Geometric Stability Analysis for Periodic Solutions of the Swift-Hohenberg Equation

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Abstract. In this paper we describe invariant geometrical structures in the phase space of the Swift-Hohenberg equation in a neighborhood of its periodic stationary states. We show that in spite of the fact that these states are only marginally stable (*i.e.*, the linearized problem about these states has continuous spectrum extending all the way up to zero), there exist finite dimensional invariant manifolds in the phase space of this equation which determine the long-time behavior of solutions near these stationary solutions. In particular, using this point of view, we obtain a new demonstration of Schneider's recent proof that these states are nonlinearly stable.

1. Introduction

In this paper, we study the non-linear stability of space-periodic, time-independent solutions of the Swift-Hohenberg equation

$$\partial_t u \;=\; \bigl(\varepsilon^2 - (1+\partial_x^2)^2\bigr)u - u^3 \;. \tag{1.1}$$

Here, $u(x,t)$ is defined on $\mathbf{R}\times\mathbf{R}^+$ and takes real values and $\varepsilon \geq 0$ is a small parameter. The Eq.(1.1) has stationary solutions $u(x,t) = u_{\varepsilon,\omega}(x)$ which are of the form

$$u_{\varepsilon,\omega}(x) \;=\; \sum_{n\in\mathbf{Z}} u_{\varepsilon,\omega,n} e^{i\omega n x} \;. \tag{1.2}$$

The non-linear stability problem addresses the question of the time evolution of initial data which are close to $u_{\varepsilon,\omega}$, and stability in this context means that the solution converges to $u_{\varepsilon,\omega}$ as $t\to\infty$. The range of possible values of ω is given by $\varepsilon^2 > (1-\omega^2)^2$ when ω is close to 1. To simplify the exposition we shall concentrate on the case $\omega = 1$, and omit henceforth the index ω.

In a very interesting paper, G. Schneider [Sch] has solved this problem, and the present work relies heavily on his ideas. Our aim is to simplify somewhat the exposition of [Sch] and to extend the result by giving a more precise asymptotic analysis, using the description of the asymptotic behavior in terms of a continuous renormalization group and invariant manifolds as introduced in [W], see below.

The *existence* of solutions of the form Eq.(1.2) is a well-established fact, (see *e.g.* [CE]) and we repeat here only those points of the discussion which are needed in the sequel. The equation for the stationary solution is $F(u,\varepsilon) = 0$, where

$$F(u,\varepsilon) \;\equiv\; \bigl(\varepsilon^2 - (1+\partial_x^2)^2\bigr)u - u^3 \;. \tag{1.3}$$

The equation $F = 0$ has the trivial solution $u = 0$, $\varepsilon = 0$. Linearizing around this solution, we see that $\mathrm{D}F$ equals

$$\mathrm{D}F \;=\; -(1+\partial_x^2)^2 \oplus 0 \;,$$

acting on some weighted subspace of $L^2(\mathbf{R})\oplus\mathbf{R}$. The null space of $\mathrm{D}F$ is spanned by

$$\{\cos x, \sin x\}\oplus 0 \quad \text{and} \quad 0\oplus 1 \;, \tag{1.4}$$

and thus, bifurcation theory suggests the existence of solutions of the form of Eq.(1.2), when $\varepsilon \neq 0$. This is indeed what happens (*cf.* [CR], [CE]), and the higher frequency terms in Eq.(1.2) are generated from the basis Eq.(1.4) by the non-linearity u^3. The method clearly extends to similar polynomial non-linearities. An explicit calculation shows that $F(u_\varepsilon,\varepsilon) = 0$ for

$$u_\varepsilon(x) \;=\; \varepsilon\frac{2}{\sqrt{3}}\cos(x) + \varepsilon^2 h_\varepsilon(x) \;, \tag{1.5}$$

and $h_\varepsilon(x) = h_\varepsilon(x + 2\pi)$. Thus, the function u_ε equals $u_{\varepsilon,1}$ of Eq.(1.2). We have broken the translation invariance of the problem by the choice of cos in Eq.(1.5), instead of, say, sin.

We next pass to the *linear stability analysis* of the solution u_ε. This is again a classical subject, initiated by Eckhaus [E], which we summarize for convenience, see also [CE]. Linearizing Eq.(1.1) around the solution u_ε we are led to study the operator $L_\varepsilon = \big(\varepsilon^2 - (1+\partial_x^2)^2\big) - 3u_\varepsilon^2$, that is,

$$\big(L_\varepsilon v\big)(x) \;=\; \big(\varepsilon^2 - 3u_\varepsilon^2(x)\big)v(x) - (1+\partial_x^2)^2 v(x) \;.$$

Because u_ε is a 2π periodic function, it is most convenient to work in Floquet coordinates (*i.e.*, with Bloch waves). To fix the notation, we give some details: Begin by introducing the following representation for $f \in L^2(\mathbf{R})$:

$$\begin{aligned} f(x) \;&=\; \int dk e^{-ikx}\hat f(k) \;=\; \sum_{m\in\mathbf{Z}} \int_{-1/2}^{1/2} d\ell\, e^{-imx} e^{-i\ell x} \hat f(m+\ell) \\ &=\; \int_{-1/2}^{1/2} d\ell\, e^{-i\ell x} \tilde f_\ell(x) \;, \end{aligned}$$

where

$$\tilde f_\ell(x) \;=\; \sum_{m\in\mathbf{Z}} e^{-imx}\hat f(m+\ell) \;. \tag{1.6}$$

Properties of $\tilde f$. Note first that $\tilde f_\ell$ is 2π periodic. Furthermore, the definition of $\tilde f_\ell(x)$ can be extended to all $\ell \in \mathbf{R}$ by the definition

$$\tilde f_{\ell+1}(x) \;=\; e^{-ix}\tilde f_\ell(x) \;.$$

We next observe that if f has a smooth, rapidly decaying Fourier transform, then $\tilde f_\ell(x)$ will also be a smooth function of ℓ and x. If f, g are in $L^2(\mathbf{R})$, then it follows from the definition of $\tilde f_\ell$ that

$$(fg)_\ell^{\sim}(x) \;=\; \int_{-1/2}^{1/2} dk\, \tilde f_{\ell-k}(x)\, \tilde g_k(x) \;. \tag{1.7}$$

We finally note that if s is a 2π periodic function, then

$$\tilde s_\ell(x) \;=\; \delta(\ell) s(x) \;. \tag{1.8}$$

It is now easy to see that

$$\big(L_\varepsilon v\big)_\ell^{\sim}(x) \;=\; \big(\varepsilon^2 - (1+(i\ell+\partial_x)^2)^2\big)\tilde v_\ell(x) - 3(u_\varepsilon^2 v)_\ell^{\sim}(x) \;.$$

In the language of condensed matter physics, ℓ is the quasi-momentum in the "Brillouin zone" $[-\frac12, \frac12]$ and L_ε leaves the subspace $\mathcal{F}_\ell$ of functions with quasi-momentum ℓ invariant. Using the properties just described, we get

$$\big(L_\varepsilon v\big)_\ell^{\sim}(x) \;=\; \big(\varepsilon^2 - (1+(i\ell+\partial_x)^2)^2\big)\tilde v_\ell(x) - 3u_\varepsilon^2(x)\cdot\tilde v_\ell(x) \;\equiv\; \big(L_{\varepsilon,\ell} v_\ell\big)(x) \;. \tag{1.9}$$

To fix the notation, we repeat the calculation done by Eckhaus, *cf.* also [CE], [M]. We denote $c(x) = \cos(x)$, $s(x) = \sin(x)$. The method of Eckhaus consists in projecting the eigenvalue problem for $L_{\varepsilon,\ell}$ onto the subspace spanned by the "bifurcating directions" c and s. Observe that, modulo higher frequency terms, we have $c^3 = \frac{3}{4}c$, $c^2 s = \frac{1}{4}s$, and therefore the projection of $L_{\varepsilon,\ell}$ onto this subspace is described by the matrix

$$\begin{pmatrix} -4\ell^2 - \ell^4 - 2\varepsilon^2 + \mathcal{O}(\varepsilon^4) & -4i\ell^3 \\ 4i\ell^3 & -4\ell^2 - \ell^4 \end{pmatrix} + \mathcal{O}(\varepsilon^4) \begin{pmatrix} \mathcal{O}(\ell^2) & \mathcal{O}(\ell) \\ \mathcal{O}(\ell) & \mathcal{O}(\ell^2) \end{pmatrix} .$$

The eigenvalues of this matrix are

$$\begin{aligned} \lambda^0_{\ell,0} &= -\big(4 + \mathcal{O}(\varepsilon^2)\big)\ell^2 + \mathcal{O}(\ell^3) , \\ \lambda^0_{\ell,1} &= -2\big(\varepsilon^2 + \mathcal{O}(\varepsilon^4)\big) - \big(4 + \mathcal{O}(\varepsilon^2)\big)\ell^2 + \mathcal{O}(\ell^3) + \mathcal{O}(\ell^4 + \varepsilon^4) . \end{aligned}$$

Thus, the restriction of $L_{\varepsilon,\ell}$ on the subspace spanned by c and s has its spectrum in the left half-plane. Note that the corresponding eigenvectors are $s + \mathcal{O}(\ell + \varepsilon)$ and $c + \mathcal{O}(\ell + \varepsilon)$. Extending this calculation to the full space, one shows in the same way [E, CE, M] that

Theorem 1.1. *For sufficiently small $\varepsilon > 0$ the operators $L_{\varepsilon,\ell}$, with $\ell \in [-\frac{1}{2}, \frac{1}{2}]$ are selfadjoint on the Sobolev space H^4, have compact resolvent and a spectrum satisfying*

$$\begin{aligned} \lambda_{\ell,0}(\varepsilon^2) &= -\big(4 + \mathcal{O}(\varepsilon^2)\big)\ell^2 + \mathcal{O}(\ell^3) \equiv -c_0(\varepsilon^2)\ell^2 + \mathcal{O}(\ell^3) , \\ \lambda_{\ell,1}(\varepsilon^2) &= -2\big(\varepsilon^2 + \mathcal{O}(\varepsilon^4)\big) - \big(4 + \mathcal{O}(\varepsilon^2)\big)\ell^2 + \mathcal{O}(\ell^3) , \\ \lambda_{\ell,j} &\leq -(1 - j^2)^2 + \mathcal{O}(\varepsilon^2) , \quad j = 2, 3, \dots . \end{aligned} \tag{1.10}$$

Notation. Since we mostly concentrate on the branch 0, we shall abbreviate $\lambda_\ell = \lambda_{\ell,0}(\varepsilon^2)$. The eigenfunction corresponding to λ_ℓ is

$$\varphi_{\varepsilon,\ell}(x) = \text{const.}\,\big(u'_\varepsilon(x) + i\ell g_\varepsilon(x) + h_{\varepsilon,\ell}(x)\ell^2 \big) , \tag{1.11}$$

where u_ε is the stationary solution, and both g_ε and $h_{\ell,\varepsilon}$ are 2π periodic. If we choose the constant to normalize the L^2 norm of $\varphi_{\varepsilon,\ell}$ to 1, then $\varphi_{\varepsilon,\ell} = \pi^{-1/2}\sin(x) + \mathcal{O}(\varepsilon + |\ell|)$.

We can now formulate the main question of this paper: Having seen that the solution u_ε is linearly (marginally) stable, is it true that this solution is *stable* under the non-linear evolution? The answer will be affirmative. As pointed out by Schneider [Sch], the result is not obvious, since the leading non-linear term does not have a sign. Indeed, the non-linear evolution equation for a (small) perturbation of u_ε is

$$\partial_t v = -(1 + \partial_x^2)^2 v + \varepsilon^2 v - 3u_\varepsilon^2 v - 3u_\varepsilon v^2 - v^3 ,$$

where we recall that u_ε is of order ε, and approximately equal to $\mathcal{O}(\varepsilon)\cos(x)$. Reducing again to quasi-momentum ℓ, and using Eq.(1.8), we get the equation

$$\partial_t \tilde{v}_\ell = L_{\varepsilon,\ell}\tilde{v}_\ell - 3u_\varepsilon (v^2)^{\sim}_\ell - (v^3)^{\sim}_\ell , \tag{1.12}$$

and it is the term $3u_\varepsilon(v^2)^\sim_\ell$ which does not have a sign. The saving grace will be the *diffusive* behavior suggested by the spectrum (in particular the branch λ_ℓ). At first sight, the non-linearities seem to be too singular for diffusion to dominate a potential divergence. Indeed, it is well known that, *e.g.*, the equation

$$\partial_t u \;=\; \partial_x^2 u + u^3 \;,$$

has solutions which blow up in finite time [L], and the quadratic term makes things even worse. The beautiful observation of Schneider[Sch] is, however, that the problem Eq.(1.12) is rather of a form reminiscent of

$$\partial_t v \;=\; \partial_x^2 v - \partial_x^2 (v^2 + v^3) \;, \tag{1.13}$$

which is good enough for convergence [CEE, BK, BKL].

In later sections we examine in detail the form of the non-linear terms in Eq.(1.12), but here we explain briefly why these terms are similar to the non-linear terms in Eq.(1.13). The derivatives in the non-linearity have their origin in the symmetries of the problem, and they are easier to understand in momentum space. In fact, Eq.(1.13) is a good approximation to Eq.(1.12) only in the low-momentum (small ℓ) regime, but this is sufficient since for ℓ outside a neighborhood of $\ell = 0$, the stationary solutions are linearly stable, (and not only *marginally* stable) and the form of the non-linearity is unimportant.

To understand the low-momentum behavior of Eq.(1.1), note first that the Swift-Hohenberg equation Eq.(1.1)—and, incidentally, other equations with coordinate independent right hand side—has a *circle of fixed points* generated by translations. If we now study the Eq.(1.12) at $\ell = 0$, this corresponds to studying the Swift-Hohenberg equation in the space of functions of period 2π. In this space, say $L^2([0, 2\pi])$, the linear operator in Eq.(1.12) has pure point spectrum with a simple eigenvalue at 0 and all other eigenvalues real and strictly negative. In this case, as Schneider notes, the center manifold theorem can be applied, and there exists a 1-dimensional center manifold. We also see immediately that the eigenvector corresponding to the 0 eigenvalue is $\partial_x u_\varepsilon$, *i.e.*, it is tangent to the circle of fixed points generated by translations. In fact, since any fixed point sufficiently close to the origin must lie in the center manifold, we see that *the center manifold coincides with the 1-dimensional circle of fixed points*. Thus the non-linearity in the equation, when restricted to the center manifold, *must vanish*. This shows that the effective non-linearity in Eq.(1.12), when evaluated at $\ell = 0$, must vanish and this accounts for one derivative in Eq.(1.13). More precisely, we see that the effective non-linearity in Eq.(1.12) is bounded by $\mathcal{O}(\ell)$, as is the non-linearity in Eq.(1.13). The second derivative of the non-linearity in Eq.(1.13) arises because of "momentum conservation." Since $\varphi_{\varepsilon,\ell}$ is a smooth function of ℓ, the linear term in Eq.(1.11) must of the form $i\ell g_\varepsilon$, with g_ε independent of ℓ. Since the interaction is local in x, one sees upon working out the integrals that all terms proportional to ℓ in the non-linearity cancel exactly, see Eq.(A.3). Thus, the low momentum behavior of Eq.(1.12) is as if the non-linearity was differentiated twice—*i.e.*, exactly as in Eq.(1.13).

Our main result is that this intuitive argument correctly predicts that the leading order asymptotics are diffusive, and that furthermore, the higher order asymptotics are controlled by a sequence of finite dimensional invariant manifolds. Thus, our approach provides some insight into how finite dimensional geometrical structures can arise from a problem with continuous spectrum.

Stability Theorem 1.2. *Fix $n \geq 1$ and $\delta > 0$. There exists a Hilbert space, $\mathcal{H}(n)$, such that if the initial conditions of (1.12) lie in a sufficiently small neighborhood of the origin in $\mathcal{H}(n)$, then there exists an $n+1$ dimensional, invariant manifold in the extended phase space $P(n) = \mathbf{R}^+ \times \mathcal{H}(n)$ of (1.12), and any sufficiently small solution of this equation which is not on this manifold approaches it at a rate $\mathcal{O}(t^{-(n+1-\delta)/2})$. In particular, if $n = 1$, small solutions of (1.12) have the asymptotic form:*

$$v(x,t) \;=\; \frac{A}{\sqrt{t}} e^{-x^2/4t} + \mathcal{O}(\frac{1}{t^{3/4-\delta}}) \;.$$

Remark. In Sections 2 and 3, we will make clear precisely what the Hilbert spaces $\mathcal{H}(n)$ are and what we mean by "sufficiently small."

The remainder of the paper is devoted to a proof of the Stability Theorem 1.2.

2. Formulating the Stability Theorem 1.2 in terms of scaling variables

In this section, we transform the problem to a rescaled dynamical system. In the next section, we will cast the dynamical system thus obtained into an invariant manifold problem.

The idea of the proof is to focus on the "central branch" of the spectrum, $\lambda_\ell = \lambda_{\ell,0}(\varepsilon^2)$, which is only marginally stable. The relevant part of the spectrum for the long-time asymptotics is only the part in a small neighborhood of $\ell = 0$, a fact we exhibit by an appropriate rescaling of the dependent and independent variables. This rescaling has the disadvantage that it introduces a singular perturbation in the variables corresponding to the "stable branches" of the spectrum, $\lambda_{\ell,n}(\varepsilon^2)$, $n \geq 1$, because the corresponding modes decay extremely fast, when rescaled (at least on a linear level). However, invariant manifold theory has long been used to treat singular perturbation problems, and we are able to use it for that purpose here as well. In addition, these invariant manifolds will provide us with a geometric description of the long-time asymptotics of solutions near the stationary states.

Our method generalizes to other problems of similar spectral nature, see the example of a cylindrical domain given in [W2].

Henceforth, we fix $\varepsilon > 0$, and omit it from most subscripts. Since $L_\ell = L_{\varepsilon,\ell}$ is self-adjoint, we can define the (orthogonal) spectral projections P_ℓ and $P_\ell^\perp$, which project onto the central branch and its complement.

Remark. We know that for $|\ell|$ sufficiently small, say $|\ell| < \ell_0/2$, one has

$$\mathrm{spec}(P_\ell L_\ell P_\ell) \;=\; -c_0(\varepsilon^2)\ell^2 + \mathcal{O}(\ell^3) \;,$$

and that this is the eigenvalue closest to 0 in $\mathrm{spec}(L_\ell)$. We continue this projection smoothly to larger ℓ even if it cannot be guaranteed to be a projection onto the highest eigenvalue. But note that for those values of ℓ the spectrum of L_ℓ can be shown to be strictly bounded away from 0, see, *e.g.*, [CE, page 102].

To study the non-linearity, and to show the mechanism leading to the result which is analogous to Eq.(1.13), we write the Eq.(1.12) in more detail:

$$\begin{aligned}\partial_t \tilde{v}_\ell(x) &= \big(L_{\varepsilon,\ell}\tilde{v}_\ell\big)(x) - 3u_\varepsilon(x)\int_{-1/2}^{1/2} dk\, \tilde{v}_{\ell-k}(x)\tilde{v}_k(x) \\ &\quad - \int_{-1/2}^{1/2} dk_1\, dk_2\, \tilde{v}_{\ell-k_1-k_2}(x)\tilde{v}_{k_1}(x)\tilde{v}_{k_2}(x) \\ &\equiv \big(L_\ell\tilde{v}_\ell\big)(x) - \big(F_2(\tilde{v})\big)_\ell(x) - \big(F_3(\tilde{v})\big)_\ell(x)\ .\end{aligned} \tag{2.1}$$

We now decompose the Eq.(2.1) by projecting onto P_ℓ and $P_\ell^\perp$. If $f \in L^2$, we let $\tilde{f}_\ell^{\mathrm{c}} = P_\ell \tilde{f}_\ell$, and $\tilde{f}_\ell^\perp = P_\ell^\perp \tilde{f}_\ell$. Similarly, $L_\ell^{\mathrm{c}} = P_\ell L_\ell P_\ell$ and $L_\ell^\perp = P_\ell^\perp L_\ell P_\ell^\perp$. Then we get

$$\partial_t \tilde{v}_\ell^{\mathrm{c}}(x) = L_\ell^{\mathrm{c}} \tilde{v}_\ell^{\mathrm{c}}(x) - \big(P_\ell F_2(\tilde{v})_\ell\big)(x) - \big(P_\ell F_3(\tilde{v})_\ell\big)(x)\ , \tag{2.2}$$

and a similar equation for $\tilde{v}_\ell^\perp$:

$$\partial_t \tilde{v}_\ell^\perp(x) = L_\ell^{\mathrm{c}} \tilde{v}_\ell^\perp(x) - \big(P_\ell^\perp F_2(\tilde{v})_\ell\big)(x) - \big(P_\ell^\perp F_3(\tilde{v})_\ell\big)(x)\ . \tag{2.3}$$

We next split the first equation into a piece corresponding to small $|\ell|$, *i.e.*, $|\ell| < \ell_0$ and another corresponding to large ℓ. Since we want to construct invariant manifolds, we need some smoothness in this construction and we choose a smooth cutoff χ satisfying

$$\chi(\ell) = \begin{cases} 1, & \text{if } |\ell| \le \ell_0\ , \\ 0, & \text{if } |\ell| > 2\ell_0\ , \end{cases}$$

and of course $\ell_0 < \frac{1}{2}$. In fact, we shall choose $\ell_0 > 0$ so small that P_ℓ is the projection onto the central eigenspace for all $\ell \in [-\ell_0, \ell_0]$. Let φ_ℓ denote the normalized eigenvector which spans the range of P_ℓ (for $|\ell| < \ell_0$, and smoothly continued for ℓ beyond that value). Then $\tilde{v}_\ell^{\mathrm{c}}$ can be written as $\tilde{v}_\ell^{\mathrm{c}} = V(\ell)\varphi_\ell$, where it is understood that V is really a function of v. We also let Π_ℓ denote the operation $\Pi_\ell f_\ell = \langle \varphi_\ell | f_\ell \rangle$, where $\langle\cdot\rangle$ is the scalar product in $\mathcal{F}_\ell$. This operation extracts the coefficient V and therefore Eq.(2.2) can be written as

$$\partial_t V(\ell) = \lambda_\ell V(\ell) - \Pi_\ell P_\ell F_2(\tilde{v})_\ell - \Pi_\ell P_\ell F_3(\tilde{v})_\ell\ . \tag{2.4}$$

Defining $V^<(\ell) = \chi(\ell)V(\ell)$, and $V^>(\ell) = (1-\chi(\ell))V(\ell)$, the Eq.(2.4) can be rewritten as

$$\begin{aligned}\partial_t V^<(\ell) &= \lambda_\ell V^<(\ell) - \big(f^{\mathrm{c}}(V^<, V^>, \tilde{v}^\perp)\big)(\ell)\ , \\ \partial_t V^>(\ell) &= \lambda_\ell V^>(\ell) - \big(f^{\mathrm{s}}(V^<, V^>, \tilde{v}^\perp)\big)(\ell)\ ,\end{aligned} \tag{2.5}$$

where

$$\big(f^{\mathrm{c}}(V^<, V^>, \tilde{v}^\perp)\big)(\ell) = \chi(\ell)\Big(\Pi_\ell P_\ell F_2\big(\tilde{v}\big)_\ell + \Pi_\ell P_\ell F_3\big(\tilde{v}\big)_\ell\Big)\ ,$$

$$\big(f^{\mathrm{s}}(V^<, V^>, \tilde{v}^\perp)\big)(\ell) = \big(1-\chi(\ell)\big)\Big(\Pi_\ell P_\ell F_2\big(\tilde{v}\big)_\ell + \Pi_\ell P_\ell F_3\big(\tilde{v}\big)_\ell\Big)\ ,$$

and

$$\tilde{v}_\ell(x) \;=\; (V^<(\ell) + V^>(\ell)) \cdot \varphi_\ell(x) + \tilde{v}_\ell^\perp(x)\;.$$

Note that since $V^>$ is supported outside $[-\ell_0, \ell_0]$, both it and $\tilde{v}^\perp$ decay exponentially (at least at the linear level) and hence will be irrelevant for the asymptotics of $V^<$, as we shall show. With this in mind, we introduce a new coordinate, $V^{\rm s}$, which combines the "irrelevant" pieces, $V^{\rm s} = (V^>, \tilde{v}^\perp)$. Then the Eq.(2.5) combined with Eq.(2.3) takes the more suggestive form

$$\begin{aligned} \partial_t V^<(\ell) \;&=\; \lambda_\ell V^<(\ell) - \bigl(f(V^<, V^{\rm s})\bigr)(\ell)\;,\\ \partial_t V^{\rm s} \;&=\; \mathcal{L}_b^{(0)} V^{\rm s} + g(V^<, V^{\rm s})\;, \end{aligned} \tag{2.6}$$

and we know that the spectrum of the linear operator $\mathcal{L}_b^{(0)}$ is contained in $(-\infty, -\sigma^{\rm s})$, for some $\sigma^{\rm s} > 0$.

In order to proceed further, we analyze the non-linear terms in Eq.(2.6) in more detail. In particular, we concentrate on the most critical terms, namely those in f of Eq.(2.6) which depend only on $V^<$. We decompose $f(V^<, V^{\rm s}) = f_2^{(0)}(V^<) + f_3^{(0)}(V^<) + f_4^{(0)}(V^<, V^{\rm s})$, where $f_2^{(0)}$ collects the terms which are homogeneous of degree 2 in $V^<$ and $f_3^{(0)}$ those of degree 3. One gets

$$\begin{aligned} \bigl(f_2^{(0)}(V^<)\bigr)(\ell) \;&=\; 3\chi(\ell)\int dx\,\overline{\varphi}_\ell(x) u_\varepsilon(x)\int_{-1/2}^{1/2} dk\,\varphi_k(x)\varphi_{\ell-k}(x) V^<(k) V^<(\ell-k)\\ &\equiv\; 3\chi(\ell)\int_{-1/2}^{1/2} dk\,K_2(\ell,k) V^<(k) V^<(\ell-k)\;,\\ \bigl(f_3^{(0)}(V^<)\bigr)(\ell) \;&=\; \chi(\ell)\int dx\,\overline{\varphi}_\ell(x)\int_{-1/2}^{1/2} dk_1\,dk_2\,\varphi_{k_1}(x)\varphi_{k_2}(x)\varphi_{\ell-k_1-k_2}(x)\\ &\qquad\times V^<(k_1)V^<(k_2)V^<(\ell-k_1-k_2)\\ &\equiv\; \chi(\ell)\int_{-1/2}^{1/2} dk_1\,dk_2\,K_3(\ell,k_1,k_2) V^<(k_1)V^<(k_2)V^<(\ell-k_1-k_2)\;. \end{aligned} \tag{2.7}$$

At this point, we make use of the diffusive nature of the problem for $V^<$, by introducing scaling variables as in [W]. This will give us a more precise description of the convergence process than the one obtained in [Sch]. We rescale the variables in Eq.(2.6) as follows: We first fix, once and for all, a (large) constant $t_0 > 0$. Then we define

$$\begin{aligned} V^<(\ell,t) \;&=\; w^{\rm c}\bigl(\mathrm{sign}(\ell)\sqrt{|\Lambda_\ell|(t+t_0)}\,, \log(t+t_0)\bigr)\;,\\ V^{\rm s}(\ell,t) \;&=\; w^{\rm s}\bigl(\mathrm{sign}(\ell)\sqrt{|\Lambda_\ell|(t+t_0)}\,, \log(t+t_0)\bigr)/(t+t_0)^{1/2}\;, \end{aligned} \tag{2.8}$$

where $\Lambda_\ell = \lambda_\ell$ for $|\ell| < \ell_0/2$ and is monotonically extended beyond that region in such a way that it is parabolic for large $|\ell|$. (This artifact is needed because we have no guarantee that λ_ℓ

itself is monotone.) Note that if λ_ℓ were equal to $-$const. ℓ^2, this scaling would amount to the usual "diffusive" rescaling. Our choice takes into account higher order corrections produced by higher order terms in λ_ℓ. If we let now $p = \text{sign}(\ell)\sqrt{|\Lambda_\ell|(t+t_0)}$, and $\tau = \log(t+t_0)$, then Eq.(2.6) implies that w^{c} and w^{s} obey the following equations:

$$\begin{aligned}
\partial_\tau w^{\text{c}} &= (-p^2 - \tfrac{1}{2}p\partial_p)w^{\text{c}} \\
&\quad + e^\tau \Big(f_2(w^{\text{c}}, e^{-\tau/2}) + f_3(w^{\text{c}}, e^{-\tau/2}) + f_4(w^{\text{c}}, w^{\text{s}}e^{-\tau/2}, e^{-\tau/2}) \Big) , \\
e^{-\tau}\partial_\tau w^{\text{s}} &= M_{\exp(-\tau/2)} w^{\text{s}} + \tfrac{1}{2}e^{-\tau}w^{\text{s}} - \tfrac{1}{2}e^{-\tau}p\partial_p w^{\text{s}} + e^{\tau/2}g(w^{\text{c}}, w^{\text{s}}e^{-\tau/2}, e^{-\tau/2}) ,
\end{aligned} \tag{2.9}$$

where f_2, f_3, f_4 and M in Eq.(2.9) are defined below. If

$$pe^{-\tau/2} = p(t+t_0)^{-1/2} = \text{sign}(\ell)\sqrt{|\Lambda_\ell|} ,$$

and if we denote the inverse transformation by

$$\ell = \Phi(pe^{-\tau/2}) ,$$

where Φ is the inverse function of $x \mapsto \text{sign}(x)\sqrt{|\Lambda_x|}$, then, given a function $w = w(\ell, t)$, we define the nonlinearity

$$\begin{aligned}
\big[f_2(w, e^{-\tau/2})\big](p) &= \big[f_2^{(0)}(w(\cdot, e^\tau))\big](\Phi(pe^{-\tau/2})) \\
&= \big[f_2^{(0)}(w(\cdot, t+t_0))\big](\Phi(p(t+t_0)^{-1/2})) .
\end{aligned}$$

(Note that $\Phi(x) = x\big(1 + \mathcal{O}(x)\big)$.) Analogous definitions apply to f_3 and f_4. The operator M will be described in detail in Eq.(2.13).

Remark. The non-linearities $f_2, \ldots$ depend on the choice of t_0. If we consider the initial value problem for the Swift-Hohenberg equation, the "smallness" assumption on the perturbation of the periodic state is to be understood with respect to a choice of a (sufficiently large) t_0. As we will see, however, the nonlinear terms can be bounded, independent of t_0, for all $t_0 \geq T > 0$.

To this change of variables will correspond the following (non-exhaustive) list of substitutions in the integrals in Eq.(2.7): Let $a, b \in [-\frac{1}{2}, \frac{1}{2}]$. Then

$$\begin{aligned}
\chi(\ell)\int_a^b dk &\to \chi\big(\Phi(pe^{-\tau/2})\big)e^{-\tau/2}\int_{e^{\tau/2}\Phi^{-1}(a)}^{e^{\tau/2}\Phi^{-1}(b)} dq\, \Phi'(qe^{-\tau/2}) , \\
\varphi_\ell &\to \varphi_{\Phi(pe^{-\tau/2})} , \\
\varphi_{k-\ell} &\to \varphi_{\Gamma(p,q,\tau)} , \\
V(k,t) &\to w(p,\tau) , \\
V(\ell-k,t) &\to w(\Delta(p,q,\tau)) .
\end{aligned} \tag{2.10}$$

Here, we define

$$\begin{aligned}\Gamma(p,q,\tau) &= \Phi(pe^{-\tau/2}) - \Phi(qe^{-\tau/2}) ,\\ \Delta(p,q,\tau) &= e^{\tau/2}\Phi^{-1}\big(\Phi(pe^{-\tau/2}) - \Phi(qe^{-\tau/2})\big) .\end{aligned} \tag{2.11}$$

It follows at once from the definition of Φ that

$$\begin{aligned}\Gamma(p,q,\tau) &= e^{-\tau/2}(p-q)\cdot\big(1+\gamma(p,q,\tau)\big) ,\\ \Delta(p,q,\tau) &= (p-q)\cdot\big(1+\varkappa(p,q,\tau)\big) ,\end{aligned} \tag{2.12}$$

where $\varkappa$ and γ are bounded and smooth.

We next discuss in detail the *spectrum* of $M_{\exp(-\tau/2)}$, which is just the rescaled linear operator for the "stable" part of w, *cf.* Eq.(2.6). Recall first that $V^{\mathrm{s}} = (V^{>}, \tilde{v}^{\perp})$. This introduces a natural decomposition of $w^{\mathrm{s}} = (w_1^{\mathrm{s}}, w_2^{\mathrm{s}})$, as well as of $M_{\exp(-\tau/2)} = M_{\exp(-\tau/2),1} \oplus M_{\exp(-\tau/2),2}$. From the definition of the first component, we get

$$\big(M_{\exp(-\tau/2),1} f_1^{\mathrm{s}}\big)(p,\tau) = \left(\varepsilon^2 - \big(1+(i+i\Phi(pe^{-\tau/2}))^2\big)^2 - \mathcal{K}\big(\Phi(pe^{-\tau/2})\big)\right) f_1^{\mathrm{s}}(p,\tau) , \tag{2.13}$$

where $\mathcal{K}(\ell)$ is a kernel given by

$$\mathcal{K}(\ell) = 3\int dx\, \overline{\varphi}_\ell(x) u_\varepsilon^2(x)\varphi_\ell(x) .$$

(Recall that φ_ℓ really depends on ε as well and should be written $\varphi_{\varepsilon,\ell}$.) Since V^{s} has support bounded away from $\ell = 0$, say $|\ell| > \ell_0/2$, we see that $w_1^{\mathrm{s}}(p,\tau)$ will have support in $|p|e^{-\tau/2} > \sqrt{|\Lambda_{\ell_0/3}|}$, and the spectrum of $M_{\exp(-\tau/2),1}$ is seen to be contained in $\{\sigma | \mathrm{Re}\,\sigma \le \sigma_0 < 0\}$, for some σ_0 and for all $\tau > 0$.

A very similar argument detailed in Appendix B shows that the spectrum of $M_{\exp(-\tau/2),2}$ is also contained in such a set. Thus, *the linear evolution generated by $M_{\exp(-\tau/2)}$ contracts exponentially*. See Lemma B.6 below for details.

We next consider the operator $L = (-p^2 - \frac{1}{2}p\partial_p)$, which appears in the first component of Eq.(2.9). The detailed study of the semi-group generated by L will be given in Appendix B. Here, we discuss its properties on an informal level. The Fourier transform of L is $\partial_x^2 + \frac{1}{2}x\partial_x + \frac{1}{2}$, which is conjugate to the harmonic oscillator $H_0 = \partial_x^2 - x^2/16 + 1/4$ by the (unbounded!) transformation T, of multiplication by $\exp(x^2/8)$. In formulas: $L = T^{-1}H_0T$. Therefore, H_0 has (say, on L^2), discrete spectrum $\mu_j = -j/2$, $j = 0, 1, \dots$. It is this spectrum which leads to a nice interpretation of the convergence properties of the Swift-Hohenberg equation. The eigenvalues of L are unchanged by the transformation T, (and the eigenfunctions are multiplied by a Gaussian), so to each eigenvalue μ of L there corresponds a decay rate $e^{\tau\mu}$ in the linear problem. Because of the transformation of variables from t to τ, this decay rate becomes $(t+t_0)^\mu$ in the original problem Eq.(2.6). In other words: Neglecting the non-linearities in Eq.(2.9) and setting $w^{\mathrm{s}} = 0$, (and ignoring potential problems related to the unbounded operator T) we have a solution

$$w^{\mathrm{c}}(p,\tau) = \sum_{m=0}^{\infty} w_m e^{-\tau m/2}\mathcal{H}_m(2p) , \tag{2.14}$$

where $\mathcal{H}_m$ is the m^{th} eigenfunction of L. In the original variables, this means that

$$V^{<}(\ell,t) \;=\; \sum_{m=0}^{\infty} w_m (t+t_0)^{-m/2}\mathcal{H}_m\big(2\ell(t+t_0)^{1/2}(1+\mathcal{O}(|\ell|^{1/2}))\big)\ . \tag{2.15}$$

Thus, *to each m there corresponds a specific rate ($\mu_m = -m/2$) of decay* for a part of the function $V^{<}$. Note that a change of t_0 just corresponds to a rearrangement of the series. (This is not contradictory, since a change of t_0 also changes the initial condition, and hence the solution whose asymptotics we are computing.) In particular, the slowest rate of decay is associated with $\mathcal{H}_0$, which is Gaussian, and thus, at least at the linear level, a "generic" perturbation of the stationary state will decay like $\exp(-c\ell^2 t)$, for some $c > 0$. In terms of the original independent variables (x,t), it decays like $t^{-1/2}\exp(-x^2/(4tc))$, as $t \to \infty$. This means that at this level, the periodic stationary states are stable, and that perturbations of them decay like solutions of the linear heat equation. The invariant manifold theory guarantees that this behavior persists in the non-linear problem, and in fact it tells us more. We will see that in suitable spaces we can construct a sequence of manifolds $\mathcal{M}_j$ of dimension $j = 1, 2, \ldots$, such that any solution of Eq.(2.9) approaches a solution on $\mathcal{M}_j$ at a rate $e^{\tau\mu_{j-1}}$, or again reverting to the original (x,t) variables, at a rate $\mathcal{O}((t+t_0)^{\mu_{j-1}})$. In the case at hand, this is $\mathcal{O}((t+t_0)^{-j/2})$. Thus, in principle, we can analyze finer and finer details of the asymptotics of perturbations of the stationary state by considering the behavior of the solution on these *finite dimensional* manifolds.

3. Casting the Stability Theorem 1.2 into an invariant manifold theorem

At the end of the preceding section, we have seen that the spectrum of the linear part of Eq.(2.9) has the following nature: The component w^{c} satisfies a differential equation whose linear part has eigenvalues $\mu_j = -j/2$, $j = 0, 1, \ldots, N$, *provided* we work on a space of sufficiently smooth and rapidly decaying functions. The evolution of w^{s} is governed by an equation with an even more stable spectrum.

The invariant manifold theorem will show in which sense the built-in scalings of Eq.(2.14) survive the addition of non-linearities. While this presents no conceptual problems at all—and this is the beauty of the present approach—some care is of course needed in the application of the invariant manifold theorem. Another point which might be overlooked is the following: The invariant manifold theorem does *not* say that the representation of the full non-linear problem is the same as in Eq.(2.15), but with slowly varying w_j. Rather, we will show that on the complement of a dimension $j-1$ surface in the function space, the solutions decay at least like $t^{-j/2}$, (for every $j \geq 1$), provided the initial data are sufficiently small and smooth.

In order to apply the invariant manifold method to the problem, we need bounds on the non-linearities and bounds on the semi-group generated by L. While the factor of $t = \exp(\tau)$ in front of f_2 in Eq.(2.9) might look like a disaster, we will see that by working in appropriate function spaces, and taking advantage of the nature of the nonlinear term, this factor will disappear. Its presence is in part due to the fact that we chose to work in "momentum" space, rather than "position" space, because the linear problem is most naturally studied in Floquet variables. If we rewrote these terms in position space (*i.e.*, in the original (x,t) variables), they would look much less singular.

We will work in Sobolev spaces, and we define

$$H_{q,r} = \{v \mid (1-\partial_p^2)^{r/2}(1+p^2)^{q/2}v \in L^2\} , \tag{3.1}$$

equipped with the corresponding norm $\|\cdot\|_{q,r}$. The function w^{c} will be an element of $H_{q,r}$.

The function w^{s} has two components. The first component comes from the central branch of the spectrum of the linear operator (1.9), and will also be in $H_{q,r}$. The second component comes from the stable branches of the spectrum, and it depends on both p, and x. It will be an element of the space:

$$\begin{aligned} H_{q,r,\nu} = \{w = w(p;x) \mid w(p;x) = w(p;x+2\pi),& \\ (1-\partial_x^2)^{\nu/2}(1-\partial_p^2)^{r/2}(1+p^2)^{q/2}w \in L^2(\mathbf{R}\times[-\pi,\pi])\} .& \end{aligned}$$

By a slight abuse of notation, we will denote by $\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}}$ the sum of the $H_{q,r}$ norm of the first component of w^{s} and the $H_{q,r,\nu}$ norm of the second component, and by $\|w^{\mathrm{s}}\|_{q,r,\nu}$, we will mean the $H_{q,r,\nu}$ norm of just the second component. We will also use $\mathcal{H}_{q,r,\nu}$ to denote the space of all functions with finite $\mathcal{H}_{q,r,\nu}$ norm.

The non-linearities satisfy the following bounds:

Proposition 3.1. *For every $q \ge 2$ and every $r \ge 0$ there is a constant C for which*

$$\begin{aligned} \|e^{\tau} f_2(w, e^{-\tau/2})\|_{q-1,r} &\le C\|w\|_{q,r}^2 , \\ \|e^{\tau} f_3(w, e^{-\tau/2})\|_{q,r} &\le C\|w\|_{q,r}^3 , \end{aligned} \tag{3.2}$$

for all $\tau > 0$.

Proposition 3.2. *For every $q \ge 2$ and every $r \ge 0$ there is a constant C for which*

$$\begin{aligned} &\|e^{\tau} f_4(w^{\mathrm{c}}, w^{\mathrm{s}}e^{-\tau/2}, e^{-\tau/2})\|_{q,r} \\ &\qquad \le Ce^{\tau/2}\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}} \left(e^{-\tau/2}\|w^{\mathrm{c}}\|_{q,r} + e^{-\tau}\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}}\right) \\ &\qquad\qquad \times\left(1 + e^{-\tau/2}\|w^{\mathrm{c}}\|_{q,r} + e^{-\tau}\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}}\right) , \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\|e^{\tau/2} g(w^{\mathrm{c}},\, w^{\mathrm{s}}e^{-\tau/2}, e^{-\tau/2})\|_{\mathcal{H}_{q,r,\nu}} \\ &\qquad \le Ce^{\tau}\left(e^{-\tau/2}\|w^{\mathrm{c}}\|_{q,r} + e^{-\tau}\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}}\right)^2 \\ &\qquad\qquad \times\left(1 + e^{-\tau/2}\|w^{\mathrm{c}}\|_{q,r} + e^{-\tau}\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}}\right) , \end{aligned} \tag{3.4}$$

for all $\tau > 0$.

Remark. Note that every factor of $\|w^{\mathrm{c}}\|_{q,r}$ is multiplied by $e^{-\tau/2}$ and every factor of $\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}}$ is multiplied by $e^{-\tau}$.

Remark. As we pointed out above, the nonlinear terms depend on the constant t_0. However, the bounds in the two preceeding propositions are independent of t_0. More precisely, for any $T > 0$, the the constants C in both propositions can be chosen so that the estimates in (3.2)–(3.4) hold for all $t_0 \geq T$.

The proofs will be given in Appendix A. Note that one loses a power of p in the first estimate of Eq.(3.2), but of course, one "gains" the square of the function.

We will regain the "lost" power of p by examining in detail the semi-group generated by L. We denote by P_N the projection onto the space spanned by the N eigenvalues $\{\mu_j = -j/2\}_{j=0,\dots,N-1}$ of L. We define $Q_N = 1 - P_N$. (We verify in Appendix B that these projections are defined.) On the space corresponding to Q_N, we expect the norm of the semi-group generated by L to decay like $\exp(\tau\mu_N)$. This is indeed the case.

Theorem 3.3. *For every $\varepsilon > 0$, there is a constant N_0 and a function $r(N,q)$ such that for every $N \geq N_0$, every $q \geq 1$ and every $r \geq r(N,q)$, there is a $C = C(q,r,N) < \infty$ such that*

$$\left\| e^{\tau L} Q_N v \right\|_{q,r} \leq \frac{C(q,r,N)}{\sqrt{a(\tau)}} e^{-\tau(|\mu_N| - \varepsilon)} \|v\|_{q-1,r} , \tag{3.5}$$

where $a(\tau) = 1 - e^{-\tau}$ and $L = -p^2 - \frac{1}{2} p \partial_p$

The proof will be given in Appendix B.

We also need an estimate on the linear evolution generated by $M_{\exp(-\tau/2)}$. Let U_τ be the solution of

$$e^{-\tau} \partial_\tau U_\tau = M_{\exp(-\tau/2)} U_\tau ,$$

with initial condition $U_0 = 1$. (Compare with the linear part of (2.9).) Then, in Appendix B, we prove

Theorem 3.4. *If $w_0 \in \mathcal{H}_{q,r,\nu}$, then there exists $c_0 > 0$, such that for all $\tau \geq 0$,*

$$\|U_\tau w_0\|_{\mathcal{H}_{q,r,\nu}} \leq \exp(-e^{c_0 \tau/2}) \|w_0\|_{\mathcal{H}_{q,r,\nu}} .$$

With the help of the bounds Proposition 3.1–Theorem 3.4, we can now reformulate the problem in terms of invariant manifolds. The Eq.(2.9) can be written as an *autonomous system* by defining $\eta = (t + t_0)^{-1/2} = e^{-\tau/2}$:

$$\begin{aligned} \partial_\tau w^{\mathrm{c}} &= L w^{\mathrm{c}} + \eta^{-2} \left(f_2(w^{\mathrm{c}}, \eta) + f_3(w^{\mathrm{c}}, \eta) + f_4(w^{\mathrm{c}}, w^{\mathrm{s}}\eta, \eta) \right) , \\ \eta^2 \partial_\tau w^{\mathrm{s}} &= M_\eta w^{\mathrm{s}} + \eta^{-1} g(w^{\mathrm{c}}, w^{\mathrm{s}}\eta, \eta) , \\ \partial_\tau \eta &= -\tfrac{1}{2}\eta . \end{aligned} \tag{3.6}$$

We will construct an invariant manifold tangent at the origin to the eigenspace corresponding to the N largest eigenvalues of L, and the η direction. We subdivide the center variable w^{c} according to the projection Q_N defined earlier, where N is fixed once and for all. Define

$$x_1 = (1 - Q_N) w^{\mathrm{c}}, \quad x_2 = Q_N w^{\mathrm{c}}, \quad x_3 = w^{\mathrm{s}} . \tag{3.7}$$

Note that the variable x_1 is in a finite dimensional space, while x_2 and x_3 are in infinite dimensional Hilbert spaces. The system of equations Eq.(3.6) now takes the form

$$\begin{aligned} \partial_\tau x_1 &= A_1 x_1 + N_1(x_1, \eta, x_2, x_3) , \\ \partial_\tau \eta &= -\tfrac{1}{2}\eta , \\ \partial_\tau x_2 &= A_2 x_2 + N_2(x_1, \eta, x_2, x_3) , \\ \eta^2 \partial_\tau x_3 &= A_{3,\eta} x_3 + N_3(x_1, \eta, x_2, x_3) . \end{aligned} \tag{3.8}$$

Here $A_1 = (1 - Q_N)L$, $A_2 = Q_N L$, and $A_{3,\eta} = M_\eta$.

Remark. In view of later developments, we consider x_1 and η to be the "interesting" variables and x_2 and x_3 the "slaved" variables, hence the new order of the variables.

Remark. Eq.(3.8) is a very singular perturbation problem, because of the factor of η^2 in front of the derivative of x_3. What is more, since $\eta(\tau) = e^{-\tau/2}$, it becomes steadily more singular in precisely the limiting regime in which we are interested. Nonetheless, we will see that the invariant manifold theorem provides just the tool we need to understand this limit. Singular perturbation problems of this type do not seem to have been studied much, but they do arise naturally in other contexts, such as the study of parabolic equations in cylindrical domains ([W2]).

We shall call Eq.(3.8) the *full system*. To simplify the notation, we shall omit the dependence on η in $A_{3,\eta}$. Consider the spectra of A_1, A_2, A_3. From what we have seen earlier, we find that

$$\begin{aligned} \text{spec}(A_1) &= \{0, -1/2, -1, \ldots, -(N-1)/2\} , \\ \text{spec}(A_2) &\subseteq [-\infty, -N/2] , \\ \text{spec}(\eta^{-2} A_3) &= [-\infty, -c/\eta^2] , \end{aligned} \tag{3.9}$$

where c is some positive constant. Thus, we expect to apply a pseudo center manifold theorem to "slave" the variables x_2, x_3 to the variables x_1 and η. While there are certain technical difficulties associated with the very singular perturbation, in Appendix C, we demonstrate the following Proposition:

Proposition 3.5. *Fix $N > 0$. There exist $r > 0$, $q \geq 1$, and $\nu > 1/2$, such that the system of equations (3.8) has an invariant, $N+1$-dimensional manifold, given in a neighborhood of the origin by the graph of a pair of functions*

$$\begin{aligned} h_2^* &: \mathbf{R}^N \times \mathbf{R} \to H_{q,r} , \\ h_3^* &: \mathbf{R}^N \times \mathbf{R} \to \mathcal{H}_{q,r,\nu} . \end{aligned}$$

We next turn to the task of showing that the invariant manifold we found for Eq.(3.6) actually attracts solutions at an exponential rate.

Notation. It is useful to introduce the notation $\xi = (x_1, \eta)$ for the two relevant variables.

Consider a solution of the form $\big(w^{\rm c}(\tau), w^{\rm s}(\tau)\big)$ of Eq.(3.6), with $w^{\rm c}(\tau) = \big(x_1(\tau), x_2(\tau)\big)$ as in Eq.(3.7), and $w^{\rm s}(\tau) = x_3(\tau)$. We wish to show that

$$\big(\,\xi(\tau)\,,\, x_2(\tau)\,,\, x_3(\tau)\,\big) \;\longrightarrow\; \big(\,\xi(\tau)\,,\, h_2^*(\xi(\tau))\,,\, h_3^*(\xi(\tau))\,\big)\,,$$

as $\tau \to \infty$, and furthermore, that it does so at an exponential rate, given essentially by the least negative eigenvalue, μ_N, of the operator A_2.

Proposition 3.6. *Fix $N > 0$. For every δ satisfying $0 < \delta$ there is an $\varepsilon_0 > 0$ such that if the solution of Eq.(3.6) remains in a neighborhood of the origin of size ε_0 one has the following bound: There is a $C^* < \infty$ for which*

$$\big\|x_2(\tau) - h_2^*(\xi(\tau))\big\|_{q,r} + \|x_3(\tau) - h_3^*(\xi(\tau))\|_{\mathcal{H}_{q,r,\nu}} \;\le\; C^* e^{-(|\mu_N|-\delta)\tau}\,,$$

as $\tau \to \infty$.

Proof. This proof is relatively standard, see *e.g.*, Carr [C]. Let

$$z(\tau) \;=\; \begin{pmatrix} x_2(\tau) - h_2^*(\xi(\tau)) \\ x_3(\tau) - h_3^*(\xi(\tau)) \end{pmatrix} \;\equiv\; \begin{pmatrix} z_2(\tau) \\ z_3(\tau) \end{pmatrix}\,.$$

Then we have

$$\dot z \;=\; \begin{pmatrix} A_2 z_2 + \hat N_2(\xi, z_2, z_3) \\ \eta^{-2} A_3 z + \eta^{-2} \hat N_3(\xi, z_2, z_3) \end{pmatrix}\,, \tag{3.10}$$

where, with the notation of Eq.(3.8),

$$\hat N_j(\xi, z_2, z_3) \;=\; N_j(\xi, z_2 + h_2^*(\xi), z_3 + h_3^*(\xi)) \;-\; N_j(\xi, h_2^*(\xi), h_3^*(\xi))\,,$$

for $j = 2, 3$. The only novelty in Eq.(3.10) w.r.t. [C] is the factor of η^{-2} in the "3"-component which is the reason for our repeating his arguments. But we can integrate Eq.(3.10) explicitly and get

$$\begin{aligned} z_2(\tau) \;&=\; e^{\tau A_2} z_2(0) + \int_0^\tau d\sigma\, e^{(\tau-\sigma)A_2} \hat N_2\big(\xi(\sigma), z_2(\sigma), z_3(\sigma)\big)\,, \\ z_3(\tau) \;&=\; e^{(\eta(\tau)^{-2}-\eta(0)^{-2})A_3} z_3(0) + \int_0^\tau d\sigma\, \frac{1}{\eta(\sigma)^2} e^{(\eta(\tau)^{-2}-\eta(\sigma)^{-2})A_3} \hat N_3\big(\xi(\sigma), z_2(\sigma), z_3(\sigma)\big)\,. \end{aligned}$$

We assume $\eta(0) > 0$, since we are interested in the case $\eta(0) = t_0^{-1/2}$, and we have chosen the scaling factor t_0 to be a positive, finite constant. Note also that ξ remains in a neighborhood of the origin, as $\tau \to \infty$. From the bounds on the non-linear terms we see that if the solution satisfies

$$\|x_2(\tau)\|_{q,r} + \|x_3(\tau)\|_{\mathcal{H}_{q,r,\nu}} \;\le\; \rho\,,$$

for all $\tau \geq 0$, then, with $\nu_N = N/2$, the modulus of the N^{th} eigenvalue μ_N of L, we have

$$\begin{aligned} \|z_2(\tau)\|_{q,r} &\leq e^{-\tau\nu_N}\|z_2(0)\|_{q,r} + C\varepsilon\int_0^\tau d\sigma e^{-(\tau-\sigma)\nu_N}\big(\|z_2(\sigma)\|_{q,r} + \|z_3(\sigma)\|_{\mathcal{H}_{q,r,\nu}}\big)\ , \\ \|z_3(\tau)\|_{\mathcal{H}_{q,r,\nu}} &\leq e^{(\eta(\tau)^{-2}-\eta(0)^{-2})\nu_N}\|z_3(0)\|_{\mathcal{H}_{q,r,\nu}} \\ &\quad + C\varepsilon\int_0^\tau d\sigma\,\frac{1}{\eta(\sigma)^2}e^{-(\eta(\tau)^{-1}-\eta(\sigma)^{-1})\nu_N}\big(\|z_2(\sigma)\|_{q,r} + \|z_3(\sigma)\|_{\mathcal{H}_{q,r,\nu}}\big)\ . \end{aligned} \tag{3.11}$$

In deriving these inequalities, we used the inequalities

$$\begin{aligned} \|e^{\tau A_2}\hat{N}_2(\xi,z_2,z_3)\|_{q,r} &\leq e^{-\tau\nu_N}\|\hat{N}_2(\xi,z_2,z_3)\|_{q-1,r}\ , \\ \|e^{\rho A_3}\hat{N}_3(\xi,z_2,z_3)\|_{\mathcal{H}_{q,r,\nu}} &\leq e^{-\rho\nu_N}\|\hat{N}_3(\xi,z_2,z_3)\|_{\mathcal{H}_{q,r,\nu}}\ , \end{aligned}$$

which follow from the bounds of Appendix B. If we now fix $\delta > 0$ and define

$$\begin{aligned} C_2(\tau) &= \sup_{0\leq\tau'\leq\tau} e^{\tau'(\nu_N-\delta)}\|z_2(\tau')\|_{q,r}\ , \\ C_3(\tau) &= \sup_{0\leq\tau'\leq\tau} e^{\tau'(\nu_N-\delta)}\|z_3(\tau')\|_{\mathcal{H}_{q,r,\nu}}\ , \end{aligned}$$

then the Eq.(3.11) leads to the inequality

$$\begin{aligned} C_2(\tau) &\leq K_1 + K_2\varepsilon\big(C_2(\tau)+C_3(\tau)\big)\int_0^\tau d\sigma\, e^{-(\tau-\sigma)\delta}\ , \\ C_3(\tau) &\leq K_3 + K_4\varepsilon\big(C_2(\tau)+C_3(\tau)\big)\int_0^\tau d\sigma\,\frac{1}{\eta(\sigma)^2}e^{(\eta(\tau)^{-2}-\eta(\sigma)^{-2})\nu_N}e^{(\tau-\sigma)(\nu_N-\delta)}\ . \end{aligned}$$

If we insert into these integrals the definitions

$$\eta(\sigma) = \exp(-\sigma/2)\eta(0)\ , \quad \eta(\tau) = \exp(-\tau/2)\eta(0)\ ,$$

we find that both integrals are uniformly bounded in $\tau \geq 0$ if $\eta(0)$ is in a compact subinterval of $(0,1)$. The proof of Proposition 3.6 is complete.

Thus, all solutions near the invariant manifold approach it exponentially fast in τ.

One can now show without difficulty that every solution approaches exponentially quickly a *particular* solution on the (approximate) invariant manifold

$$\big(x_1(\tau), \eta = 0, h_2^*(x_1(\tau),0), h_3^*(x_1(\tau),0)\big)\ .$$

This consists simply in translating the pp.21–24 of [C] into the present setting and thus there is no need to repeat this argument here.

If we combine these results with Proposition 3.5, we arrive finally at a description of the invariant manifolds which exist close to the origin for (3.8).

Theorem 3.7. *Fix $N > 0$ and $\delta > 0$. There exist $r > 0$, $q \geq 1$, and $\nu > 1/2$, such that the system of equations (3.8) has an invariant, $N+1$-dimensional manifold, given in a neighborhood of the origin by the graph of a pair of functions $h_2^* : \mathbf{R}^N \times \mathbf{R} \to H_{q,r}$, and $h_3^* : \mathbf{R}^N \times \mathbf{R} \to \mathcal{H}_{q,r,\nu}$. Any solution of (3.8) which remains in a neighborhood of the origin for all $\tau \geq 0$ approaches a solution of the $N+1$-dimensional system of ordinary differential equations*

$$\begin{aligned} \partial_\tau x_1 &= A_1 x_1 + N_1(x_1, \eta, h_2^*(x_1,\eta), h_3^*(x_1,\eta)) \,, \\ \partial_\tau \eta &= -\tfrac{1}{2}\eta \,, \end{aligned} \tag{3.12}$$

which results from restricting (3.8) to this invariant manifold. Furthermore, the rate of approach to this manifold is $\mathcal{O}(\exp(-\tau(N/2-\delta)))$.

Remark. This theorem almost suffices to prove Stability Theorem 1.2. In particular, it emphasizes that in a neighborhood of the periodic solutions of (1.1) there exists a family of invariant manifolds, M_2, M_3, ..., described in that theorem. The one remaining piece of the puzzle is to describe the behavior of solutions restricted to the invariant manifold, and that we do in the next section.

4. The projection of the non-linearity onto zero momentum

We have already shown that there exists a (smooth) invariant manifold, parameterized by $(\xi, h_2^*(\xi), h_3^*(\xi))$, where $\xi = (x_1, \eta)$. This manifold satisfies the equation Eq.(3.8), which, in the case of $N = 1$, *i.e.*, in the case of a two-dimensional invariant manifold amounts to

$$\begin{aligned} \partial_\tau x_1 &= N_1\big(x_1, \eta, h_2^*(\xi), h_3^*(\xi)\big) \,, \\ \partial_\tau \eta &= -\tfrac{1}{2}\eta \,, \\ \partial_\tau\big(h_2^*(\xi)\big) &= A_2 h_2^*(\xi) + N_2\big(x_1, \eta, h_2^*(\xi), h_3^*(\xi)\big) \,, \\ \eta^2 \partial_\tau\big(h_3^*(\xi)\big) &= A_3 h_3^*(\xi) + N_3\big(x_1, \eta, h_2^*(\xi), h_3^*(\xi)\big) \,. \end{aligned} \tag{4.1}$$

Note that because $N = 1$ the operator A_1 equals zero (which is the highest eigenvalue of L).

To understand the dynamics inside this invariant manifold, we now state and prove the following proposition, which is based on Schneider's beautiful observation: Let $\tilde{N}_1(x_1, \eta)$ be the r.h.s. of the first equation in (4.1), *i.e.*, $\partial_\tau x_1 = \tilde{N}_1(x_1, \eta)$.

Proposition 4.1. *There is an $x_{1,0} > 0$ such that $\tilde{N}_1(x_1, 0) = 0$, for all $|x_1| < x_{1,0}$.*

Thus, the non-linearity *vanishes identically* at "infinite time," which corresponds to $\eta = 0$. Before proving Proposition 4.1, we show that it implies the following important

Theorem 4.2. *If $x_1(0)$ is sufficiently close to 0, then there are a constant $C < \infty$ and an x_1^* such that*

$$|x_1(\tau) - x_1^*| < Ce^{-\tau/2} \,. \tag{4.2}$$

Proof. Using the fact that $\eta(\tau) = e^{-\tau/2}$, we can rewrite the equation for x_1 as

$$\partial_\tau x_1 = \tilde{N}_1(x_1, e^{-\tau/2}) \,. \tag{4.3}$$

Since $\tilde{N}_1$ is a smooth (at least $\mathcal{C}^{1+\alpha}$) function with $\tilde{N}_1(x_1, 0) = 0$ in some neighborhood of the origin, there exists a constant $C_N > 0$, such that $|\tilde{N}_1(x_1, e^{-\tau/2})| \leq C_N \exp(-\tau/2)$, for $|x_1|$ sufficiently small. Integrating (4.3) and applying this estimate yields:

$$
\begin{aligned}
|x_1(\tau_\mathrm{f}) - x_1(\tau_\mathrm{i})| &= \left| \int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} d\sigma\, \tilde{N}_1(x(\sigma), e^{-\sigma/2}) \right| \\
&\leq C_N \int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} d\sigma\, e^{-\sigma/2} = 2C_N e^{-\tau_\mathrm{i}/2}(1 - e^{(\tau_\mathrm{i}-\tau_\mathrm{f})/2}) \ .
\end{aligned}
$$

This estimate immediately implies the behavior claimed in Theorem 4.2.

Proof of Proposition 4.1. The basic idea is to relate $\tilde{N}_1(x_1, 0)$ to the non-linear term of *another problem*, which is known to be 0. This other problem is the center manifold equation for the perturbations of a stationary solution of Eq.(1.1) *restricted to a space of* 2π*-periodic functions*. In this case, the equation analogous to Eq.(1.12) is

$$
\partial_t v = L_\mathrm{per} v + F(v) \ ,
$$

where $F(v)$ collects the non-linear terms in v. The spectrum of L_per is pure point, with a simple zero eigenvalue, and all others negative, and bounded away from 0. The eigenvector with 0 eigenvalue is u'_ε, where u_ε is given by Eq.(1.5). If we call $x_{1,\mathrm{per}}$ the coordinate in the u'_ε direction, then there exists a one-dimensional center manifold, tangent to this direction and given as the graph of a function $H(x_{1,\mathrm{per}})$. A very nice observation by Schneider is that this center manifold must coincide with the translates of the stationary state u_ε, which is formed of fixed points of the Swift-Hohenberg Eq.(1.1). Hence, on this center manifold we must have $\dot{x}_{1,\mathrm{per}} = 0$. Using this information, the equations for this center manifold take a particularly simple form. Let P_per denote the projection onto u'_ε and let $Q_\mathrm{per} = 1 - P_\mathrm{per}$. Then the preceding discussion implies that the flow $\psi_{t,\mathrm{per}}$ is the identity on $x_{1,\mathrm{per}}$, and hence the equations for the invariant manifold read:

$$
\dot{x}_{1,\mathrm{per}} = P_\mathrm{per} F(x_{1,\mathrm{per}}, H(x_{1,\mathrm{per}})) = 0 \ , \tag{4.4}
$$

$$
\begin{aligned}
H(x_{1,\mathrm{per}}) &= \int_{-\infty}^{0} d\tau\, e^{-Q_\mathrm{per} L_\mathrm{per}} Q_\mathrm{per} F(x_{1,\mathrm{per}}, H(x_{1,\mathrm{per}})) \\
&= -\big(Q_\mathrm{per} L_\mathrm{per}\big)^{-1} Q_\mathrm{per} F(x_{1,\mathrm{per}}, H(x_{1,\mathrm{per}})) \ . \qquad (4.5)
\end{aligned}
$$

We now wish to use this information to prove Proposition 4.1. The rough idea is to show that

$$
\tilde{N}_1(x_1, 0) = P_\mathrm{per} F(x_{1,\mathrm{per}}, H(x_{1,\mathrm{per}})) \ , \tag{4.6}
$$

and this quantity vanishes by Eq.(4.4). More precisely, we shall show:

Proposition 4.3. *The cubic term in* x_1 *of* $\tilde{N}_1(x_1, \eta)$ *coincides in the limit* $\eta \to 0$ *with the cubic term in* x_1 *of* $P_\mathrm{per} F(x_{1,\mathrm{per}}, H(x_{1,\mathrm{per}}))$. *All other terms in* $\tilde{N}_1$ *go to 0 as* $\eta \to 0$.

Remark. Since $P_\mathrm{per} F\big(x_{1,\mathrm{per}}, H(x_{1,\mathrm{per}})\big) = 0$, this proves Eq.(4.6) and thus Proposition 4.1.

Proof. The proof of Proposition 4.3 will be given in Appendix D.

5. Completion of the proof of Stability Theorem 1.2

We now consider exactly how the results of the previous two sections about the behavior of solutions in, and near, the invariant manifold translate back into statements about solutions in terms of the original variables. We will focus specifically on the case considered in the previous section in which the invariant manifold is two-dimensional, with coordinates (x_1, η), but the results can be immediately extended to the case of a manifold of arbitrary dimension.

Suppose we have a solution $w_\tau = w_\tau^{\mathrm{c}} + w_\tau^{\mathrm{s}}$, of the system (3.6), which remains in a neighborhood of the origin for all $\tau \geq 0$. This will be the case if its initial condition is sufficiently small in $H_{q,r} \oplus \mathcal{H}_{q,r,\nu}$. We measure the size of w in the norm $|||\cdot|||$, which is the sum of the $H_{q,r}$ norm of w^{c}, and the $\mathcal{H}_{q,r,\nu}$ norm of w^{s}. By the results of Theorem 3.7, we know that there exists a solution, w_τ^{inv}, on the invariant manifold such that

$$|||w_\tau - w_\tau^{\mathrm{inv}}||| \leq Ce^{-\tau(1/2-\delta)} , \tag{5.1}$$

with $\delta > 0$. In addition, from Theorem 4.2, we know that there exists some w^*, which lies in the invariant manifold for which

$$|||w_\tau^{\mathrm{inv}} - w^*||| \leq Ce^{-\tau/2} . \tag{5.2}$$

Here, w^* is the function whose coordinates in the invariant manifold representation is just the limiting point x_1^* in Theorem 4.2, *i.e.*, $w^* = \bigl(x_1^*, 0, h_2^*(x_1^*, 0), h_3^*(x_1^*, 0)\bigr)$. Combining (5.1) and (5.2), we see that for solutions that remain near the origin, there exists a function w^*, for which

$$|||w_\tau - w^*||| \leq Ce^{-\tau(1/2-\delta)} . \tag{5.3}$$

Our final task is now to untangle the various changes of variables which we made in the original equation. If we first "undo" the rescaling in (2.8), we see that the solution $v(\ell, t)$, corresponding to $w(\cdot, \tau) = w_\tau$ is

$$\begin{aligned} v(\ell,t) &= w^{\mathrm{c}}(\mathrm{sign}(\ell)\sqrt{|\Lambda_\ell|(t+t_0)}, \log(t+t_0)) \\ &+ \frac{1}{(t+t_0)^{1/2}} w^{\mathrm{s}}(\mathrm{sign}(\ell)\sqrt{|\Lambda_\ell|(t+t_0)}, \log(t+t_0)) \\ &\equiv v^{\mathrm{c}}(\ell,t) + v^{\mathrm{s}}(\ell,t) . \end{aligned} \tag{5.4}$$

One can make a corresponding decomposition of v^*, the solution corresponding to w^*.

First consider v^{c}. From (5.3), one has

$$\|w_\tau^{\mathrm{c}} - w_\tau^{*,\mathrm{c}}\|_{q,r}^2 = \int dp \, |(1-\partial_p^2)^{r/2}(1+p^2)^{q/2}(w^{\mathrm{c}}(p,\tau) - w^{*,\mathrm{c}}(p,\tau))|^2 \leq Ce^{-\tau(1-2\delta)} . \tag{5.5}$$

According to (5.4), $w^{\mathrm{c}}(\ell,\tau) = v^{\mathrm{c}}(\Phi^{-1}(pe^{-\tau/2}),t)$, so substituting this expression—and the analog for $w^{*,\mathrm{c}}$—into (5.5) one finds that the left hand side of that inequality is equal to:

$$\begin{aligned}
&\int dp\,|(1-\partial_p^2)^{r/2}(1+p^2)^{q/2}\big(v^{\mathrm{c}}(\Phi^{-1}(pe^{-\tau/2}),t)-v^{*,\mathrm{c}}(\Phi^{-1}(pe^{-\tau/2}),t)\big)|^2\\
&\qquad\geq \int dp\,|(1+p^2)^{q/2}\big(v^{\mathrm{c}}(\Phi^{-1}(pe^{-\tau/2}),t)-v^{*,\mathrm{c}}(\Phi^{-1}(pe^{-\tau/2}),t)\big)|^2 \qquad (5.6)\\
&\qquad\geq \int d\ell\,(t+t_0)^{1/2}\Phi'(\ell)\,|(1+(t+t_0)(\Phi(\ell))^2)^{q/2}\big(v^{\mathrm{c}}(\ell,t)-v^{*,\mathrm{c}}(\ell,t)\big)|^2\,,
\end{aligned}$$

where in the last integral we changed the integration variable to $\ell = \Phi^{-1}(pe^{-\tau/2}) = \Phi^{-1}(p(t+t_0)^{-1/2})$.

Remark. We dropped the derivatives with respect to p in the second line of (5.6) for simplicity—one could retain them at the expense of complicating the following expressions.

Since $\Phi(x) \approx x$, for x small, and is equal to a constant times x for $|x|$ large (due to the definition of Λ_ℓ), we see that combining (5.5) and (5.6) and recalling that $t_0 > 0$, one finds:

$$\int d\ell\,|(1+\ell^2)^{q/2}(v^{\mathrm{c}}(\ell,t)-v^{*,\mathrm{c}}(\ell,t))|^2 \leq Ct^{-3/2(1-2\delta)}\,. \qquad (5.7)$$

Analogous estimates hold for the "stable" part of the solution. Proceeding as above, one can show that

$$\sum_n (1+n^2)^\nu \int d\ell\,|(1+\ell^2)^{q/2}(v^{\mathrm{s}}(\ell,t)-v^{*,\mathrm{s}}(\ell,t))|^2 \leq Ct^{-5/2(1-2\delta)}\,. \qquad (5.8)$$

Thus, the "stable" part of a solution near the origin approaches the solution v^* on the invariant manifold faster than the "center" part of the solution. (An effect that is entirely in accord with one's intuition.)

We next take a closer look at the solution w^* (or v^*) on the invariant manifold. From the computation in the previous section, we know that since the eigenfunction in the x_1 direction is $\exp(-p^2)$, *cf.* Eq(2.14), we have $w^*(p) = c^*\exp(-p^2) + h_3^*(c^*\exp(-p^2))$. If we now rewrite this in terms of the $v(\ell,t)$ variables, we find

$$v^*(\ell,t) = c^* e^{-\Lambda_\ell t} + t^{-1/2} h_3^*(c^* e^{-\Lambda_\ell t})\,. \qquad (5.9)$$

Thus, if $v(\ell,t)$ is a solution of (1.12) (in the unscaled variables), we see from (5.7)–(5.9) that in the $L^2((1+\ell^2)^{q/2}d\ell)$ norm,

$$v(\ell,t) = c^* e^{-\Lambda_\ell t} + \mathcal{O}(t^{-1/2(1-2\delta)})\,. \qquad (5.10)$$

But we know from Section 2 that $\Lambda_\ell = \ell^2 + \mathcal{O}(\ell^3)$ for ℓ small, and $\Lambda_\ell = c\ell^2$, for $|\ell|$ large, so one finds by an easy and explicit estimate that

$$\int d\ell\,|(1+\ell^2)^{q/2}(e^{-\Lambda_\ell t} - e^{-\ell^2 t})|^2 \leq Ct^{-1/2}\,. \qquad (5.11)$$

Combining (5.10) and (5.11) one has

Proposition 5.1. *If v is a solution of (1.12) with sufficiently small initial condition (in $H_{q,r} \oplus H_{q,r,\nu}$), then*

$$\left(\int |(1+\ell^2)^{q/2}(v(\ell,t) - c^* e^{-\ell^2 t})|^2 d\ell\right)^{1/2} \leq Ct^{-1/4(1-2\delta)} .$$

Note that if we transform back to the (x,t) variables, this implies the asymptotic estimate in Stability Theorem 1.2, and hence the proof of that theorem is complete.

A. Bounds on the non-linearities

In this section, we prove Proposition 3.1 and Proposition 3.2. We begin by studying the kernels $K_2(\ell,k)$, and $K_3(\ell,k)$ introduced in (2.7).

Lemma A.1. *There is a constant C such that*

$$|K_2(\ell,k)| \leq C\varepsilon \max\bigl((|k|^2 + |\ell|^2), 1\bigr) .$$

Proof. By the definition of Eq.(2.7), we have

$$K_2(\ell,k) = \int dx\, \overline{\varphi}_\ell(x) u_\varepsilon(x) \varphi_k(x) \varphi_{\ell-k}(x) . \tag{A.1}$$

Since u_ε and φ_k are both uniformly bounded, we have immediately that $|K_2(k,\ell)| \leq C\varepsilon$. The crucial observation of Schneider[Sch] is that because of Eq.(1.11), repeated here for convenience

$$\varphi_{\varepsilon,\ell}(x) = u'_\varepsilon(x) + i\ell g_\varepsilon(x) + h_{\varepsilon,\ell}(x)\ell^2 , \tag{A.2}$$

(with real g_ε), K_2 has an expansion

$$\int dx\, u_\varepsilon(x)(u'_\varepsilon(x))^3 + u_\varepsilon(x)\bigl(u'_\varepsilon(x)\bigr)^2\bigl(-i\ell + ik + i(\ell - k)\bigr) + \varepsilon\mathcal{O}(\ell^2 + k^2) . \tag{A.3}$$

Note that the first term vanishes because u is a symmetric function and hence $u(u')^3$ is odd, and the term which is linear in k and ℓ vanishes as well, because of momentum conservation, so the proof of Lemma A.1 is complete.

Remark. Note that a similar calculation immediately shows that the kernel K_3 satisfies:

$$|K_3(\ell,k_1,k_2)| \leq C\varepsilon .$$

We now need the following auxiliary result:

Lemma A.2. *If ρ_2 and ρ_3 are in $H_{q,r}$, and if $\rho_1 = \rho_1(p,p')$ is a $\mathcal{C}^r$ function, then*

$$\Xi(p) \;=\; \int dp'\, \rho_1(p,p')\rho_2(\Delta(p,p',\tau))\rho_3(p')$$

is in $H_{q,r}$ and

$$\|\Xi\|_{q,r} \;\leq\; C\|\rho_1\|_{\mathcal{C}^r}\|\rho_2\|_{q,r}\|\rho_3\|_{q,r}\;.$$

Proof. Recall from Eq.(2.12) that $\Delta(p,p',\tau) \approx p-p'$, so we are really estimating a slightly distorted convolution. If $\Delta(p,p',\tau) = (p-p')$, the proof is easy using the definition of the norms. In the present case, where $\Delta(p,p',\tau)$ is not trivial, the result follows in a similar way by "undoing" part of the variable transformation which led from the variables ℓ, k to the variables p, p'. To simplify matters, we consider only the somewhat easier problem of bounding

$$\int dp'\, \Phi'(pe^{-\tau/2})\rho_2(\Delta(p,p',\tau))\rho_3(p')\;. \tag{A.4}$$

Using the definition of $\Delta(p,p',\tau)$ this is equal to

$$\int dp'\, \Phi'(pe^{-\tau/2})\rho_2\big(e^{\tau/2}\Phi^{-1}\big(\Phi(pe^{-\tau/2})-\Phi(p'e^{-\tau/2})\big)\big)\,\rho_3\big(e^{\tau/2}\Phi^{-1}\big(\Phi(p'e^{-\tau/2})\big)\big)\;. \tag{A.5}$$

Changing variables to $k=\Phi(e^{-\tau/2}p)$ and $\ell=\Phi(e^{-\tau/2}p')$, we get

$$\int d\ell\, e^{\tau/2}\rho_2\big(e^{\tau/2}\Phi^{-1}(k-\ell)\big)\,\rho_3\big(e^{\tau/2}\Phi^{-1}(\ell)\big)\;. \tag{A.6}$$

We now define a function Ψ_τ by

$$\Psi_\tau(e^{\tau/2}x) \;=\; e^{\tau/2}\Phi^{-1}(x)\;,$$

and note that from $\Phi(x)=x\cdot\big(1+\mathcal{O}(x)\big)$ it follows that $\Psi_\tau(y)=y\cdot\big(1+\mathcal{O}(e^{-\tau/2}y)\big)$. We can rewrite Eq.(A.6) as

$$\int d\ell\, e^{\tau/2}\rho_2\big(\Psi_\tau(e^{\tau/2}(k-\ell))\big)\,\rho_3\big(\Psi_\tau(e^{\tau/2}\ell)\big)\;. \tag{A.7}$$

We define next $\hat\rho_j(k)=\rho_j\circ\Psi_\tau$, and we see that Eq.(A.7) is equal to

$$\int d\ell\, \hat\rho_2\big(k-\ell\big)\,\hat\rho_3\big(\ell\big)\;. \tag{A.8}$$

Thus, we can bound the $H_{q,r}$ norm of Eq.(A.4) by $\|\hat\rho_2\|_{q,r}\|\hat\rho_3\|_{q,r}$, and, since Ψ_τ is uniformly close to the identity for all τ, this is in turn bounded by const. $\|\rho_2\|_{q,r}\|\rho_3\|_{q,r}$. This proves

Lemma A.2 in this special case. The extension to the general case is easy and is left to the reader.

We now have the necessary tools to attack the proofs of Proposition 3.1 and Proposition 3.2.

Proof of Proposition 3.1. If we write out the transformation leading to f_2, *i.e.*, from Eq.(2.7) to Eq.(2.9), we get, using Eq.(2.10),

$$\begin{aligned} e^\tau \cdot \bigl(f_2(w, e^{-\tau/2})\bigr)(p) \;&=\; e^\tau 3\chi\bigl(\Phi(pe^{-\tau/2})\bigr) \int_{-P(\tau)}^{P(\tau)} dp'\, e^{-\tau/2}\Phi'(p'e^{-\tau/2}) \\ &\times K_2\bigl(\Phi(pe^{-\tau/2}), \Phi(p'e^{-\tau/2})\bigr) w(\Delta(p,p',\tau))w(p')\;, \end{aligned} \tag{A.9}$$

where

$$P(\tau) \;=\; \Phi^{-1}(\tfrac{1}{2})e^{\tau/2} \;\approx\; \tfrac{1}{2}e^{\tau/2}\;.$$

We bound $|K_2\bigl(\Phi(pe^{-\tau/2}), \Phi(p'e^{-\tau/2})\bigr)|$ by $C\varepsilon|\Phi(pe^{-\tau/2})^2 + \Phi(p'e^{-\tau/2})^2|$ using Lemma A.1. Since the expressions $\Phi(pe^{-\tau/2})$, and $\Phi(p'e^{-\tau/2})$, in Eq.(A.9) are bounded, and $\Phi(x) = x(1+\mathcal{O}(x))$, we can extract another factor of $e^{-\tau/2}$ and get a bound on $e^\tau f_2$ of the form

$$\begin{aligned} &\text{const. } e^{\tau/2}\chi\bigl(\Phi(pe^{-\tau/2})\bigr) \int_{-P(\tau)}^{P(\tau)} dp'\, \bigl(\bigl|\Phi(pe^{-\tau/2})\bigr| + \bigl|\Phi(p'e^{-\tau/2})\bigr|\bigr) \cdot |w(\Delta(p,p',\tau))w(p')| \\ &\quad\le\; \text{const. } \chi\bigl(\Phi(pe^{-\tau/2})\bigr) \int_{-\infty}^{\infty} dp'\, |p+p'||w(\Delta(p,p',\tau))w(p')|\;. \end{aligned} \tag{A.10}$$

If w is in $H_{q,r}$, then with the aid of Lemma A.2, we can estimate the $H_{q-1,r}$ norm of Eq.(A.9) by $C\|w\|_{q,r}^2$. Note further, that from the above discussion it is also clear that $e^\tau f_2(w^{\mathrm{c}}, e^{-\tau/2})(p)$ is also a smooth function of $e^{-\tau/2}$.

Remark. The factors $|p|, |p'|$ are responsible for the loss of one power in the norm estimate of Proposition 3.1. It is only in the study of the flow within the invariant manifold that we will need the second order bound of Lemma A.1.

Remark. Note that the nonlinear terms depend (implicitly) on the constant t_0 which entered the definition of the new temporal variable τ. However, all the estimates above (as well as those which follow in the proof of Proposition 3.2) are independent of this constant.

The bound on f_3 is similar, but no additional regularization is needed, since there are *two* integrations, each of which contributes a factor $e^{-\tau/2}$. We leave this to the reader. The proof of the asserted bounds of Eq.(3.2) is complete.

We now turn to the estimates of the nonlinear terms f_4 and g. Because these terms involve the w^{s}, we begin with a discussion of the appropriate function space for these components. These were defined in Section 3, but we repeat them here for convenience. Recall that $w^{\mathrm{c}} \in H_{q,r}$, while $w^{\mathrm{s}} \in H_{q,r} \oplus H_{q,r,\nu}$, where

$$\begin{aligned} H_{q,r,\nu} \;=\; \{w = w(p;x) \mid w(p;x) = w(p;x+2\pi),& \\ (1-\partial_x^2)^{\nu/2}(1-\partial_p^2)^{r/2}(1+p^2)^{q/2}w \in L^2(\mathbf{R}\times[-\pi,\pi])\}\;.& \end{aligned}$$

The fact that w^{s} is an element of the direct sum of two spaces reflects the fact (see the paragraph preceding (2.6), and then (2.8)) that it has two components, the first of which comes from the central branch of the spectrum of L_ℓ, but with ℓ localized away from zero, and the second component coming from the stable branches of the spectrum of L_ℓ. In a slight abuse of notation we will denote by $\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}}$ the sum of the $H_{q,r}$ norm of the first component of w^{s} and the $H_{q,r,\nu}$ norm of the second component, and by $\|w^{\mathrm{s}}\|_{q,r,\nu}$, we will mean the $H_{q,r,\nu}$ norm of just the second component.

Remark. An easy fact which will be useful later is that if we expand $w(p;x) \in H_{q,r,\nu}$ in a Fourier series with respect to x,

$$w(p;x) \;=\; \sum_{n=-\infty}^{\infty} e^{inx}\hat{w}_n(p) \;,$$

then the $H_{q,r,\nu}$ norm of w is equivalent to the norm

$$\|w\|^2_{H_{q,r,\nu}} \;=\; \sum_{n=-\infty}^{\infty} (1+n^2)^\nu \|\hat{w}_n\|^2_{q,r} \;. \tag{A.11}$$

Thus we will use the two norms interchangeably.

Now consider

$$e^\tau f_4(w^{\mathrm{c}}, w^{\mathrm{s}}e^{-\tau/2}, e^{-\tau/2}) \;. \tag{A.12}$$

We shall concentrate on the most “dangerous” piece which is the quadratic term with one factor of w^{c} and one of w^{s}. Other terms are “less dangerous” in the sense that they contain either more factors of w^{s} each of which contributes a small factor of $e^{-\tau/2}$, or more convolutions which again contribute a factor of $e^{-\tau/2}$. The quadratic piece of (A.12) has the form

$$\begin{aligned} & e^\tau 3\chi\big(\Phi(pe^{-\tau/2})\big) \int dx\, \bar{\varphi}_{\Phi(pe^{-\tau/2})}(x)\, u_\varepsilon(x) \\ & \times \int_{-P(\tau)}^{P(\tau)} dp'\, e^{-\tau/2}\Phi'(p'e^{-\tau/2}) w^{\mathrm{c}}\big(\Delta(p,p',\tau)\big) \\ & \times \varphi_{\Gamma(p,p',\tau)}(x)\, e^{-\tau/2} w^{\mathrm{s}}(p';x) \;. \end{aligned} \tag{A.13}$$

As we mentioned above, w^{s} has two components—one in $H_{q,r}$, and one in $H_{q,r,\nu}$. The contribution from the component in $H_{q,r}$ is bounded by the same techniques used to control f_3—note that it is not necessary to extract any additional factors of $e^{-\tau/2}$, since we get one from the integration, and one from the fact that each factor of w^{s} is multiplied by $e^{-\tau/2}$. Thus, we restrict our attention to the component of w^{s} in $H_{q,r,\nu}$, which is where the new ingredients are necessary.

Interchanging the order of the x and p' integrals, we use Lemma A.2, with

$$\begin{aligned} \rho_1(p,p') \;&=\; \sup_x \big|3\chi\big(\Phi(pe^{-\tau/2})\big)\Phi'(p'e^{-\tau/2})\, \varphi_{\Phi(pe^{-\tau/2})}(x)\, \varphi_{\Gamma(p,p',\tau)}(x) u_\varepsilon(x)\big| \;, \\ \rho_2(r) \;&=\; |w^{\mathrm{c}}(r)| \;, \\ \rho_3(p') \;&=\; \Big|\int dx\, w^{\mathrm{s}}(p';x)\Big| \;. \end{aligned}$$

Since $\varphi_\ell(x)$ and $u_\varepsilon(x)$ are smooth, 2π-periodic functions of x, and $\|\rho_1\|_{\mathcal{C}^r}$ is bounded, the Lemma A.2 implies that the $H_{q,r}$ norm of (A.13) is bounded by

$$C\|w^{\mathrm{c}}\|_{q,r}\, \| \int dx\, w^{\mathrm{s}}(\cdot;x)\|_{q,r} \,. \tag{A.14}$$

The $H_{q,r}$ norm of the integral can be bounded by

$$\sup_x \|w^{\mathrm{s}}(\cdot;x)\|_{q,r} \,\le\, C\|w^{\mathrm{s}}\|_{H_{q,r,\nu}} \,, \tag{A.15}$$

provided $\nu > 1/2$, where we used Sobolev's inequality to estimate the supremum over x. Inserting (A.15) into (A.14) yields the bound claimed in (3.3).

The remaining terms in f_4 can be bounded in a similar fashion, but as noted above, they will tend to 0 as $\tau \to \infty$. In fact, they will be bounded by $C\varepsilon e^{-\tau/2}$.

Proof of Eq.(3.4) of Proposition 3.2. We finally bound the non-linear term

$$e^{\tau/2} g(w^{\mathrm{c}}, w^{\mathrm{s}}e^{-\tau/2}, e^{-\tau/2}) \,. \tag{A.16}$$

In bounding $e^{\tau/2}g(w^{\mathrm{c}}, w^{\mathrm{s}}e^{-\tau/2}; e^{-\tau/2})$, recall that just as w^{s} did, this expression will have two components—one in $H_{q,r}$, and one in $H_{q,r,\nu}$. The component in $H_{q,r}$ is bounded using exactly the same techniques used to control the term f_4 above, so we concentrate here on explaining the new ingredients necessary to bound the component in $H_{q,r,\nu}$.

As in the bound on f_4, the potentially largest terms are those of minimal order, because each additional order provides a factor of $e^{-\tau/2}$. So we look at the terms which are quadratic and which are of order $w^{\mathrm{c}}w^{\mathrm{c}}$, $w^{\mathrm{c}}w^{\mathrm{s}}$, and $w^{\mathrm{s}}w^{\mathrm{s}}$, respectively. The first term leads us to study

$$e^{\tau/2} P_p^{\perp}\left(u_\varepsilon(x) \int_{-1/2}^{1/2} dp'\, V^{\mathrm{c}}(p-p')V^{\mathrm{c}}(p')\varphi_{p'}(x)\,\varphi_{p-p'}(x)\right) . \tag{A.17}$$

Rescaling as in (2.8), we see we must bound

$$\begin{aligned} P^{\perp}_{\Phi(pe^{-\tau/2})}\Bigg(u_\varepsilon(x) \int_{-P(\tau)}^{P(\tau)} dp'\, \Phi'(p'e^{-\tau/2}) \\ \times \varphi_{\Phi(p'e^{-\tau/2})}(x)\,\varphi_{\Gamma(p,p',\tau)}(x)\, w^{\mathrm{c}}(p')\, w^{\mathrm{c}}(\Delta(p,p',\tau))\Bigg) . \end{aligned} \tag{A.18}$$

Note that the prefactor of $e^{\tau/2}$ has disappeared due to the factor of $e^{-\tau/2}$ which we gain as usual from the change of variables.

Since the projection $P_\ell^{\perp}$ has bounded norm and is a smooth function of ℓ, we can discard this factor at the price of introducing an overall constant in the estimate. Note next that the square of the $H_{q,r,\nu}$ norm of the remaining expression is equal to:

$$\begin{aligned} \Bigg\| \int_{-P(\tau)}^{P(\tau)} dp'\, \Phi'(p'e^{-\tau/2}) w^{\mathrm{c}}(p') w^{\mathrm{c}}(\Delta(p,p',\tau)) \\ \times \|u_\varepsilon(x)\,\varphi_{\Phi(p'e^{-\tau/2})}(x)\,\varphi_{\Gamma(p,p',\tau)}(x)\,\|^2_{H^\nu(dx)} \Bigg\|^2_{H_{q,r}(dp)} , \end{aligned} \tag{A.19}$$

where the H^ν norm is the H^ν-Sobolev norm of the quantity

$$u_\varepsilon(x)\, \varphi_{\Phi(p'e^{-\tau/2})}(x)\, \varphi_{\Gamma(p,p',\tau)}(x)\ ,$$

considered as a function of x, and the $H_{q,r}$ norm is the norm of the resulting function of p. Since $u_\varepsilon(x)\, \varphi_{\Phi(p'e^{-\tau/2})}(x)\, \varphi_{\Gamma(p,p',\tau)}(x)$ is a smooth function of x, p', and p, there exists a smooth, bounded function $\psi(p,p')$, such that

$$\psi(p,p') \;=\; \|u_\varepsilon(x)\, \varphi_{\Phi(p'e^{-\tau/2})}(x)\, \varphi_{\Gamma(p,p',\tau)}(x)\, \|_{H^\nu(dx)}\ . \tag{A.20}$$

But now, by Lemma A.2, we can conclude that (A.19) is bounded by

$$\| \int_{-P(\tau)}^{P(\tau)} dp'\, \Phi'(p'e^{-\tau/2}) w^{\mathrm{c}}(p') w^{\mathrm{c}}(\Delta(p,p',\tau))\psi(p,p')\|^2_{H_{q,r}(dp)} \;\le\; C\|\Phi'\psi\|^2_{\mathcal{C}^r}\|w^{\mathrm{c}}\|^4_{q,r}\ . \tag{A.21}$$

We next consider the quadratic term in g which contains one factor of w^{c} and one factor of w^{s}. In this case, the analog of (A.18) is

$$\begin{aligned} e^{-\tau/2} P^{\perp}_{\Phi(pe^{-\tau/2})}\Bigg(u_\varepsilon(x) \int_{-P(\tau)}^{P(\tau)} dp'\, \Phi'(p'e^{-\tau/2}) \\ \times\, \varphi_{\Gamma(p,p',\tau)}(x)\, w^{\mathrm{c}}(\Delta(p,p',\tau)) w^{\mathrm{s}}(p';x) \Bigg)\ . \end{aligned} \tag{A.22}$$

Note that in this case, we pick up an extra factor of $e^{-\tau/2}$, in comparison with (A.18), since each factor of w^{s} is multiplied by this exponential.

Once again, we must contend with the fact that w^{s} has two components. However, the component in $H_{q,r}$ behaves exactly as in the estimates leading to (3.2), so we concentrate on the component in $H_{q,r,\nu}$.

As above, the projection operator can be dropped at the cost of an overall constant, and we are left with the task of bounding the $H_{q,r,\nu}$ norm of the remainder. The square of this norm is equal to

$$\begin{aligned} &\Bigg\| \int_{-P(\tau)}^{P(\tau)} dp'\, \Phi'(p'e^{-\tau/2}) w^{\mathrm{c}}(\Delta(p,p',\tau)) \\ &\qquad \times \|u_\varepsilon(x)\, \varphi_{\Phi(p'e^{-\tau/2})}(x)\, \varphi_{\Gamma(p,p',\tau)}(x)\, w^{\mathrm{s}}(p';x)\|^2_{H^\nu(dx)}\Bigg\|^2_{H_{q,r}(dp)} \\ &\qquad \le\; C\|\Phi'\|_{\mathcal{C}^r}\, \|w^{\mathrm{c}}\|^2_{H_{q,r}} \\ &\qquad\quad \times \Bigg\| \|u_\varepsilon(x)\, \varphi_{\Phi(p'e^{-\tau/2})}(x)\, \varphi_{\Gamma(p,p',\tau)}(x)\, w^{\mathrm{s}}(p';x)\|^2_{H^\nu(dx)}\Bigg\|^2_{H_{q,r}(dp')}\ , \end{aligned} \tag{A.23}$$

by Lemma A.2. Note that the pair of norms on the last factor is equivalent to computing the square of the $H_{q,r,\nu}$ norm of

$$u_\varepsilon(x)\, \varphi_{\Phi(p'e^{-\tau/2})}(x)\, \varphi_{\Gamma(p,p',\tau)}(x)\, w^{\mathrm{s}}(p';x)\ . \tag{A.24}$$

Since u_ε, Φ, φ_Γ, and Δ are all smooth, bounded functions, we see just by writing out the definition of the norm that this is bounded by

$$C\|w^{\mathrm{s}}\|^2_{H_{q,r,\nu}} \ . \tag{A.25}$$

If we estimate the term quadratic in w^{s} in a similar fashion, and combine this estimate with that in (A.21) we see that the quadratic terms in $e^{-\tau/2}g(w^{\mathrm{c}}, w^{\mathrm{s}}e^{-\tau/2}; e^{-\tau/2})$ are bounded in $\mathcal{H}_{q,r,\nu}$, by

$$C(\|w^{\mathrm{c}}\|_{H_{q,r}} + e^{-\tau/2}\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}})^2 \ . \tag{A.26}$$

Analogous estimates of the cubic terms lead to a bound

$$Ce^{-\tau/2}(\|w^{\mathrm{c}}\|_{H_{q,r}} + e^{-\tau/2}\|w^{\mathrm{s}}\|_{\mathcal{H}_{q,r,\nu}})^3 \ , \tag{A.27}$$

where the additional factor of $e^{-\tau/2}$ comes from the additional convolution. Combining (A.26) and (A.27) leads to the estimate in (3.4) and completes the proof of Proposition 3.2.

B. Bounds on the linear operators

In this Appendix, we give bounds on the semi-group generated by L and on the linear evolution defined by $M_{\exp(-\tau/2)}$.

B.1. Bound on the semi-group generated by L

We consider the semi-group whose generator is $L = \partial_x^2 + \frac{1}{2}x\partial_x + \frac{1}{2}$. Note that in this section, for ease of use, we define L in the Fourier transformed variables, compared to Section 2. Fourier transformation is an isomorphism from $H_{q,r}$ (in the p-variables) to $H_{r,q}$ (in the x-variables), so establishing estimates on the semigroup associated to $\partial_x^2 + \frac{1}{2}x\partial_x + \frac{1}{2}$ in the space $H_{r,q}(dx)$ will immediately imply estimates on the representation of L in the p-variables in the space $H_{q,r}(dp)$. In order to avoid confusion, in what follows we will denote by $|\cdot|_{q,r}$ the norm on $H_{r,q}(dx)$. With this notation, the norms $\|\cdot\|_{q,r}$ and $|\cdot|_{q,r}$ resp. the spaces $H_{q,r}(dp)$ and $H_{r,q}(dx)$ are equivalent.

The integral kernel of the semigroup generated by L is given by [GJ]

$$\left(e^{\tau L}v\right)(x) \;=\; \frac{1}{\sqrt{4\pi a(\tau)}}\int dz\, e^{-z^2/(4a(\tau))}v(e^{\tau/2}(x+z)) \ ,$$

where $a(\tau) = 1 - e^{-\tau}$. If we denote by T the operator of multiplication by $\exp(x^2/8)$ and by H_0 the harmonic oscillator Hamiltonian $H_0 = \partial_x^2 - x^2/16 + 1/4$, (note the unconventional sign!), then

$$L \;=\; T^{-1}H_0T \ .$$

Thus, the two operators L and H_0 are “the same,” but they act on two quite different spaces. If the $\{\varphi_j\}_{j\geq 0}$ are the eigenfunctions of H_0, then the $\psi_j = T^{-1}\varphi_j$ are the eigenfunctions of L, with the same eigenvalues $\mu_j = -j/2$. We let $P_n f = \sum_{j\le n} \psi_j (\psi_j, f)_q$, where $(\cdot,\cdot)_q$ is the scalar product

$$(f,g)_q = (Tf, T(1-L_0)^q g) = (Tf, (1-H_0)^q Tg)\ .$$

We next show that for $n < q-2$, the operator P_n is bounded in $H_{r,q}(dx)$. First of all, the eigenfunctions φ_j are bounded by $\mathcal{O}(1)|x|^j e^{-x^2/8}$ at large x. Therefore, we also have $\psi_j = T^{-1}\varphi_j \in H_{r,q}(dx)$, since it decays exponentially. Finally,

$$(\psi_j, f)_q = (T\psi_j, (1-H_0)^q Tf) = |1-\mu_j|^q (\varphi_j, Tf)\ ,$$

and the last scalar product is bounded if $f \in H_{r,q}(dx)$ when $r > j+2$, since, with a weight function $W(x) = (1+x^2)^{1/2}$,

$$\begin{aligned} |(\varphi_j, Tf)| &\le C\big|(W^j, f)\big| \le C\big|(W^{-1}, W^{j+1} f)\big| \\ &\le C\|W^{j+1} f\|_2 \le C|f|_{0,r}\ . \end{aligned}$$

Thus P_n is defined. We let $Q_n = 1 - P_n$ (in $H_{r,q}(dx)$).

Theorem B.1. *For every $\varepsilon > 0$, there are an m_0 and a function $r(m,q)$ such that for every $m \ge m_0$, every $q \ge 1$ and every $r \ge r(m,q)$, there is a $C = C(q,r,m) < \infty$ such that*

$$|e^{\tau L} Q_m v|_{q,r} \le \frac{C(q,r,m)}{\sqrt{a(\tau)}} e^{-\tau|\mu_m| + \tau\varepsilon} |v|_{q-1,r}\ . \tag{B.1}$$

Remark. The function $r(m,q)$ is of order $\mathcal{O}(m+q)$.

Proof. To explain the strategy of the proof, we need some notation. Let $P_n^{(0)}$ denote the projection in $H_{0,q}(dx)$ onto the subspace spanned by $\{\varphi_j\}_{j\le n}$ and let $Q_n^{(0)} = 1 - P_n^{(0)}$. Then, formally, $TQ_n = Q_n^{(0)} T$, and $LQ_n = T^{-1} H_0 Q_n^{(0)} T$. This suggests that L restricted to Q_n has no spectrum in the half-plane $\{z \mid \mathrm{Re}\, z > -|\mu_{n+1}|\}$, and thus one can understand the decay in Eq.(B.1). The square-root singularity at $\tau = 0$ is related to our gain in smoothness. The problem is that $TQ_n = Q_n^{(0)} T$ is ill-defined. However, it will be well defined if we localize near $x = 0$. In that region, the heuristic argument will be seen to be valid, whereas in the complement of such a region, when $|x| > R$, decay will be shown by direct methods, using the explicit form of the integral kernel.

We study first the quantity $\chi_R e^{\tau L}$, where χ_R is a smooth characteristic function which vanishes for $|x| < R$ and is equal to 1 for $|x| > 4R/3$. Thus we study a region far from the origin. Our bound is

Proposition B.2. *For every $q \ge 1$ and every $r \ge 0$ there exists a $C(q,r) < \infty$ such that for all $v \in H_{r,q}(dx)$ one has*

$$|\chi_R e^{\tau L} v|_{q,r} \le \frac{C(q,r)}{\sqrt{a(\tau)}} e^{\tau q/2} \left(e^{-\tau r/2} + e^{-3R^2/16} \right) |v|_{q-1,r}\ , \tag{B.2}$$

$$|\chi_R e^{\tau L} v|_{q,r} \le C(q,r) e^{\tau q/2} \left(e^{-\tau r/2} + e^{-3R^2/16} \right) |v|_{q,r}\ . \tag{B.3}$$

Corollary B.3. *For every* $q \geq 1$ *and every* $r \geq 0$ *there exists a* $C(q,r) < \infty$ *such that for all* $v \in H_{r,q}(dx)$ *one has*

$$|e^{\tau L} v|_{q,r} \leq \frac{C(q,r)}{\sqrt{a(\tau)}} e^{\tau q/2} |v|_{q-1,r} , \tag{B.4}$$

$$|e^{\tau L} v|_{q,r} \leq C(q,r) e^{\tau q/2} |v|_{q,r} . \tag{B.5}$$

Remarks. The improvement over [W] is that we "gain" a derivative in x. The corollary follows easily by repeating the proof of Proposition B.2 with $R = 0$.

Proof. We let $\mathrm{D} = \partial_x$ and denote, as before, by W the operator of multiplication by $(1+x^2)^{1/2}$. Then

$$|\chi_R e^{\tau L} w|_{q,r}^2 \quad \text{and} \quad \sum_{q' \leq q} \|W^r \mathrm{D}^{q'} \chi_R e^{\tau L} w\|_2^2$$

are equivalent. We shall only consider the term with the highest derivative, because only there is the issue of regularization important. Thus we are led to bound

$$X^2 = \|W^r \mathrm{D}^q \chi_R e^{\tau L} w\|_2^2 .$$

Since $L = \partial_x^2 + \frac{1}{2} x \partial_x + \frac{1}{2}$, a quick calculation shows that

$$\mathrm{D}^q e^{\tau L} = e^{\tau q/2} e^{\tau L} \mathrm{D}^q .$$

The diverging factor exp$(\tau q/2)$ will appear in the final bound. Note now that

$$\bigl(e^{\tau L} \mathrm{D}^q v\bigr)(x) = \frac{1}{\sqrt{4\pi a(\tau)}} e^{\tau/2} \int dz\, e^{-z^2/(4a(\tau))} \bigl(\mathrm{D}^q v\bigr)\bigl(e^{\tau/2}(x+z)\bigr) , \tag{B.6}$$

which upon integrating by parts becomes

$$\frac{1}{\sqrt{4\pi a(\tau)}} \int dz\, \frac{z}{2a(\tau)} e^{-z^2/(4a(\tau))} \bigl(\mathrm{D}^{q-1} v\bigr)\bigl(e^{\tau/2}(x+z)\bigr) .$$

Use now the Schwarz inequality in the form (for positive f and g),

$$\begin{aligned} \|f * g\|_2^2 &= \int dx \int dz_1\, dz_2\, f(z_1) f(z_2) g(x - z_1) g(x - z_2) \\ &\leq \int dz_1\, dz_2\, f(z_1) f(z_2) \|g(\cdot - z_1)\|_2 \|g(\cdot - z_2)\|_2 \\ &= \left(\int dz\, f(z) \|g(\cdot - z)\|_2 \right)^2 . \end{aligned}$$

This leads to a bound

$$X \le \frac{e^{\tau q/2}}{\sqrt{4\pi a(\tau)}} \int_{R_1 \cup R_2} dz \, \frac{|z|}{2a(\tau)} e^{-z^2/(4a(\tau))} \| W^r \chi_R \big(\mathrm{D}^{q-1} w\big)\big(e^{\tau/2}(\cdot + z)\big)\|_2 \\ \equiv X_1 + X_2 \,, \tag{B.7}$$

where we let $R_1 = \{x \ : \ |x| < 7R/8\}$ and $R_2 = \mathbf{R} \setminus R_1$. To be more precise, we define χ_R by the scaling of a fixed function: $\chi_R(x) = \chi(x/R)$. If $R \to \infty$, then $\partial_x \chi_R(x) = \mathcal{O}(R^{-1})$ and therefore it is uniformly bounded.

Lemma B.4. *(Lemma A.2 of [W]). One has the bounds*

$$\| W^r \chi_R(\cdot) v(e^{\tau/2}(\cdot + z))\|_2^2 \le \begin{cases} Ce^{-r\tau} |v|_{0,r}^2 \,, & \textit{if } |z| \le 7R/8, \\ C(1+z^2)^r |v|_{0,r}^2 \,, & \textit{if } |z| > 7R/8. \end{cases} \tag{B.8}$$

Proof of Lemma B.4. Consider first the case $|z| \le 7R/8$. Since $|x| > R$ on the support of χ_R, we have $|x+z| \ge |x|/8$ and hence

$$(1+x^2)/\big(1 + (e^{\tau/2}|x+z|)^2\big) \le \text{ const. } e^{-\tau} \,.$$

Using this, we bound

$$\int_{R_1} dx \, (1+x^2)^r |\chi_R(x) v(e^{\tau/2}(x+z))|^2 \\ = \int_{R_1} dx \, \frac{(1+x^2)^r}{\big(1+(e^{\tau/2}|x+z|)^2\big)^r} \cdot \big(1 + (e^{\tau/2}|x+z|)^2\big)^r |v(e^{\tau/2}(x+z))|^2 \\ \le \text{ const. } e^{-\tau r} e^{-\tau/2} |v|_{0,r}^2 \le \text{ const. } e^{-\tau r} |v|_{0,r}^2 \,.$$

In the second case, we get

$$\int_{R_2} dx \, (1+x^2)^r |\chi_R(x) v(e^{\tau/2}(x+z))|^2 \\ = e^{-\tau/2} \int dy \, \frac{\big(1+(e^{-\tau/2}y - z)^2\big)^r}{(1+y^2)^r} (1+y^2)^r |v(y)|^2 \\ \le \text{ const. } e^{-\tau/2}(1+z^2)^r |v|_{0,r}^2 \le \text{ const. } (1+z^2)^r |v|_{0,r}^2 \,.$$

The proof of Lemma B.4 is complete.

Continuing the proof of Proposition B.2, we first bound the integral over R_1 in Eq.(B.7). We get from the first alternative of Lemma B.4,

$$X_1 = \frac{1}{\sqrt{4\pi a(\tau)}} e^{\tau q/2} \int_{R_1} dz \, \frac{|z|}{\sqrt{2a(\tau)}} e^{-z^2/(4a(\tau))} \| W^r \chi_R \big(\mathrm{D}^{q-1} w\big)\big(e^{-\tau/2}(\cdot + z)\big)\|_2 \\ \le \text{ const. } \frac{1}{\sqrt{4\pi a(\tau)}} e^{\tau q/2} \int_{R_1} dz \, \frac{|z|}{\sqrt{2a(\tau)}} e^{-z^2/(4a(\tau))} e^{-\tau r/2} |w|_{q-1,r} \\ \le \text{ const. } \frac{1}{\sqrt{4\pi a(\tau)}} e^{\tau(q/2 - r/2)} |w|_{q-1,r} \,.$$

Similarly, using the second alternative in Eq.(B.8), we get

$$
\begin{aligned}
X_2 &= \frac{1}{\sqrt{4\pi a(\tau)}} e^{\tau q/2} \int_{R_2} dz \, \frac{|z|}{\sqrt{2a(\tau)}} e^{-z^2/(4a(\tau))} \|W^r \chi_R \big(\mathrm{D}^{q-1} w\big)(e^{\tau/2}(\cdot + z))\|_2 \\
&\le \text{ const. } \frac{1}{\sqrt{4\pi a(\tau)}} e^{\tau q/2} \int_{R_2} dz \, (1+z^2)^{r/2} \frac{|z|}{\sqrt{2a(\tau)}} e^{-z^2/(4a(\tau))} |w|_{q-1,r} \, . \\
&\le \text{ const. } \frac{1}{\sqrt{4\pi a(\tau)}} e^{\tau q/2} e^{-3R^2/16} |w|_{q-1,r} \, ,
\end{aligned}
$$

since $3/16 < (7/8)^2/4$. Note that the constants above depend on r and q, but can be chosen uniformly for all $R \ge 1$. The proof of Eq.(B.2) is complete. Omitting the integration by parts in Eq.(B.6), the assertion Eq.(B.3) follows in the same way. The proof of Proposition B.2 is complete.

We next study $e^{\tau L} Q_n (1-\chi_R) w$. We have the following bound

Proposition B.5. *For every $\varepsilon > 0$, $q \ge 1$, and every $r \ge 0$ there is a $C(\varepsilon, q, r) < \infty$ such that*

$$
|e^{\tau L} Q_n (1-\chi_R) w|_{q,r} \le \frac{C(\varepsilon, q, r)}{\sqrt{a(\tau)}} e^{-|\mu_{n+1}|\tau + \tau\varepsilon} e^{R^2/6} |w|_{q-1,r} \, . \tag{B.9}
$$

Proof. Recall that $T = e^{x^2/8}$ and that $L = T^{-1} H_0 T$. The operator $T(1-\chi_R)$ is bounded and $\|T(1-\chi_R)\| \le$ const. $e^{R^2/6}$. Therefore we have

$$
\begin{aligned}
Q_n T(1-\chi_R) &= (1-P_n) T(1-\chi_R) = T(1-\chi_R) - T P_n^{(0)} (1-\chi_R) \\
&= T(1-P_n^{(0)})(1-\chi_R) = T Q_n^{(0)} (1-\chi_R) \, ,
\end{aligned}
$$

where $Q_n^{(0)}$ is the orthogonal projection onto the complement of the subspace spanned by the first n eigenvalues of H_0 in $H_{q,0}$. It is easy to see that on $H_{r,q}(dx)$, the operator $(1+x^2)^{1/2}(1-H_0)^{-1/2}$ is bounded. Thus, we get, using the spectral properties of H_0 (on $Q_n^{(0)}$),

$$
\begin{aligned}
|e^{\tau H_0} T Q_n (1-\chi_R) w|_{q,r} &= \tau^{-1/2} \\
&\quad \times \; |(1-H_0)^{-1/2} e^{\tau H_0} (\tau(1-H_0))^{1/2} Q_n^{(0)} T (1-\chi_R) w|_{q,r} \\
&\le \text{ const. } \tau^{-1/2} |e^{\tau H_0} (\tau(1-H_0))^{1/2} Q_n^{(0)} T (1-\chi_R) w|_{q-1,r} \\
&\le \text{ const. } \tau^{-1/2} e^{-\tau|\mu_{n+1}| + \tau\varepsilon} |T(1-\chi_R) w|_{q-1,r} \\
&\le \text{ const. } \tau^{-1/2} e^{-\tau|\mu_{n+1}| + \tau\varepsilon} e^{R^2/6} |w|_{q-1,r} \, .
\end{aligned}
\tag{B.10}
$$

The proof of Proposition B.5 is complete.

End of proof of Theorem B.1. We first rewrite $e^{\tau L} Q_n$ as

$$
e^{\tau L} Q_n = e^{\tau L/2} Q_n e^{\tau L/2} = e^{\tau L/2} Q_n \chi_R e^{\tau L/2} + e^{\tau L/2} Q_n (1-\chi_R) e^{\tau L/2} \, .
$$

The second term can be bounded by Proposition B.5 and Eq.(B.5) as

$$\begin{aligned} |e^{\tau L/2} Q_n (1-\chi_R) e^{\tau L/2} w|_{q,r} &\leq \frac{C}{\sqrt{a(\tau)}} e^{R^2/6 - \tau|\mu_{n+1}|/4} |e^{\tau L/2} w|_{q-1,r} \\ &\leq \frac{C}{\sqrt{a(\tau)}} e^{R^2/6 - \tau|\mu_{n+1}|/4} e^{\tau q/4} |w|_{q-1,r} \ . \end{aligned}$$

This quantity is bounded by

$$\frac{C}{\sqrt{a(\tau)}} e^{-\tau n/8} |w|_{q-1,r} \ , \tag{B.11}$$

provided n is much larger than q and $R^2/6 < \tau n/16$. The first term can be bounded by Eq.(B.5) and Eq.(B.2) as

$$\begin{aligned} |e^{\tau L/2} Q_n \chi_R e^{\tau L/2} w|_{q,r} &\leq \frac{C}{\sqrt{a(\tau)}} e^{\tau q/4} |\chi_R e^{\tau L/2} w|_{q-1,r} \\ &\leq \frac{C}{\sqrt{a(\tau)}} e^{\tau q/2} \left(e^{-\tau r/2} + e^{-3R^2/16} \right) |w|_{q-1,r} \\ &\leq \frac{C}{\sqrt{a(\tau)}} e^{-\tau n/8} |w|_{q-1,r} \ , \end{aligned} \tag{B.12}$$

provided $r \geq n/4 + q$ and $3R^2/16 \geq \tau(n/8 + q/2)$. Note that the conditions on R from the first and second term are compatible

Combining Eqs.(B.11)–(B.12), we get

$$|e^{\tau L} Q_n w|_{q,r} \leq \frac{C}{\sqrt{a(\tau)}} e^{-\tau n/8} |w|_{q-1,r} \ . \tag{B.13}$$

It remains to improve the decay rate from $n/8$ to $|\mu_{m+1}|$. The idea is to just take $n = 8(m+1)$. Then we find

$$e^{\tau L} Q_m = e^{\tau L} Q_n Q_m + e^{\tau L} P_m Q_m + e^{\tau L} (P_n - P_m) Q_m \ . \tag{B.14}$$

The first term is bounded by Eq.(B.13), and $m/8 > -|\mu_{n+1}|$. The second term vanishes and the third is diagonalized explicitly:

$$e^{\tau L} (P_n - P_m) Q_m = T^{-1} e^{-\tau H_0} T (P_n - P_m) Q_m = T^{-1} e^{-\tau H_0} (P_n^{(0)} - P_m^{(0)}) T Q_m \ .$$

We are operating here on the finite dimensional subspace spanned by the eigenvectors φ_{m+1}, $\dots, \varphi_n$, and there the technique of Eq.(B.10) yields a bound

$$\frac{C}{\sqrt{a(\tau)}} \sqrt{\tau |\mu_{m+1}|} e^{-|\mu_{m+1}|\tau} \ .$$

Combining this with the bound on the first term in Eq.(B.14), we complete the proof of Theorem B.1.

B.2. The linear evolution generated by $M_{\eta,2}$

In this section, we deal with the problem of giving bounds on the linear evolution generated by the operator $M_{\eta,2}$, which is defined by

$$M_{\eta,2} \;=\; M_{\eta,2,0} \,\oplus\, \bigoplus_{n=2}^{\infty} M_{\eta,2,n} \;,$$

where

$$M_{\eta,2,n} \;=\; \Big(\big(\varepsilon^2 - (1+(in+i\Phi(p\eta))^2\big)^2 - \mathcal{K}\big(\Phi(p\eta)\big)\Big) - \eta^2 \tfrac{1}{2} p\partial_p \;.$$

We want to bound the solution $U_{n,\tau}$ of the equation

$$e^{-\tau}\partial_\tau U_{n,\tau} \;=\; M_{\exp(-\tau/2),2,n} U_{n,\tau} \;, \tag{B.15}$$

with $U_{n,0}=1$. Recall the definition of $L=-p^2-\frac{1}{2}p\partial_p$, and rewrite $M_{\exp(-\tau/2),2,n}$ as

$$\begin{aligned}
M_{\exp(-\tau/2),2,n} \;&=\; \Big(\varepsilon^2 - \big(1+(in+i\Phi(pe^{-\tau/2}))^2\big)^2 - \mathcal{K}(\Phi(pe^{-\tau/2}))\Big) - e^{-\tau}\tfrac{1}{2}p\partial_p \\
&=\; \Big(\varepsilon^2 - \big(1+(in+i\Phi(pe^{-\tau/2}))^2\big)^2 - \mathcal{K}(\Phi(pe^{-\tau/2}))\Big) + e^{-\tau}p^2 + e^{-\tau}L \\
&=\; X_n(pe^{-\tau/2}) + e^{-\tau}L \;,
\end{aligned}$$

where $X_n(\xi)=\varepsilon^2-\big(1+(in+i\Phi(\xi))^2\big)^2-\mathcal{K}(\Phi(\xi))+\xi^2$. We want to solve Eq.(B.15):

$$e^{-\tau}\partial_\tau U_{n,\tau} \;=\; (e^{-\tau}L + X_n(pe^{-\tau/2}))U_{n,\tau} \;,$$

with initial condition $U_{n,0}=1$. Observe now that X_n is an operator of multiplication by a function of $p\eta$. Since the commutator $[p^m,-p^2-\frac{1}{2}p\partial_p]$ is equal to $\frac{m}{2}p^m$, we find $[h(p),L]=\frac{1}{2}ph'(p)$, and, furthermore,

$$e^{h(p)}L \;=\; (L+\tfrac{1}{2}ph'(p))e^{h(p)} \;.$$

It follows that the solution of Eq.(B.15) is

$$U_{n,\tau} \;=\; e^{(e^\tau-1)X_n(pe^{-\tau/2})}e^{\tau L} \;,$$

as one can check by explicit computation.

From the explicit form of X_n, (in particular, the factor of $-n^4$), and the estimates derived in Theorem B.1, we see that for any $x_n \in H_{q,r}$, we have

$$\|U_{n,\tau}x_n\|_{q,r} \;\le\; C\exp(-c_0(e^\tau-1)n^4)e^{\tau q/2}\|x_n\|_{q,r} \;.$$

Combining this with the Remark of (A.11), we immediately obtain

Lemma B.6. *If U_τ satisfies*

$$e^{-\tau}\partial_\tau U_\tau \;=\; M_{\exp(-\tau/2),2} U_\tau \;,$$

with $U_0 = 1$, then there exist a $C(r,q,\nu) > 0$, and a $c_0 > 0$ such that for any $w \in H_{q,r,\nu}$,

$$\|U_\tau w\|_{q,r,\nu} \;\leq\; C \exp(-e^{c_0\tau/2}) \|w\|_{q,r,\nu} \;. \tag{B.16}$$

To complete the proof of Theorem 3.4, we also need an estimate of the semigroup generated by $M_{\eta,1}$. This is simply obtained, however, because from (2.13) we see that $M_{\eta,1} \;=\; M_{\exp(-\tau/2),2,1}$, restricted to functions whose Fourier transform is supported away from the origin. Using this fact, and the explicit formula given above for $M_{\exp(-\tau/2),2,n}$, we see immediately that for any $w_1 \in H_{q,r}$, there exists a constant $c_1 > 0$, such that if $U_{\tau,1}$ is the semigroup generated by $M_{\eta,1}$ one has

$$\|U_{\tau,1} w_1\|_{q,r} \;\leq\; C e^{-c_1\tau} \|w_1\|_{q,r} \;.$$

C. The pseudo center manifold theorem for the singular system Eq.(3.8)

In this section, we prove Theorem 3.7. Before we start with the proof, we wish to point out in which sense we are here confronted with a new problem, which does not allow for a straightforward application of results from the literature. If we write the system Eq.(3.8) in the form

$$\begin{aligned}
\partial_\tau x_1 \;&=\; A_1 x_1 + N_1(x_1,\eta,x_2,x_3) \;, \\
\partial_\tau \eta \;&=\; -\tfrac{1}{2}\eta \;, \\
\partial_\tau x_2 \;&=\; A_2 x_2 + N_2(x_1,\eta,x_2,x_3) \;, \\
\partial_\tau x_3 \;&=\; \eta^{-2} A_{3,\eta} x_3 + \eta^{-2} N_3(x_1,\eta,x_2,x_3) \;,
\end{aligned} \tag{C.1}$$

then, in view of the spectral properties of Eq.(3.9), there is a “gap” between the “central” part (corresponding to x_1 and η) and the “stable” part (corresponding to x_2, x_3). The problem is that we are really dealing with a singular perturbation because the non-linearity in the equation for x_3 also diverges as $\eta \downarrow 0$. This problem would be more easily overcome if A_2 were bounded. In that case, for sufficiently small η, the spectra of A_2 and $\eta^{-2}A_3$ would not overlap, and we could define first an invariant manifold by “eliminating” x_3, and then the true invariant manifold by eliminating x_2 from the equations obtained after elimination of x_3. However, since the spectra overlap for all values of η, we resort to a strategy which consists of a converging sequence of alternate eliminations of x_2 and x_3.

To define these successive eliminations, we consider two equivalent representations of Eq.(3.8), one being Eq.(C.1) above and the other being

$$\begin{aligned} \partial_t x_1 &= \eta^2\bigl(A_1 x_1 + N_1(x_1,\eta,x_2,x_3)\bigr) , \\ \partial_t \eta &= -\tfrac{1}{2}\eta^3 , \\ \partial_t x_2 &= \eta^2\bigl(A_2 x_2 + N_2(x_1,\eta,x_2,x_3)\bigr) , \\ \partial_t x_3 &= A_{3,\eta} x_3 + N_3(x_1,\eta,x_2,x_3) . \end{aligned} \tag{C.2}$$

We shall again omit the index η from A_3. We obtain Eq.(C.2) from Eq.(C.1) by rescaling the evolution parameter of the autonomous system as $t + t_0 = \exp(\tau)$. (Note that time is really given by $1/\eta^2 - t_0$, while we view t and τ as the evolution parameters of the vector fields.) We will call $\Phi_\tau^{\text{center}}$ the flow corresponding to Eq.(C.1) and Φ_t^{stable} the flow corresponding to Eq.(C.2). A simple inspection of the definition of these flows yields the useful identity:

$$\Phi^{\text{center}}_{\tau=\log\,(y+t_0)}(\xi,x) \;=\; \Phi^{\text{stable}}_{t=y}(\xi,x) , \tag{C.3}$$

where

$$\xi \;=\; (x_1,\eta), \quad x \;=\; (x_2,x_3) . \tag{C.4}$$

We shall use the relations (C.4) throughout. The identity (C.3) holds for all x_1, x_2, x_3 and for $\eta \ge 0$. Note that the initial conditions are given for the parameter $t = 0$ and the parameter $\tau = \log(t_0)$, and that $\eta(0) = t_0^{-1/2}$. Thus, $\eta(0)$ is small if the parameter t_0 has been chosen sufficiently large. (The bounds on the nonlinearities are uniform in $t_0 \ge t_0^*$ as follows from the calculations.)

Let h_0 be a function of ξ. This function will always be an approximate invariant manifold for one of two problems. To define these problems, we first introduce two effective non-linearities

$$\begin{aligned} F_j(h_0;\xi,x_2) &= N_j\bigl(x_1,\eta,x_2,h_0(\xi)\bigr) , \quad \text{for } j = 1,2 , \\ G_j(h_0;\xi,x_3) &= N_j\bigl(x_1,\eta,h_0(\xi),x_3\bigr) , \quad \text{for } j = 1,3 . \end{aligned}$$

We then define two equations (corresponding to the two different time scales Eq.(C.1) and Eq.(C.2) of the same problem Eq.(3.8)): The first equation will be called the *center system*:

$$\begin{aligned} \partial_\tau x_1 &= A_1 x_1 + F_1(h_0;\xi,x_2) , \\ \partial_\tau \eta &= -\tfrac{1}{2}\eta , \\ \partial_\tau x_2 &= A_2 x_2 + F_2(h_0;\xi,x_2) . \end{aligned} \tag{C.5}$$

Similarly, we define the *stable system*

$$\begin{aligned} \partial_t x_1 &= \eta^2 A_1 x_1 + \eta^2 G_1(h_0;\xi,x_3) , \\ \partial_t \eta &= -\tfrac{1}{2}\eta^3 , \\ \partial_t x_3 &= A_3 x_3 + G_3(h_0;\xi,x_3) . \end{aligned} \tag{C.6}$$

Assume now that h_2 and h_3 are two given functions of x_1 and η. We define a map

$$\mathcal{F} \;:\; \begin{pmatrix} h_2 \\ h_3 \end{pmatrix} \mapsto \begin{pmatrix} h_2' \\ h_3' \end{pmatrix} \;,$$

through the following construction: We let $h_2'(\xi)$ be the function whose graph is the invariant manifold for the center system Eq.(C.5) with non-linearity $F_j(h_3; \xi, x_2)$, and similarly we let $h_3'(\xi)$ be the function whose graph is the invariant manifold for the stable system Eq.(C.6) with non-linearity $G_j(h_2; \xi, x_3)$. Our main result here is

Proposition C.1. *The map $\mathcal{F}$ has a fixed point (h_2^*, h_3^*). This fixed point provides an invariant manifold for the system Eq.(3.8).*

Remark. We shall in fact show that $\mathcal{F}$ is a contraction in a suitable function space. In particular, we show that $\mathcal{F}^n(0,0)$, the n-fold iterate of $\mathcal{F}$, converges to the limit (h_2^*, h_3^*). The intuitive approach behind this construction is that the $\mathcal{F}^n(0,0)$ provide a sequence of successive approximations to invariant manifolds for the Eqs.(C.6) and (C.5), in which the non-linearities at the n^{th} step are given by the approximate solutions for the invariant manifold problem of the other equation: The non-linearities are then $F_j(h_3^{(n-1)}; \xi, x_2)$ (in Eq.(C.5)) and $G_j(h_2^{(n-1)}; \xi, x_3)$ (in Eq.(C.6)).

Proof. That the systems of equations (C.5) and (C.6) have invariant manifolds follows from our estimates (given in Appendix B) on the semi-group generated by the linear operators A_2 and A_3, and our estimates on the non-linear terms. (For expositions of this theory that are particularly relevant in the present context, see *e.g.*, [H, M, G].) The functions h_2^* and h_3^* whose graphs define the invariant manifolds satisfy well known integral equations, see below.

Fix $h = (h_2, h_3)$ and consider the Eq.(C.5). We want to find the function $h_2'(h; \xi)$ which eliminates x_2. To construct h_2', we first consider the equation

$$\begin{aligned} \partial_\tau x_1 \;&=\; A_1 x_1 + F_1\big(h_3; \xi, h_2(\xi)\big) \;, \\ \partial_\tau \eta \;&=\; -\tfrac{1}{2}\eta \;. \end{aligned} \tag{C.7}$$

This is a differential equation on a finite dimensional space and we let $\Psi_\tau^2(\xi; h)$ denote the corresponding flow. (Of course, the η-component of this problem can be explicitly integrated.) We can then formulate the problem of finding the invariant manifold which eliminates x_2 from Eq.(C.6) by looking at the map defined by $h \mapsto \mathcal{F}_2(h)$ where

$$\mathcal{F}_2(h) \;=\; \int_{-\infty}^{0} d\tau\, e^{-A_2\tau} F_2\big(h_3; \Psi_\tau^2(\xi; h), h_2(\Psi_\tau^2(\xi; h))\big) \;. \tag{C.8}$$

(A particularly clear derivation of these equations can be found in [G].) In a similar way, we define the flow $\Psi_\tau^3(\xi; h)$ for the equation

$$\begin{aligned} \partial_t x_1 \;&=\; \eta^2 A_1 x_1 + \eta^2 G_1\big(h_2; x_1, h_3(\xi)\big) \;, \\ \partial_t \eta \;&=\; -\tfrac{1}{2}\eta^3 \;, \end{aligned} \tag{C.9}$$

and the map

$$\mathcal{F}_3(h) \;=\; \int_{-\infty}^{0} dt\, e^{-A_3 t} G_3\big(h_2; \Psi_t^3(\xi;h), h_3(\Psi_t^3(\xi;h))\big)\;. \tag{C.10}$$

We now specify the function spaces in which we work. Recall that $x_1 \in \mathbf{R}^N$, $\eta \in \mathbf{R}$ and that $\xi \in \mathbf{R}^{N+1}$. We let $\mathcal{E}^{\mathrm{c}} = \mathbf{R}^N \oplus \mathbf{R}$ with the usual Euclidean norm. We also assume that $\mathcal{E}^2$ and $\mathcal{E}^3$ are the Banach spaces in which the x_2 and x_3 live. In our problem, these Banach spaces are the Hilbert spaces $H_{q,r}$ and $H_{q,r,\nu}$, but since we believe the present theory of singular vector fields may have further applications, we consider the more general case for the moment (see, for example, [W2]). These Banach spaces should have the $\mathcal{C}^k$ extension property [BF]. The functions h_2 and h_3 will be Lipshitz functions from a ball of radius r in $\mathcal{E}^2$ and $\mathcal{E}^3$, respectively. They satisfy $h_j(0) = 0$ and are tangent at the origin to $\mathcal{E}^j$, for $j = 2, 3$. Thus, we define the metric spaces, for $j = 2, 3$:

$$\mathcal{H}_{j,\sigma} \;=\; \big\{ h_j : \mathcal{E}^{\mathrm{c}} \to \mathcal{E}^j \;\;\big|\;\; h_j(0) = 0, \|h_j(\xi) - h_j(\tilde\xi)\|_{\mathcal{E}^j} \;\le\; \sigma \|\xi - \tilde\xi\| \big\}\;.$$

We also define a distance

$$\rho_{\mathcal{H}_{j,\sigma}}(h_j, \tilde h_j) \;=\; \sup_{\xi \neq 0} \frac{\|h_j(\xi) - \tilde h_j(\xi)\|_{\mathcal{E}_j}}{\|\xi\|}\;, \tag{C.11}$$

and introduce the notation

$$\rho_{\mathcal{H}_\sigma}(h, \tilde h) \;=\; \rho_{\mathcal{H}_{2,\sigma}}(h_2, \tilde h_2) + \rho_{\mathcal{H}_{3,\sigma}}(h_3, \tilde h_3)\;.$$

Standard results about the existence and uniqueness of solutions of systems of differential equations now imply that

$$\big\|\Psi_\tau^2(\xi;h) - \Psi_\tau^2(\tilde\xi;h)\big\| \;\le\; C e^{\beta_2|\tau|}\|\xi - \tilde\xi\|\;, \tag{C.12}$$

while

$$\big\|\Psi_\tau^2(\xi;h) - \Psi_\tau^2(\xi;\tilde h)\big\| \;\le\; C e^{\beta_2|\tau|}\rho_{\mathcal{H}_\sigma}(h, \tilde h)\;, \tag{C.13}$$

for any $\beta_2 > (N-1)/2$. Analogous estimates hold for the flow Ψ^3, though in that case one can choose any exponential growth rate $\beta_3 > 0$, provided $|\eta|$ is sufficiently small. This is due to the presence of the factor of $\eta^2 A_1$ in the first equation of Eq.(C.6).

With this in mind we define two more metric spaces (for $j = 2, 3$):

$$\begin{aligned}
\mathcal{K}_{j,\beta_j,D_j} \;=\; \big\{ \Psi_\tau : \mathbf{R}^+ \times \mathcal{E}^{\mathrm{c}} \times \mathcal{H}_{2,\sigma} \times \mathcal{H}_{3,\sigma} \to \mathcal{E}^{\mathrm{c}} \;\;\big|\;\; \\
& \Psi_0(\xi;h) = \xi, \Psi_\tau(0;h) = 0, \Psi_\tau \text{ is } \mathcal{C}^1 \text{ in } \tau, \\
& \|\Psi_\tau(\xi,h) - \Psi_\tau(\tilde\xi,\tilde h)\| \\
& \qquad \le\; D_j e^{\beta_j|\tau|}\big(\|\xi - \tilde\xi\| + \rho_{\mathcal{H}_\sigma}(h,\tilde h)\|\xi\|\big)\big\}\;,
\end{aligned}$$

with a corresponding Lipshitz metric

$$d_j(\Psi, \tilde{\Psi}) \;=\; \|\Psi - \tilde{\Psi}\|_{\mathcal{K}_j}\;, \tag{C.14}$$

where

$$\|\Psi\|_{\mathcal{K}_j} \;=\; \sup_{t\geq 0}\ \sup_{\substack{\xi\in\mathcal{E}^c\\ \xi\neq 0}} \frac{e^{\beta_j t}\|\Psi_t(\xi)\|}{\|\xi\|}\;.$$

These spaces are modeled on those used in [EW].

Remark. Since we are interested in *local* invariant manifolds, we will assume that the non-linear terms have been cut off outside a ball of radius r in each of their arguments. Since in the applications of this paper all our functions are elements of Hilbert spaces, we can assume that there exist smooth cut-off functions which are equal to 1 inside a ball of radius $r/2$ and are equal to zero outside a ball of radius r, and we multiply each of the non-linear terms in Eq.(3.8) by such a cutoff. For example, in Eq.(C.6), we certainly need to cutoff the function η^2 by $\eta^2\chi(\eta)$ (where χ is the cutoff function) to avoid blowup problems.

Given this setup, we show that the map $\mathcal{F}$ is a contraction of $\mathcal{H}_{2,\sigma}\times\mathcal{H}_{3,\sigma}$. In terms of the notation given above $\mathcal{F}$ is now defined as $\mathcal{F}(h) = \big(\mathcal{F}_2(h), \mathcal{F}_3(h)\big)$. One must first show that $\mathcal{F}$ maps this space to itself. This step is however an easy variant of the argument which shows that $\mathcal{F}$ is a contraction, and we leave it as an exercise to the reader. To show that $\mathcal{F}$ is a contraction, we use the maps (C.8) and (C.10). Then we see that the "j" component, $j = 2, 3$, of $\mathcal{F}(h_2, h_3)(\xi) - \mathcal{F}(\tilde{h}_2, \tilde{h}_3)(\xi)$ is given by

$$\Delta_j \;=\; \int_{-\infty}^{0} d\tau\, e^{-A_j\tau}\Big(U_j(h;\xi,\tau) - U_j(\tilde{h};\xi,\tau)\Big)\;, \tag{C.15}$$

where

$$\begin{aligned}
U_2(h;\xi,\tau) \;&=\; F_2\big(h_3;\Psi_\tau^2(\xi;h), h_2\big(\Psi_\tau^2(\xi;h)\big)\big)\\
&=\; N_2\big(\Psi_\tau^2(\xi;h), h_2\big(\Psi_\tau^2(\xi;h)\big), h_3\big(\Psi_\tau^2(\xi;h)\big)\big)\;,\\
U_3(h;\xi,\tau) \;&=\; G_3\big(h_2;\Psi_\tau^3(\xi;h), h_3\big(\Psi_\tau^3(\xi;h)\big)\big)\\
&=\; N_3\big(\Psi_\tau^3(\xi;h), h_2\big(\Psi_\tau^3(\xi;h)\big), h_3\big(\Psi_\tau^3(\xi;h)\big)\big)\;,
\end{aligned}$$

cf. Eqs.(C.5), (C.6). Consider now Δ_2. From the estimates on the non-linear term N_2 in Eq.(3.8), we see that $\mathcal{F}_2$ is a multi-linear function of its arguments. Thus, we can estimate the difference in the integrand of Δ_2 by the sum of the differences in the arguments of $\mathcal{F}_2$, multiplied by the Lipshitz constant of $\mathcal{F}_2$. Because we have cutoff $\mathcal{F}_2$ outside a ball of radius r, this Lipshitz constant can be made arbitrarily small by making r sufficiently small. Thus, calling this Lipshitz constant $\ell_2(r)$, we see from the estimates on $e^{A_2 t}$ which follow from the results of Appendix B and from Eqs.(C.11)–(C.14) that

$$
\begin{aligned}
\|\Delta_2\|_{\mathcal{E}^2} \le\ & \int_0^\infty d\tau\, \frac{C}{\sqrt{\tau}} e^{-N\tau/2} \ell_2(r) \Big(\rho_{\mathcal{H}_\sigma}(h,\tilde h) \\
& + \|\Psi_\tau^2(\xi,h) - \Psi_\tau^2(\xi,\tilde h)\| + \|h_2\big(\Psi_\tau^2(\xi,h)\big) - \tilde h_2\big(\Psi_\tau^2(\xi,\tilde h)\big)\|_{\mathcal{E}^2} \Big) \\
\le\ & \int_0^\infty d\tau\, \frac{C}{\sqrt{\tau}} e^{-N\tau/2} \ell_2(r) \Big(\rho_{\mathcal{H}_\sigma}(h,\tilde h) \\
& + \rho_{\mathcal{H}_\sigma}(h,\tilde h) C e^{\beta_2 \tau} + \rho_{\mathcal{H}_\sigma}(h,\tilde h) C e^{\beta_2 \tau} \Big) \le \text{ const. } \ell_2(r) \rho_{\mathcal{H}_\sigma}(h,\tilde h)\ .
\end{aligned}
$$

Thus, we have shown that $\mathcal{F}$ is a contraction.

We next consider the manifold $\mathcal{M}$ given by $(\xi, h_2^*(\xi), h_3^*(\xi))$—where x_1 is in a small neighborhood of 0 and η is in a small positive interval $0 \le \eta \le \eta_0$. We want to show that $\mathcal{M}$ is indeed an invariant manifold for the full system Eq.(3.8). From this it follows, since the flows Φ^{stable} and Φ^{center} are equivalent, up to rescaling of time, that $\mathcal{M}$ is also an invariant manifold for the Eqs.(C.1) and (C.2). If we set $x_2 = h_2^*(\xi)$ and $x_3 = h_3^*(\xi)$, then the third equation of Eq.(C.2) is satisfied because the third equation, when restricted to the manifold $x_2 = h_2^*$ is just the second equation of the stable system Eq.(C.6). with non-linearity $G_3(h_2^*;\ldots)$. To see that the remaining equations are satisfied just note that the first, second and fourth equations in the full system Eq.(3.8) become, after rescaling of time,

$$
\begin{aligned}
\dot x_1 &= A_1 x_1 + N_1(x_1,\eta,x_2,x_3)\ , \\
\dot\eta &= -\tfrac{1}{2}\eta^3\ , \\
\dot x_2 &= A_2 x_2 + N_2(x_1,\eta,x_2,x_3)\ ,
\end{aligned}
$$

and if we set $x_2 = h_2^*$ and $x_3 = h_3^*$, we see that we are just on the invariant manifold for the center system Eq.(C.5). Hence, we have found the invariant manifold for the full system Eq.(3.8).

D. The vanishing of the non-linearity at zero momentum

In this appendix, we prove Proposition 4.3. This proof is essentially a scaling argument. We shall study the nonlinearity $N_1(x_1,\eta,x_2,x_3)$ and we restrict it to the invariant manifold, *i.e.*, we replace it by $\tilde N_1(x_1,\eta)$ and let η go to 0. In particular, we shall show that only one term survives, namely the one which is cubic in x_1^3, and all others go to 0 as $\eta \to 0$.

To prove this, we will analyze the nonlinearities N_j term by term, using their definitions as given in Eqs.(3.6) and (3.8). Recall again that $A_1 = 0$ since we are considering here the projection onto the first eigenvalue of L. In Eq.(3.6), the nonlinearities are given by the terms f_2, f_3, f_4, and g, and these have been bounded in Proposition 3.1 and Proposition 3.2. Recall finally that every factor of w^{c} contributes a factor of $e^{-\tau/2} = \eta$ and every factor of w^{s} contributes a factor of $e^{-\tau} = \eta^2$ to these bounds.

Begin by considering the contribution from f_2. According to Eq.(A.9), we can extract *another factor* of η from Eq.(A.10), by using the quadratic nature of K_2, *cf.* Lemma A.1.

Using Proposition 3.1 and Proposition 3.2, we see that the only contributions from f_3, f_4, and g which do not vanish as $\eta \to 0$ are those of the type $(w^{\rm c})^3$ in f_3, of the type $w^{\rm s}(w^{\rm c})^2$ in f_4, and of the type $(w^{\rm c})^2$ in g.

We start by analyzing f_3. If we write it out, we find

$$\begin{aligned}\eta^{-2}\bigl(f_3(w^{\rm c})\bigr)(p) \;=\; & \eta^{-2}\chi\bigl(\Phi(p\eta)\bigr)\int dx\,\bar\varphi_{\Phi(p\eta)}(x)\\ & \times\eta^2\int_{\eta^{-1}\Phi(-1/2)}^{\eta^{-1}\Phi(1/2)} dp_1\,dp_2\,\Phi'(p_1\eta)\Phi'(p_2\eta)\\ & \times\varphi_{\Phi(p_1\eta)}(x)\,\varphi_{\Phi(p_2\eta)}(x)\,\varphi_{\Phi(p\eta)-\Phi(p_1\eta)-\Phi(p_2\eta)}(x)\\ & \times w^{\rm c}(p_1)\,w^{\rm c}(p_2)\,w^{\rm c}\bigl(\eta^{-1}\Phi^{-1}(\Phi(p\eta)-\Phi(p_1\eta)-\Phi(p_2\eta))\bigr)\;,\end{aligned}$$

cf. Eq.(2.11). Upon taking $\eta \to 0$, this converges to

$$\chi(0)\int dx\,\bar\varphi_0(x)\varphi_0^3(x)\int dp_1\,dp_2\,w^{\rm c}(p_1)w^{\rm c}(p_2)w^{\rm c}(p-p_1-p_2)\;. \tag{D.1}$$

Analogous arguments can be used to discuss the "surviving" terms of f_4 and g. We just summarize the steps analogous to the calculation of f_3. One gets, as $\eta \to 0$,

$$\eta^{-2}\bigl(f_4(w^{\rm c},w^{\rm s}\eta,\eta)\bigr)(p)\;\to\;6\chi(0)\int dx\,\bar\varphi_0(x)u_\varepsilon(x)\varphi_0(x)\int dp'\,w^{\rm c}(p')w^{\rm s}(p-p';x)\;, \tag{D.2}$$

and

$$\eta^{-1}\bigl(g(w^{\rm c},w^{\rm s}\eta,\eta)\bigr)(p)\;\to\;-3u_\varepsilon(x)\varphi_0^2(x)\int dp'\,w^{\rm c}(p-p')w^{\rm c}(p')\;. \tag{D.3}$$

We next study these limiting expressions in the basis $\{\psi_n(p)\}_{n=0}^\infty$ of eigenfunctions of $L=-p^2-\frac12 p\partial_p$. Then we can write $w^{\rm c}(p)$ as

$$x_1\psi_0(p)+\sum_{n=1}^\infty x_2^{(n)}\psi_n(p)\;. \tag{D.4}$$

The crucial remark is now that *on the invariant manifold, $x_2^{(n)}$ will be replaced by $h_2^{*,(n)}$*, and similarly $w^{\rm s}$ will be equal to h_3^*. We now compute the limiting forms of h_2^* and h_3^*, and then we substitute these values in Eqs.(D.1)–(D.3). Consider the equation for h_3^*. Then from Eq.(C.9), we have

$$\begin{aligned}\partial_t x_1 \;&=\; \eta^2 G_1(h_2^*;x_1,h_3^*(\xi))\;,\\ \partial_t\eta \;&=\; -\tfrac12\eta^3\;,\end{aligned}$$

because we are considering the case $N=1$ where the linear part vanishes. We also have from Eq.(C.10),

$$h_3^*(x_1,\eta) \;=\; \int_{-\infty}^0 dt\, e^{-A_{3,\eta}t} G_3\bigl(h_2^*;\Psi_t^3(x_1,\eta;h^*),h_3^*(\Psi_t^3(x_1,\eta;h^*))\bigr)\;. \tag{D.5}$$

Now, when $\eta=0$, we have $\Psi_t^3(\xi;h)=\Psi_t^3(x_1,0;h)=x_1$, and this reduces to

$$\begin{aligned} h_3^*(x_1,0) \;&=\; \int_{-\infty}^0 dt\, e^{-A_{3,0}t} G_3\bigl(h_2^*;x_1,0,h_3^*(x_1,0)\bigr) \\ &=\; -A_{3,0}^{-1} G_3\bigl(h_2^*;x_1,0,h_3^*(x_1,0)\bigr)\;. \end{aligned} \tag{D.6}$$

Note next that for $\eta=0$ we have $A_{3,0}=M_0$, *cf.* Eq.(3.8), and this means $A_{3,0}=Q_{\rm per}L_{\rm per}$. We denote by $\xi_n(x)$ the eigenfunctions and by σ_n the eigenvalues of $Q_{\rm per}L_{\rm per}$. Using Eq.(1.9) and Theorem 1.1, we see that $\sigma_n=\lambda_{\ell=0,n-1}$ and therefore they are given by $\sigma_1=-\mathcal{O}(\varepsilon^2)$ and $\sigma_n\approx-(1-(n-1)^2)^2$, when $n\neq1$. Then the $n^{\rm th}$ component (in this basis) of h_3^* (at $\eta=0$) is given by

$$h_3^{*,(n)}(p) \;=\; -\sigma_n^{-1}\cdot\bigl(-3\int dx\,\bar\xi_n(x)u_\varepsilon(x)\varphi_0^2(x)\bigr)\int dp'\,w^{\rm c}(p-p')w^{\rm c}(p')\;, \tag{D.7}$$

since all other terms vanish in the limit $\eta\to0$. We next substitute the value Eq.(D.4) for $w^{\rm c}$ and set $x_2=h_2^*$ in Eq.(D.7), and get

$$\begin{aligned} h_3^{*,(n)}(p) \;&=\; -x_1^2\sigma_n^{-1}\cdot\bigl(-3\int dx\,\bar\xi_n(x)u_\varepsilon(x)\varphi_0^2(x)\bigr) \\ &\times\Bigl(\int dp'\,\psi_0(p')\,\psi_0(p-p')+\mathcal{O}(x_1h_2^*+(h_2^*)^2)\Bigr)\;. \end{aligned}$$

Next, we replace $w^{\rm s}$ in Eq.(D.2) with h_3^*, and in that same equation make the substitution for $w^{\rm c}$ that we used above, and we find:

$$\begin{aligned} &18x_1^3\sum_{n=0}^{\infty}\sigma_n^{-1}\bigl(\int dx\,\bar\xi_n(x)u_\varepsilon(x)\varphi_0^2(x)\bigr)\bigl(\int dx'\,\bar\varphi_0(x')u_\varepsilon(x')\xi_n(x')\bigr) \\ &\quad\times\Bigl(\int dp_1\,dp_2\,\psi_0(p_1)\psi_0(p_2)\psi_0(p-p_1-p_2)+\mathcal{O}(x_1h_2^*+(h_2^*)^2)\Bigr)\;. \end{aligned} \tag{D.8}$$

Thus we see that the only terms which survive in N_1 and N_2 in the limit $\eta\to0$ result from adding together Eqs.(D.8) and (D.1). We obtain

$$\begin{aligned} X \;=\; x_1^3\Bigl\{&\int dx\,\bar\varphi_0(x)\varphi_0^3(x)+18\sum_{n=0}^{\infty}\sigma_n^{-1}\bigl(\int dx'\,\bar\xi_n(x')u_\varepsilon(x')\varphi_0^2(x')\bigr) \\ &\times\bigl(\int dx''\,\bar\varphi_0(x'')\,u_\varepsilon(x'')\,\xi_n(x'')\bigr)\Bigr\} \\ &\times\Bigl(\int dp_1\,dp_2\,\psi_0(p_1)\,\psi_0(p_2)\,\psi_0(p-p_1-p_2)\Bigr)\;. \end{aligned} \tag{D.9}$$

This coefficient will turn out to be exactly the same as that which appears below as the coefficient of the cubic terms in the center manifold in the periodic case, and since we know that in periodic case this coefficient (and indeed, the entire nonlinear term) is zero, it must vanish in the present case as well. The only remaining point in the proof of Proposition 4.3 is the computation of the coefficient of the cubic term in the equation in the center manifold in the periodic case, and we do that in the following subsection.

Remark. The above argument might seem incomplete since it ignores the $\mathcal{O}(x_1 h_2^* + (h_2^*)^2)$ error terms in (D.8). In fact, those terms vanish for x_1 small. To see why, note that our computations of the $\eta \to 0$ limit of f_2, f_3, f_4 and g apply also to the nonlinear term $N_2(x_1, \eta, h_2^*(\xi), h_3^*(\xi))$ in the equation for h_2^* in (4.1). Thus, in the $\eta \to 0$ limit h_2^* satisfies:

$$\partial_{x_1} h_2^*(x_1,0)\tilde{N}_1(x_1,0) \;=\; A_2 h_2^*(x_1,0) + N_2(x_1, 0, h_2^*(x_1,0), h_3^*(x_1,0)) \; .$$

Using the estimates on h_2^* and h_3^* derived above, we see that this equation implies $h_2^*(x_1,0) = 0$ for all x_1 sufficiently small, and hence the error terms in (D.8) vanish.

D.1. The non-linearity in the periodic case

In this subsection we compute the explicit form of the non-linearity (which we know to be 0 because the invariant manifold is made up of fixed points in this case). But this explicit form will allow us to compare it with the expression obtained in Eq.(D.9) so that the proof of Proposition 4.3 will be complete.

We start from the equation

$$\partial_\tau v \;=\; L_{\text{per}} v - 3u_\varepsilon v^2 - v^3 \; . \tag{D.10}$$

Let y_0 be the component of v in the direction of the highest eigenvalue, $\sigma_0 = 0$, of L_{per}, and y_n, the projection onto the directions ξ_n, defined after Eq.(D.6), associated to the eigenvalues σ_n. Then the invariant manifold can be written in the form

$$y_n \;=\; Y_n(y_0) \; , \quad n \;=\; 1, 2, \ldots \; . \tag{D.11}$$

Using the fact that the eigenfunction with eigenvalue 0 is u_ε', we can decompose v as:

$$v(x) \;=\; y_0 u_\varepsilon'(x) + \sum_{n=1}^{\infty} \xi_n(x) Y_n(y_0) \; , \tag{D.12}$$

the projection of Eq.(D.10) onto the invariant manifold leads to

$$\partial_\tau y_0 \;=\; -\int dx\, u_\varepsilon'(x) \left(3u_\varepsilon(x) v(x)^2 + v(x)^3\right) . \tag{D.13}$$

Note that there is no linear term because $\sigma_0 = 0$.

We are interested in the exact form of the cubic term in y_0 on the r.h.s. of Eq.(D.13). There are two contributions, one from v^3, leading to

$$-y_0^3 \int dx\, u_\varepsilon'(x)^4 \;, \tag{D.14}$$

and one from the quadratic non-linearity:

$$Y \;=\; -6y_0 \sum_{n=1}^{\infty} Y_n^{(2)}(y_0) \int dx\, u_\varepsilon'(x)\, u_\varepsilon(x)\, u_\varepsilon'(x)\, \xi_n(x) \;. \tag{D.15}$$

Here, $Y_n^{(2)}(y_0)$ is the quadratic term in y_0 of Y_n. Substituting Eq.(D.13) into the equation for Y_n, we find the perturbative result:

$$Y_n^{(2)}(y_0) \;=\; y_0^2 \cdot 3\sigma_n^{-1} \int dx\, \bar{\xi}_n(x)\, u_\varepsilon(x)\, u_\varepsilon'(x)^2 \;.$$

Inserting into Eq.(D.15), it is seen to become

$$Y \;=\; -y_0^3 18 \sum_{n=1}^{\infty} \sigma_n^{-1} \int dx\, u_\varepsilon'(x)^2\, u_\varepsilon(x)\, \xi_n(x) \int dx'\, \bar{\xi}_n(x')\, u_\varepsilon(x')\, u_\varepsilon'(x') \;. \tag{D.16}$$

Combining Eqs.(D.14) and (D.16), we get the desired result, namely that the cubic non-linearity in the periodic case coincides with the quantity X of Eq.(D.9), provided we recall that $\varphi_0 = u_\varepsilon'$. This completes the proof of Proposition 4.3.

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