	0.000001584	0.000018632	0.000034268	0.000028032	0.000016196	0.000007236	0.000002236
$Q(\delta)_{k,3}\,\delta_3$	0.000001984	0.000016320	0.000056064	0.000085504	0.000074136	0.000044336	0.000019112
$Q(\delta)_{k,4}\,\delta_4$	0.000000490	0.000006734	0.000028343	0.000064869	0.000085715	0.000071855	0.000041797
$Q(\delta)_{k,5}\,\delta_5$	0.000000075	0.000001480	0.000009045	0.000027710	0.000051325	0.000060690	0.000048180
$Q(\delta)_{k,6}\,\delta_6$	0.000000009	0.000000201	0.000001677	0.000007167	0.000017913	0.000028908	0.000031359
$(Q(\delta)\delta)_k$	0.000060642	0.000134587	0.000177957	0.000234922	0.000255255	0.000216060	0.000143369

Table 10. The vectors $Q(p)p$ and $Q(\delta)\delta$.

By computing all products $g_m g_n$, for $0 \le m \le n \le 6$, using Table 1, and adding them up according to the definition (3.26) of u, we find the following values for $2000u_k$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
g_0g_0	0.835396						
$2g_0g_1$		0.917656					
$2g_0g_2$			0.466140				
$2g_0g_3$				0.219360			
$2g_0g_4$					0.095056		
$2g_0g_5$						0.038388	
$2g_0g_6$							0.014624
g_1g_1	0.252004	0.252004	0.252004				
$2g_1g_2$		0.256020	0.256020	0.256020			
$2g_1g_3$			0.120480	0.120480	0.120480		
$2g_1g_4$				0.052208	0.052208	0.052208	
$2g_1g_5$					0.021084	0.021084	0.021084
$2g_1g_6$						0.008032	0.008032
g_2g_2	0.065025	0.065025	0.065025	0.065025	0.065025		
$2g_2g_3$		0.061200	0.061200	0.061200	0.061200	0.061200	
$2g_2g_4$			0.026520	0.026520	0.026520	0.026520	0.026520
$2g_2g_5$				0.010710	0.010710	0.010710	0.010710
$2g_2g_6$					0.004080	0.004080	0.004080
g_3g_3	0.014400	0.014400	0.014400	0.014400	0.014400	0.014400	0.014400
$2g_3g_4$		0.012480	0.012480	0.012480	0.012480	0.012480	0.012480
$2g_3g_5$			0.005040	0.005040	0.005040	0.005040	0.005040
$2g_3g_6$				0.001920	0.001920	0.001920	0.001920
g_4g_4	0.002704	0.002704	0.002704	0.002704	0.002704	0.002704	0.002704
$2g_4g_5$		0.002184	0.002184	0.002184	0.002184	0.002184	0.002184
$2g_4g_6$			0.000832	0.000832	0.000832	0.000832	0.000832
g_5g_5	0.000441	0.000441	0.000441	0.000441	0.000441	0.000441	0.000441
$2g_5g_6$		0.000336	0.000336	0.000336	0.000336	0.000336	0.000336
g_6g_6	0.000064	0.000064	0.000064	0.000064	0.000064	0.000064	0.000064
$2000u_k$	1.170034	1.584514	1.285870	0.851924	0.496764	0.262623	0.125451

Table 11. The vector $2000u$.

In order to bound the vector $10\rho^7 v$, we use again the components of g from Table 1, together with the bounds on $A_{k,7}^{(m)}$ from Table 2.

$$
\begin{array}{lcr}
10\rho^7 v_k & = & A_{k,7}^{(1)} g_1 \;+ A_{k,7}^{(2)} g_2 \;+ A_{k,7}^{(3)} g_3 \;+ A_{k,7}^{(4)} g_4 \;+ A_{k,7}^{(5)} g_5 \;+ A_{k,7}^{(6)} g_6 \\
10\rho^7 v_1 & \le & 0.000024 \le 0.000024 \\
10\rho^7 v_2 & \le & 0.000210+0.000184 \le 0.000394 \\
10\rho^7 v_3 & \le & 0.000884+0.001260+0.000776 \le 0.002920 \\
10\rho^7 v_4 & \le & 0.002040+0.004212+0.003969+0.002168 \le 0.012389 \\
10\rho^7 v_5 & \le & 0.002550+0.007200+0.009828+0.008043+0.004320 \le 0.031941 \\
10\rho^7 v_6 & \le & 0.001506 +0.005865+0.011640+0.014092+0.011340+0.006360 \le 0.050803
\end{array}
$$

Table 12. Bound on the vector $10\rho^7 v$.

Next, we consider the vector $\lambda \equiv |Q(p)p - p| + Q(\delta)\delta + u + 2v + w$. The components $w_k = 0.01\rho^{-7}\rho^{-7}A_{k,7}^{(7)}$ of w are easy to bound using Table 1 and Table 2; only two digit precision is needed. We also have to compute $|Q(p)p - p|$,u, and $2v$, using the data from Table 1 and Tables 10-12. The vector $Q(\delta)\delta$ can be found in Table 10. Putting it all together, we obtain the following.

$$
\begin{array}{llllllll}
\lambda_k & \equiv & |Q(p)p-p|_k + & (Q(\delta)\delta)_k \;+ & u_k \;+ & 2v_k \;+ & w_k & \\
\lambda_0 & \le & 0.000179 & +\,0.000061 & +\,0.000586 & +\,0.000000 & +\,0.000001 & = 0.000827 \\
\lambda_1 & \le & 0.000011 & +\,0.000135 & +\,0.000793 & +\,0.000002 & +\,0.000001 & = 0.000942 \\
\lambda_2 & \le & 0.000062 & +\,0.000178 & +\,0.000643 & +\,0.000006 & +\,0.000001 & = 0.000890 \\
\lambda_3 & \le & 0.000207 & +\,0.000235 & +\,0.000426 & +\,0.000034 & +\,0.000003 & = 0.000905 \\
\lambda_4 & \le & 0.000394 & +\,0.000256 & +\,0.000249 & +\,0.000138 & +\,0.000008 & = 0.001045 \\
\lambda_5 & \le & 0.000091 & +\,0.000217 & +\,0.000132 & +\,0.000352 & +\,0.000017 & = 0.000809 \\
\lambda_6 & \le & 0.000068 & +\,0.000144 & +\,0.000063 & +\,0.000560 & +\,0.000027 & = 0.000862
\end{array}
$$

Table 13. Bound on the vector λ.

The product of the matrix $|M|$ with the above bound on λ (multiplied by 1000) is given in the next table.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$1000\|M\|_{k,0}\lambda_0$	0.870831	0.491238	0.635136	0.822038	0.568976	0.292758	0.121569
$1000\|M\|_{k,1}\lambda_1$	0.559548	1.852914	1.965012	2.498184	1.730454	0.891132	0.370206
$1000\|M\|_{k,2}\lambda_2$	0.683520	1.856540	0.355110	0.147740	0.070310	0.035600	0.015130
$1000\|M\|_{k,3}\lambda_3$	0.899570	2.400060	0.150230	2.144850	1.086905	0.587345	0.245255
$1000\|M\|_{k,4}\lambda_4$	0.718960	1.919665	0.082555	1.255045	2.433805	0.861080	0.382470
$1000\|M\|_{k,5}\lambda_5$	0.286386	0.765314	0.032360	0.525041	0.666616	1.335659	0.283150
$1000\|M\|_{k,6}\lambda_6$	0.126714	0.338766	0.014654	0.233602	0.315492	0.301700	1.084396
$1000(\|M\|\lambda)_k$	4.145529	9.624497	3.235057	7.626500	6.872558	4.305274	2.502176

Table 14. Bound on the product $1000|M|\lambda$.

By using these bounds, and the values of δ_k from Table 1, it is now easy to check that $(|M|\lambda)_k + 3\times 10^{-5} \le \delta_k$, for $0 \le k \le 6$. This completes the proof of (N5).

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