On the structure of stationary solutions of the Navier-Stokes equations

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Abstract

We consider stationary solutions of the incompressible Navier-Stokes equations in two dimensions. We give a detailed description of the fluid flow in a half-plane by using a mathematical setup within which the idea of a change of type from an elliptic to a parabolic partial differential equation can be made precise.

1 Introduction

We consider, in $d = 2$ dimensions, the time independent incompressible Navier-Stokes equations

$$-(\mathbf{u}\cdot\nabla)\mathbf{u} + \Delta\mathbf{u} - \boldsymbol{\nabla} p = 0\ , \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0\ , \tag{2}$$

in a half-space $\Omega = \{(x,y)\in\mathbf{R}^2 \mid x \geq 1\}$. We are interested in modeling a situation where fluid enters the half-space Ω through the surface $\Sigma = \{(x,y)\in\mathbf{R}^2 \mid x = 1\}$ and where the fluid flows at infinity parallel to the x-axis at a nonzero constant speed $\mathbf{u}_\infty \equiv (1,0)$. We therefore impose the boundary conditions

$$\lim_{\substack{x\geq 0\\ x^2+y^2\to\infty}} \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty\ , \tag{3}$$

$$\mathbf{u}|_\Sigma = \mathbf{u}_\infty + \mathbf{u}_*\ , \tag{4}$$

with $\mathbf{u}_* = (u_*, v_*)$ in a certain set of vector fields $\mathcal{S}$ satisfying $u_*(y) = u_*(-y)$, $v_*(y) = -v_*(-y)$ and $\lim_{|y|\to\infty}\mathbf{u}_*(y) = \mathbf{0}$. Let $(u,v) = \mathbf{u} - \mathbf{u}_\infty$ and $\mathbf{u}_*$ in $\mathcal{S}$. From (1), (2) it is then easy to see that the discussion can be restricted to the case of functions u, v and p satisfying $u(x,y) = u(x,-y)$, $v(x,y) = -v(x,-y)$ and $p(x,y) = p(x,-y)$ for all $x \geq 1$, i.e., to flows that are symmetric with respect to the x-axis. The following theorem is our main result.

Theorem 1 *Let Σ and Ω as defined above. Then, for each $\mathbf{u}_* = (u_*, v_*)$ in a certain set of vector fields $\mathcal{S}$ to be defined later on, there exist a unique vector field $\mathbf{u} = \mathbf{u}_\infty + (u,v)$ and a unique function p satisfying the Navier-Stokes equations (1) and (2) in Ω and the boundary conditions (3) and (4). Furthermore,*

$$\left| x\ v(x, yx^{1/2})\right| \leq \text{const.}\ , \tag{5}$$

for all $(x,y)\in\Omega$, and

$$\lim_{x\to\infty} x^{1/2}\left|u(x,yx^{1/2})\right| = \frac{c}{\sqrt{4\pi}}\exp(-\frac{y^2}{4})\ , \tag{6}$$

for all $y\in\mathbf{R}$, with

$$c = \int_{\mathbf{R}} u_*(y)\ dy - \lim_{k\to 0+}\int_{\mathbf{R}} \sin(ky)\ v_*(y)\ dy\ . \tag{7}$$

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A proof of this theorem will be given in Section 5. See [1] for related results.

Remark *The set $\mathcal{S}$ in Theorem 1 will be specified in Section 5, once appropriate function spaces have been introduced.*

Remark *We consider Theorem 1 to be a first step in an effort to bridge the gap between the mathematically rigorous theory of the Navier-Stokes equations (see [7] and references therein), and the work on asymptotic expansions for solutions of the Navier-Stokes equations that addresses questions relevant for engineering [6], [4].*

Theorem 1 has the following interpretation: consider a rigid body that is placed into a uniform stream of a homogeneous incompressible fluid, filling up all of $\mathbf{R}^2$.

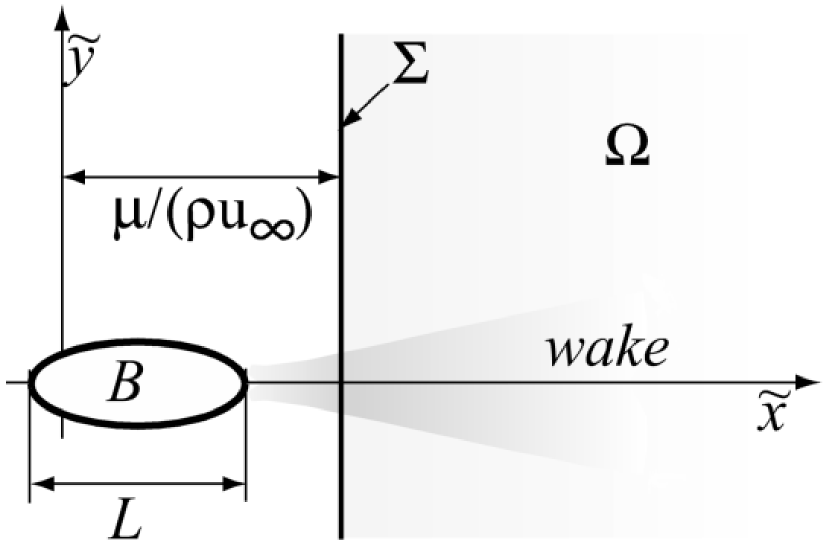

Fig.1. Stationary fluid flow around a body.

Experimentally, far away from the body, such a fluid flow appears to be close to a potential flow with the exception of a region downstream of the object, the so called wake region, within which the vorticity of the fluid is concentrated. The situation in Fig. 1 is modeled by the equations

$$-\rho\ (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}+\mu\Delta\tilde{\mathbf{u}}-\boldsymbol{\nabla}\tilde{p}=0\ , \qquad (8)$$
$$\nabla\cdot\tilde{\mathbf{u}}=0\ ,$$

in $\bar{\Omega}=\mathbf{R}^2\setminus\mathbf{B}$, subject to the boundary condition $\tilde{\mathbf{u}}|_{\partial\bar{\Omega}}=\mathbf{0}$, $\lim_{|\tilde{\mathbf{x}}|\to\infty}\tilde{\mathbf{u}}(\tilde{\mathbf{x}})=\tilde{\mathbf{u}}_\infty=(u_\infty,0)$. If we assume that the density ρ and the dynamic viscosity μ of the fluid are constant in $\bar{\Omega}$, then we can always choose a coordinate system as indicated in Fig. 1, scale to dimensionless coordinates $\mathbf{x}=(\rho u_\infty/\mu)\,\tilde{\mathbf{x}}$, introduce a dimensionless vector field $\mathbf{u}$ and a dimensionless pressure p by defining $\tilde{\mathbf{u}}(\tilde{\mathbf{x}})=u_\infty\mathbf{u}(\mathbf{x})$ and $\tilde{p}(\tilde{\mathbf{x}})=\left(\rho u_\infty^2\right)p(\mathbf{x})$. In the new coordinates equation (8) becomes equal to (1) with Σ located at $x=1$. For solutions $\tilde{\mathbf{u}}$ of (8) which are such that the corresponding scaled vector field $(\mathbf{u}-\mathbf{u}_\infty)|_\Sigma\in\mathcal{S}$ (we expect this to be all solutions of (8) for which $\mathrm{Re}=L\rho u_\infty/\mu\ll 1$, but we do not address this question here), Theorem 1 shows the existence of a parabolic wake, within which the leading order deviation from the constant flow is universal, i.e., independent of the details of the shape of the body. On a heuristic level this is a well known fact [3]. It is related to what is called a "change of type" of equation (8) from an elliptic partial differential equation to a parabolic partial differential equation. The mathematical tools that we use to prove this change of type are a version of the center manifold theorem as proved in [8], combined with matched asymptotic expansion techniques as developed in [2].

The rest of this paper is organized as follows. In Section 2 we rewrite equation (1) and (2) as a dynamical system with the coordinate parallel to the flow playing the role of time. The discussion will be formal. At the end of the discussion we get a set of integral equations. In Sections 3 and 4 we then prove that these integral equations admit a solution, and in Section 5 we finally show that this solution provides a solution for (1) and (2) with the boundary conditions (3) and (4).

2 The dynamical system

The equations (1), (2) are equivalent to

$$\begin{aligned}
\omega &= \partial_x v - \partial_y u \ , \\
0 &= -(\mathbf{u}\cdot\boldsymbol{\nabla})\omega + \Delta\omega \ , \\
0 &= \partial_x u + \partial_y v \ ,
\end{aligned} \tag{9}$$

ω being the vorticity of the fluid[1]. The main idea underlying the tools developed in this paper is to consider the coordinate parallel to the flow as a time coordinate [3]. Let $\eta = \partial_x\omega$, and $\mathbf{u} = (1,0) + (u,v)$. Then the equations (9) are equivalent to

$$\begin{aligned}
\partial_x\omega &= \eta \ , \\
\partial_x\eta &= \eta - \partial_y^2 w + q \ , \\
\partial_x u &= -\partial_y v \ , \\
\partial_x v &= \partial_y u + \omega \ ,
\end{aligned} \tag{10}$$

where

$$q = u\eta + v\partial_y\omega \ . \tag{11}$$

Let

$$\omega(x,y) = \frac{1}{2\pi}\int_{\mathbf{R}} dk \ e^{-iky}\hat{\omega}(k,x) \ ,$$

and accordingly for the other functions. For (10) we then get (for simplicity we drop the hats and use in Fourier space t instead of x for the "time"-variable) the dynamical system

$$\begin{aligned}
\dot{\omega} &= \eta \ , \\
\dot{\eta} &= \eta + k^2\omega + q \ , \\
\dot{u} &= ikv \ , \\
\dot{v} &= -iku + \omega \ ,
\end{aligned} \tag{12}$$

where

$$q = \frac{1}{2\pi}(u * \eta + v * (-ik\omega)) \ , \tag{13}$$

the dot meaning derivative with respect to t and the "star" being the convolution product. Note that u is an even, real valued function of k, and that ω, η, v and q are odd functions of k with values in $i\mathbf{R}$. The equations (12) are of the form $\dot{\mathbf{z}} = L\mathbf{z} + \mathbf{q}$, with $\mathbf{z} = (\omega, \eta, u, v)$, $\mathbf{q} = (0, q, 0, 0)$ and

$$L(k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & ik \\ 1 & 0 & -ik & 0 \end{pmatrix} .$$

The matrix $L(k)$ can be diagonalized. Namely, let $\sigma(k) \equiv \mathrm{signum}(k)$, and define Δ, Λ_+ and Λ_- by

$$\begin{aligned}
\Delta(k) &= \sqrt{1+4k^2} \ , \\
\Lambda_+(k) &= \frac{1+\Delta(k)}{2} \ , \\
\Lambda_-(k) &= \frac{1-\Delta(k)}{2} \ .
\end{aligned}$$

[1] See [9], [10] for recent ideas concerning the use of the vorticity.

Let $\mathbf{z} = S\boldsymbol{\zeta}$ with

$$S(k) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \Lambda_+ & \Lambda_- & 0 & 0 \\ -\frac{i}{k}\Lambda_- & -\frac{i}{k}\Lambda_+ & 1 & 1 \\ 1 & 1 & -i\sigma & i\sigma \end{pmatrix} .$$

Then $\dot{\boldsymbol{\zeta}} = D\boldsymbol{\zeta} + S^{-1}\mathbf{q}$ with

$$S^{-1}(k) = \begin{pmatrix} -\frac{\Lambda_-}{\Delta} & \frac{1}{\Delta} & 0 & 0 \\ \frac{\Lambda_+}{\Delta} & -\frac{1}{\Delta} & 0 & 0 \\ -\frac{i}{2}(\sigma - \frac{1}{k}) & -\frac{1}{2}\frac{i}{k} & \frac{1}{2} & \frac{1}{2}i\sigma \\ \frac{i}{2}(\sigma + \frac{1}{k}) & -\frac{1}{2}\frac{i}{k} & \frac{1}{2} & -\frac{1}{2}i\sigma \end{pmatrix} ,$$

and $D = S^{-1}LS$ a diagonal matrix with diagonal entries Λ_+, Λ_-, $|k|$, and $-|k|$. Note that $\Lambda_+(k) \geq 1$ and $\Lambda_-(k) \leq 0$ and $\Lambda_-(k) \approx -k^2$ for small values of k. Let $\boldsymbol{\zeta} = (\omega_+, \omega_-, u_+, u_-)$. Using the definitions we find that (12) is equivalent to

$$\begin{aligned} \dot{\omega}_+ &= \Lambda_+\omega_+ + \frac{1}{\Delta}q \ , \\ \dot{\omega}_- &= \Lambda_-\omega_- - \frac{1}{\Delta}q \ , \\ \dot{u}_+ &= |k|\, u_+ - \frac{1}{2}\frac{i}{k}q \ , \\ \dot{u}_- &= -|k|\, u_- - \frac{1}{2}\frac{i}{k}q \ , \end{aligned} \tag{14}$$

with q as defined in (13), with ω_+ and ω_- odd functions of k with values in $i\mathbf{R}$ and with u_+ and u_- even real valued functions of k. For convenience later on we also write $\mathbf{z} = S\boldsymbol{\zeta}$ in component form. Namely,

$$\begin{aligned} \omega &= \omega_+ + \omega_- \ , \\ \eta &= \Lambda_+\omega_+ + \Lambda_-\omega_- \ , \\ u &= -\frac{i}{k}\Lambda_-\omega_+ - \frac{i}{k}\Lambda_+\omega_- + u_+ + u_- \ , \\ v &= \omega_+ + \omega_- - i\sigma u_+ + i\sigma u_- \ . \end{aligned} \tag{15}$$

To solve (14) we convert it into an integral equation. The +-modes are unstable (remember that $\Lambda_+(k) \geq 1$) and we therefore have to integrate these modes backwards in time starting with $\omega_+(k,\infty) \equiv u_+(k,\infty) \equiv 0$ (see [8]). We get

$$\begin{aligned} \omega_+(k,t) &= -\frac{1}{\Delta}\int_t^\infty e^{\Lambda_+(t-s)}q(k,s)\ ds \ , \\ \omega_-(k,t) &= \omega_-^*(k)e^{\Lambda_-(t-1)} - \frac{1}{\Delta}\int_1^t e^{\Lambda_-(t-s)}q(k,s)\ ds \ , \\ u_+(k,t) &= \frac{1}{2}\frac{i}{k}\int_t^\infty e^{|k|(t-s)}q(k,s)\ ds \ , \\ u_-(k,t) &= u_-^*(k)e^{-|k|(t-1)} - \frac{1}{2}\frac{i}{k}\int_1^t e^{-|k|(t-s)}q(k,s)\ ds \ . \end{aligned} \tag{16}$$

In Section 4 we will show that the initial condition ω_-^* can be re-expressed in terms of the vorticity on Σ, and we will see that u_-^* adds to (u,v) a potential flow. In particular, for the case of zero vorticity, we have that $\omega_+ = \omega_- = q = u_+ = 0$ and we get that (u,v) is a pure potential flow.

In order to prove the existence of a solution for (16) we will apply the contraction mapping principle to the map $\tilde{q} = \mathcal{N}(q)$ that is formally defined by computing first $(\omega_+, \omega_-, u_+, u_-)$ from q using (16), then (ω, η, u, v) using (15) and then q by using (13). As discussed above, in direct space, one expects the vorticity to be a rapidly decaying function of y for all x, and we also assume this to be the case for $\eta = \partial_x \omega$ and for $q = u\eta + v\partial_y w$. As a consequence, in Fourier space, $q(k,t)$ ought to be smooth as a function of k (probably entire), but for our purpose it will be sufficient to assume that $k \mapsto q(k,t)$ be once continuously differentiable. The decay properties for u and v in direct space are much less obvious, and we should therefore avoid to assume any smoothness in k that goes beyond what is necessary to show that $\lim_{y\to\infty} u(x,y) = \lim_{y\to\infty} v(x,y) = 0$ in order to satisfy the boundary conditions (3), (4). We finally note that since $\Lambda_-(k) \approx -k^2$ for small k, the time evolution of ω_- is in many ways similar to that of a solution of the heat equation. This is the origin of the appearance of a wake with a parabolic structure and motivates what follows (see [5]).

Let $\alpha \geq 0$ and

$$\mu_\alpha(k,t) = \frac{1}{1 + \left(|k|\, t^{1/2}\right)^\alpha} \ . \tag{17}$$

We will consider the Banach space $\mathcal{V}_\alpha$ of even functions $f \in \mathcal{C}^0(\mathbf{R})$ equipped with the norm

$$\|f\|_\alpha = \sup_{k\in\mathbf{R}} \frac{|f(k)|}{\mu_\alpha(k,1)} \ ,$$

the Banach space $\mathcal{V}_\alpha^1$ of imaginary valued odd functions $f \in \mathcal{C}^1(\mathbf{R}, i\mathbf{R})$ equipped with the norm

$$\|f\|_\alpha^1 = \sup_{k\in\mathbf{R}} \frac{|f(k)|}{\mu_\alpha(k,1)} + \sup_{k\in\mathbf{R}} \frac{|\partial_k f(k)|}{\mu_\alpha(k,1)} \ ,$$

and the Banach space $\mathcal{B}_{\alpha,\beta}$ of continuous functions f from $[1,\infty)$ to $\mathcal{V}_\alpha^1$ equipped the norm

$$\|f\|_{\alpha,\beta} = \sup_{t\geq 1} t^\beta \|f(t^{-1/2}\cdot\,, t)\|_\alpha^1 \ .$$

Theorem 2 *Fix $\alpha > 0$. Let $u_-^* \in \mathcal{V}_{\alpha+1}$, $\omega_-^* \in \mathcal{V}_{\alpha+1}^1$, and let $\varepsilon_0 = \left\|u_-^*\right\|_{\alpha+1} + \left\|\omega_-^*\right\|_{\alpha+1}^1$. Then, $\mathcal{N}$ is well defined as a map from $\mathcal{B}_{\alpha,2}$ to $\mathcal{B}_{\alpha,2}$ and contracts, for ε_0 sufficiently small, the ball $B_\alpha(\varepsilon_0) = \left\{q \in \mathcal{B}_{\alpha,2} \mid \|q\|_{\alpha,2} \leq \varepsilon_0\right\}$ into itself.*

This theorem implies that, for small enough initial conditions ω_-^* and u_-^* on Σ, the integral equations (16) admit a unique solution.

3 Proof of Theorem 2

The proof is organized as follows: we first prove that $\mathcal{N}$ is well defined and maps, for small enough initial conditions, a ball in $\mathcal{B}_{\alpha,2}$ into itself. Then we show that $\mathcal{N}$ is a contraction on this ball.

Note that the equations (15) and (16) contain divisions of q and ω_-^* by k and we first prove bounds on these quotients. Let ε_0 be as in Theorem 2. Throughout this proof we then denote by ε a constant multiple of ε_0, i.e., $\varepsilon = \text{const.}\ \varepsilon_0$ with a constant that may be different from instance to instance.

Proposition 3 *Let $q \in \mathcal{B}_{\alpha,2}$, with $\|q\|_{\alpha,2} \leq \varepsilon$. Then,*

$$|q(k,t)| \leq \frac{\varepsilon}{t^2}\mu_\alpha(k,t) \ , \tag{18}$$

$$|\partial_k q(k,t)| \leq \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t) \ , \tag{19}$$

$$|q(k,t)| \leq \varepsilon\frac{|k|}{t^{3/2}}\ \mu_{\alpha+1}(k,t) \ . \tag{20}$$

Proof. The inequalities (18) and (19) follow from our definition of the norm of $\mathcal{B}_{\alpha,\beta}$. We now prove (20). From (19) we get for $|k| \leq 1/t^{1/2}$ the bound

$$|q(k,t)| \leq |k| \left(\sup_{|k|\leq 1/t^{1/2}} |\partial_k q(k,t)| \right) \leq |k|\frac{\varepsilon}{t^{3/2}} \leq |k|\frac{\varepsilon}{t^{3/2}}\ \mu_{\alpha+1}(k,t) \ .$$

The last inequality follows because $|k| \leq 1/t^{1/2}$ implies that $|k|\, t^{1/2} \leq 1$, and therefore $\mu_{\alpha+1}(k,t) \leq$ const.. Similarly, for $|k| > 1/t^{1/2}$ we find using (18) that

$$|q(k,t)| \leq \frac{\varepsilon}{t^2}\mu_\alpha(k,t) \leq \left(\frac{t^{1/2}|k|}{1+t^{1/2}|k|}\right)\frac{\varepsilon}{t^2}\mu_\alpha(k,t) \leq |k|\frac{\varepsilon}{t^{3/2}}\ \mu_{\alpha+1}(k,t)\ ,$$

as claimed. ■

Similarly we have:

Proposition 4 *Let $\omega_-^* \in \mathcal{V}_{\alpha+1}^1$ with $\left\|\omega_-^*\right\|_{\alpha+1}^1 \leq \varepsilon_0$. Then,*

$$\begin{aligned} \left|\omega_-^*(k)\right| &\leq \varepsilon\mu_{\alpha+1}(k,1)\ , && (21)\\ \left|\partial_k\omega_-^*(k)\right| &\leq \varepsilon\mu_{\alpha+1}(k,1)\ , && (22)\\ \left|\omega_-^*(k)\right| &\leq \varepsilon|k|\ \mu_{\alpha+2}(k,1)\ . && (23)\end{aligned}$$

Proof. The inequalities (21) and (22) follow from the definition of the norm in $\mathcal{V}_{\alpha+1}^1$. We now prove (23). From (22) we get for $|k| \leq 1$ that

$$\left|\omega_-^*(k)\right| \leq |k| \sup_{|k|\leq 1}\left|\partial_k\omega_-^*(k)\right| \leq |k|\,\varepsilon_0 \leq \varepsilon\,|k|\,\mu_{\alpha+2}(k,1)\ ,$$

and from (21) we have for $|k| > 1$ that

$$\left|\omega_-^*(k,t)\right| \leq \varepsilon_0\mu_{\alpha+1}(k,1) \leq \varepsilon_0\frac{2\,|k|}{1+|k|}\mu_{\alpha+1}(k,1) \leq \varepsilon\,|k|\,\mu_{\alpha+2}(k,1)\ ,$$

as claimed. ■

In the following two subsections we prove the following proposition:

Proposition 5 *Let $\alpha > 0$, $u_-^* \in \mathcal{V}_{\alpha+1}$, $\omega_-^* \in \mathcal{V}_{\alpha+1}^1$, and let $\varepsilon_0 = \left\|u_-^*\right\|_{\alpha+1} + \left\|\omega_-^*\right\|_{\alpha+1}^1$. Then, for all $q \in \mathcal{B}_{\alpha,2}$, with $\|q\|_{\alpha,2} \leq$ const. ε_0 we have the bounds*

$$\begin{aligned} |\mathcal{N}(q)(k,t)| &\leq \varepsilon^2\mu_\alpha(k,t)/t^2\ , && (24)\\ |\partial_k\mathcal{N}(q)(k,t)| &\leq \varepsilon^2\mu_\alpha(k,t)/t^{3/2}\ . && (25)\end{aligned}$$

The bounds (24) and (25) imply that $\|\mathcal{N}(q)\|_{\alpha,2} \leq \varepsilon^2$, and therefore $\mathcal{N}$ is well defined as a map from $\mathcal{B}_{\alpha,2}$ to $\mathcal{B}_{\alpha,2}$. Furthermore, since $\varepsilon^2 =$ const. ε_0^2, it follows that $\mathcal{N}$ maps the ball $B_\alpha(\varepsilon_0) \equiv \left\{q \in \mathcal{B}_{\alpha,2} \mid \|q\|_{\alpha,2} \leq \varepsilon_0\right\}$ into itself for ε_0 small enough.

3.1 Bound on $\mathcal{N}(q)$

Using the bounds (18)-(20) and (21)-(23) we can now estimate ω_+, ω_-, u_+ and u_-. Let $\alpha' \geq 0$ and

$$\bar{\mu}_{\alpha'}(k,t) = 1/(1+|kt|^{\alpha'})\ . \tag{26}$$

Proposition 6 *Let u_-^*, ω_-^* and q as defined above. Then,*

$$\begin{aligned} |\omega_+(k,t)| &\leq \frac{\varepsilon}{t^2}\frac{1}{\Delta}\frac{1}{\Lambda_+}\mu_\alpha(k,t)\ , && (27)\\ |\omega_-(k,t)| &\leq \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t)\ , && (28)\\ \left|\frac{1}{k}\omega_-(k,t)\right| &\leq \varepsilon\mu_{\alpha+2}(k,t)\ , && (29)\\ |\Lambda_-\omega_-(k,t)| &\leq \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)\ , && (30)\\ |u_+(k,t)| &\leq \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t)\ , && (31)\\ |u_-(k,t)| &\leq \varepsilon\bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t)\ . && (32)\end{aligned}$$

Proof. For ω_+ we have

$$|\omega_+(k,t)| \leq \frac{1}{\Delta} \sup_{s \geq t} |q(k,s)| \int_t^\infty e^{\Lambda_+(t-s)} \, ds \leq \frac{\varepsilon}{t^2} \frac{1}{\Delta} \frac{1}{\Lambda_+} \mu_\alpha(k,t) \ , \tag{33}$$

and (27) follows. For ω_- we have

$$\begin{aligned}
|\omega_-(k,t)| &\leq \left|\omega_-^*(k)\right| e^{\Lambda_-(t-1)} + \frac{1}{\Delta} \int_1^{(t+1)/2} e^{\Lambda_-(t-s)} |q(k,s)| \, ds + \frac{1}{\Delta} \int_{(t+1)/2}^t e^{\Lambda_-(t-s)} |q(k,s)| \, ds \\
&\leq \left|\omega_-^*(k)\right| e^{\Lambda_-(t-1)} + \frac{1}{\Delta} e^{\Lambda_-(t-1)/2} \int_1^{(t+1)/2} |q(k,s)| \, ds + \frac{1}{\Delta} \int_{t/2}^t |q(k,s)| \, ds \\
&\leq \left[\left|\omega_-^*(k)\right| + \frac{\varepsilon}{\Delta} |k| \, \mu_{\alpha+1}(k,1) \int_1^\infty \frac{ds}{s^{3/2}} \right] e^{\Lambda_-(t-1)/2} + \frac{\varepsilon}{\Delta} |k| \, \mu_{\alpha+1}(k, \frac{t}{2}) \frac{1}{t^{1/2}} \\
&\leq \varepsilon \, |k| \, \mu_{\alpha+2}(k,1) e^{\Lambda_-(t-1)/2} + \varepsilon \, |k| \, \frac{1}{\Delta t^{1/2}} \mu_{\alpha+1}(k, \frac{t}{2}) \leq \varepsilon \, |k| \, \mu_{\alpha+2}(k,t) \ ,
\end{aligned} \tag{34}$$

and (28) and (29) follow. For the last inequality we have used that $\sup_{k \in \mathbf{R}} \sup_{t \geq 1} \mu_{\alpha+1}(k,t/2)/(\Delta t^{1/2} \mu_{\alpha+2}(k,t))$ $\leq$ const., and furthermore that $\mu_{\alpha+2}(k,1) e^{\Lambda_-(t-1)/2} \leq \mu_{\alpha+2}(k,t)$. This is a consequence of the following proposition:

Proposition 7 *Let $\alpha' \geq \beta' \geq 0$. Then, for all $t \geq 1$ and $k \in \mathbf{R}$,*

$$\frac{1}{1+|k|^{\alpha'}} e^{\Lambda_- \frac{t-1}{2}} \left|\Lambda_-\right|^{\beta'} \leq \text{const.} \frac{1}{t^{\beta'}} \frac{1}{1+\left(|k| \, t^{1/2}\right)^{\alpha'-\beta'}} \ . \tag{35}$$

Proof. Since $\Lambda_-(k) \leq 0$, $|\Lambda_-| \leq \text{const.} \, |k|$ and $\mu_{\alpha+2}(k,2) \leq \text{const.} \ \mu_{\alpha+2}(k,1)$, (35) is obvious for $1 \leq t \leq 2$. For $t > 2$ we use that

$$\begin{aligned}
\left(1+\left(|k| \, t^{1/2}\right)^{\alpha'-\beta'}\right) e^{\Lambda_- \frac{t-1}{2}} \left|\Lambda_- t\right|^{\beta'} &\leq \left(1+\left(|k| \, t^{1/2}\right)^{\alpha'-\beta'}\right) e^{\Lambda_- \frac{t}{4}} \left|\Lambda_- t\right|^{\beta'} \\
&\leq \text{const.} \left(1+\frac{|k|^{\alpha'-\beta'}}{|\Lambda_-|^{(\alpha'-\beta')/2}} \left|\Lambda_- t\right|^{\left(\alpha'-\beta'\right)/2} \left|\Lambda_- t\right|^{\beta'} e^{\Lambda_- \frac{t}{4}}\right) \\
&\leq \text{const.} \left(1+\frac{|k|^{\alpha'-\beta'}}{|\Lambda_-|^{(\alpha'-\beta')/2}}\right) \leq \text{const.} \left(1+|k|^{\left(\alpha'-\beta'\right)/2}\right) \\
&\leq \text{const.} \left(1+|k|^{\alpha'}\right) \ ,
\end{aligned}$$

and (35) follows. ■

Note that Proposition (7) will be routinely used below without mention.

We now bound $\Lambda_-\omega_-$. We find, using the same techniques as for ω_-, that

$$\begin{aligned}
|\Lambda_-\omega_-(k,t)| &\leq \left[\left|\omega_-^*(k)\right| + \varepsilon \frac{|k|}{\Delta} \mu_{\alpha+1}(k,1) \int_1^\infty \frac{ds}{s^{3/2}} \right] |\Lambda_-| \, e^{\Lambda_- \frac{t-1}{2}} + \\
&\quad + \frac{1}{\Delta} \int_{t/2}^t e^{\Lambda_-(t-s)} |\Lambda_-| \, |q(k,s)| \, ds \\
&\leq \varepsilon \, |k| \, \mu_{\alpha+2}(k,1) \, |\Lambda_-| \, e^{\Lambda_- \frac{t-1}{2}} + \frac{1}{\Delta} \frac{\varepsilon}{t^2} \mu_\alpha(k, \frac{t}{2}) \int_{t/2}^t e^{-|\Lambda_-|(t-s)} |\Lambda_-| \, ds \\
&\leq \frac{\varepsilon}{t} |k| \, \mu_{\alpha+1}(k,t) + \frac{1}{\Delta} \frac{\varepsilon}{t^2} \mu_\alpha(k,t) \\
&\leq \frac{\varepsilon}{t^{3/2}} \mu_\alpha(k,t) + \frac{\varepsilon}{t^2} \mu_\alpha(k,t) \ ,
\end{aligned}$$

and (30) follows. We now bound u_+ and u_-. For u_+ we find

$$|u_+(k,t)| \leq \varepsilon \mu_{\alpha+1}(k,t) \int_t^\infty \frac{ds}{s^{3/2}} \leq \frac{\varepsilon}{t^{1/2}} \mu_{\alpha+1}(k,t) \ ,$$

and for u_- (proceeding as in the case of ω_-) we have

$$\begin{aligned} |u_-(k,t)| &\leq \left[\left|u_-^*(k)\right| + \varepsilon\mu_{\alpha+1}(k,1)\int_1^\infty \frac{ds}{s^{3/2}}\right] e^{-|k|\frac{t-1}{2}} + \frac{1}{|k|}\int_{t/2}^t e^{-|k|(t-s)}\left|q(k,s)\right|\ ds \\ &\leq \varepsilon\mu_{\alpha+1}(k,1)e^{-|k|\frac{t-1}{2}} + \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,\frac{t}{2}) \leq \varepsilon\bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t)\ , \end{aligned}$$

as claimed. For the last inequality we have used that $\mu_{\alpha+1}(k,t/2) \leq$ const. $\mu_{\alpha+1}(k,t)$, and furthermore that $\mu_{\alpha+1}(k,1)e^{-|k|(t-1)/2} \leq \bar{\mu}_{\alpha+1}(k,t)$. This is obvious for $1 \leq t \leq 2$, and for $t > 2$ we use that for all $\alpha' \geq 0$

$$\left(1 + |kt|^{\alpha'}\right)e^{-|k|\frac{t-1}{2}} \leq \left(1 + |kt|^{\alpha'}\right)e^{-|k|\frac{t}{4}} \leq \text{const.}\ .$$

This completes the proof of Proposition 6.

From the bounds on ω_+, ω_-, u_+ and u_- we get from (15) the following bounds on ω, η, u and v.

Proposition 8 *Let u_-^*, ω_-^* and q as defined above. Then,*

$$|\omega(k,t)| \leq \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t)\ , \tag{36}$$

$$|-ik\omega(k,t)| \leq \frac{\varepsilon}{t}\mu_\alpha(k,t)\ , \tag{37}$$

$$|\eta(k,t)| \leq \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)\ , \tag{38}$$

$$|u(k,t)| \leq \varepsilon\mu_{\alpha+1}(k,t)\ , \tag{39}$$

$$|v(k,t)| \leq \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t) + \varepsilon\bar{\mu}_{\alpha+1}(k,t)\ . \tag{40}$$

Proof. (37) immediately follows from (36). To prove (36) and (38)-(40) we apply the triangle inequality to the equations (15) and use then the bounds (27)-(32). We get

$$\begin{aligned} |\omega(k,t)| &\leq \frac{\varepsilon}{t^{3/2}}\frac{1}{\Lambda_+}\mu_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t) \leq \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t)\ , \\ |\eta(k,t)| &\leq \frac{\varepsilon}{t^{3/2}}\mu_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t) \leq \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)\ , \\ |u(k,t)| &\leq \frac{\varepsilon}{t^{3/2}}\frac{1}{\Lambda_+}\mu_{\alpha+1}(k,t)\ + \varepsilon\Lambda_+\mu_{\alpha+2}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t) + \varepsilon\bar{\mu}_{\alpha+1}(k,t) \leq \varepsilon\mu_{\alpha+1}(k,t)\ , \\ |v(k,t)| &\leq \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t) + \varepsilon\bar{\mu}_{\alpha+1}(k,t)\ , \end{aligned}$$

as claimed. ■

We can now bound the convolutions in (13):

Proposition 9 *Let u_-^*, ω_-^* and q as defined above. Then,*

$$|(u * \eta)(k,t)| \leq \frac{\varepsilon^2}{t^2}\mu_\alpha(k,t)\ , \tag{41}$$

$$|(v * (-ik\omega))(k,t)| \leq \frac{\varepsilon^2}{t^2}\mu_\alpha(k,t). \tag{42}$$

Proof. Let $k \geq 0$. Then, we have for $u * \eta$

$$\begin{aligned} |(u * \eta)(k,t)| &\leq \frac{\varepsilon^2}{t^{3/2}}\int_{-\infty}^\infty \mu_{\alpha+1}(k',t)\mu_\alpha(k-k',t)\ dk' \\ &\leq \frac{\varepsilon^2}{t^{3/2}}\left[\mu_\alpha(k/2,t)\int_{-\infty}^{k/2}\mu_{\alpha+1}(k',t)\ dk'\right. \\ &\qquad \left. + \int_{k/2}^{3k/2}\mu_{\alpha+1}(k',t)\ dk' + \mu_\alpha(k/2,t)\int_{3k/2}^\infty \mu_{\alpha+1}(k',t)\ dk'\right] \\ &\leq \frac{\varepsilon^2}{t^2}\mu_\alpha(k,t) + \frac{\varepsilon^2}{t^{3/2}}\left|k\right|\mu_{\alpha+1}(k/2,t) \leq \frac{\varepsilon^2}{t^2}\mu_\alpha(k,t)\ , \end{aligned} \tag{43}$$

and (41) follows for $k \geq 0$. Since $u * \eta$ is odd we have the same bound for $k < 0$. Similarly, we have for $v * (-ik\omega)$

$$\begin{aligned}
|(v * (-ik\omega))(k,t)| &\leq \frac{\varepsilon^2}{t^{3/2}} \int_{-\infty}^{\infty} \mu_{\alpha+1}(k,t)\mu_\alpha(k-k',t)\ dk' + \frac{\varepsilon^2}{t} \int_{-\infty}^{\infty} \bar{\mu}_{\alpha+1}(k,t)\mu_\alpha(k-k',t)\ dk' \\
&\leq \frac{\varepsilon^2}{t^2}\mu_\alpha(k,t) + \frac{\varepsilon^2}{t}\left[\mu_\alpha(k/2,t)\int_{-\infty}^{k/2} \bar{\mu}_{\alpha+1}(k',t)\ dk' \right.\\
&\quad \left. + \int_{k/2}^{3k/2} \bar{\mu}_{\alpha+1}(k',t)\ dk' + \mu_\alpha(k/2,t)\int_{3k/2}^{\infty} \bar{\mu}_{\alpha+1}(k',t)\ dk'\right] \\
&\leq \frac{\varepsilon^2}{t^2}\mu_\alpha(k,t) + \frac{\varepsilon^2}{t}|k|\,\bar{\mu}_{\alpha+1}(k/2,t)\ ,
\end{aligned} \tag{44}$$

and (42) follows. ■

Note that the bounds (43) and (44) show that $|\mathcal{N}(q)(k,t)| \leq \varepsilon^2 \mu_\alpha(k,t)/t^2$ as required.

3.2 Bound on $\partial_k \mathcal{N}(q)$

We have

$$\partial_k q(k,t) = (u * \partial_k \eta + v * (-i\omega) + v * (-ik\partial_k w))(k,t)\ , \tag{45}$$

and it is therefore sufficient to have bounds on the derivatives of η and ω to bound (45). In particular, no derivatives on u_+ or u_- are needed. We have:

Proposition 10 *Let u_-^*, ω_-^* and q as defined above. Then,*

$$|\partial_k \omega_+(k,t)| \leq \frac{1}{\Delta}\frac{1}{\Lambda_+}\frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)\ , \tag{46}$$

$$|\partial_k \omega_-(k,t)| \leq \varepsilon\mu_{\alpha+1}(k,t)\ , \tag{47}$$

$$|\Lambda_- \partial_k \omega_-(k,t)| \leq \frac{\varepsilon}{t}\mu_\alpha(k,t)\ . \tag{48}$$

Proof. Proceeding as in Section 3.1 we find that

$$\begin{aligned}
|\partial_k \omega_+(k,t)| &\leq \text{const.}\left(\frac{|k|}{\Delta}|\omega_+(k,t)| + \frac{|k|}{\Delta^2}\int_t^\infty e^{\Lambda_+(t-s)}|t-s|\ \ |q(k,s)|\ ds \right.\\
&\quad \left. + \frac{1}{\Delta}\int_t^\infty e^{\Lambda_+(t-s)}|\partial_k q(k,s)|\ ds\right) \\
&\leq \frac{\varepsilon}{t^2}\frac{1}{\Delta}\frac{1}{\Lambda_+}\mu_\alpha(k,t) + \frac{\varepsilon}{t^2}\frac{1}{\Delta}\frac{1}{\Lambda_+}\mu_\alpha(k,t)\int_t^\infty e^{\Lambda_+(t-s)}\Lambda_+|t-s|\ ds \\
&\quad + \frac{1}{\Delta}\frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)\int_t^\infty e^{\Lambda_+(t-s)}\ ds\ ,
\end{aligned}$$

and (46) follows. Similarly we have, using the triangle inequality to bound $\partial_k \omega_-$, that

$$\begin{aligned}
|\partial_k \omega_-(k,t)| &\leq \text{const.}\left(\left|\partial_k \omega_-^*(k)\right| e^{\Lambda_-(t-1)} + \left|\omega_-^*(k)\right|\frac{|k|}{\Delta}(t-1)e^{\Lambda_-(t-1)} \right.\\
&\quad + \frac{|k|}{\Delta^2}\int_1^t e^{\Lambda_-(t-s)}|q(k,s)|\ ds + \frac{|k|}{\Delta^2}\int_1^t e^{\Lambda_-(t-s)}|t-s|\ \ |q(k,s)|\ ds \\
&\quad \left. + \frac{1}{\Delta}\int_1^t e^{\Lambda_-(t-s)}|\partial_k q(k,s)|\ ds\right)\ .
\end{aligned} \tag{49}$$

We now estimate the terms on the right hand side of (49) individually. We have, proceeding in particular as in the bound (34) on ω_- to prove (50) and (51), that

$$
\begin{aligned}
\left|\partial_k \omega_-^*(k)\right| e^{\Lambda_-(t-1)} &\leq \varepsilon \mu_{\alpha+1}(k,1) e^{\Lambda_-(t-1)} \leq \varepsilon \mu_{\alpha+1}(k,t) \ , \\
\left|\omega_-^*(k)\right| \frac{|k|}{\Delta}(t-1) e^{\Lambda_-(t-1)} &\leq \varepsilon \frac{|k|^2}{\Delta} \mu_{\alpha+2}(k,1) |t-1| e^{\Lambda_-(t-1)} \\
&\leq \varepsilon \mu_{\alpha+2}(k,1)(t-1) |\Lambda_-| e^{\Lambda_-(t-1)} \leq \varepsilon \mu_{\alpha+1}(k,t) \ , \\
\frac{|k|}{\Delta^2} \int_1^t e^{\Lambda_-(t-s)} |q(k,s)| \ ds &\leq \frac{|k|}{\Delta} \frac{\varepsilon}{t^{1/2}} \mu_{\alpha+1}(k,t) \ , \qquad (50) \\
\frac{|k|}{\Delta^2} \int_1^t e^{\Lambda_-(t-s)} |t-s| \ \ |q(k,s)| \ ds &\leq \varepsilon \frac{|k|^2}{\Delta^2} \mu_{\alpha+1}(k,1)(t-1) e^{\Lambda_- \frac{t-1}{2}} \\
&\quad + \varepsilon \frac{|k|^2}{\Delta^2} \mu_{\alpha+1}(k,\frac{t}{2}) \int_{t/2}^t e^{\Lambda_-(t-s)}(t-s) \ \frac{ds}{s^{3/2}} \\
&\leq \varepsilon \frac{1}{\Delta} \mu_{\alpha+1}(k,1) |\Lambda_-| (t-1) e^{\Lambda_- \frac{t-1}{2}} \\
&\quad + \varepsilon \frac{1}{\Delta} \mu_{\alpha+1}(k,\frac{t}{2}) \int_{t/2}^t e^{\Lambda_-(t-s)} |\Lambda_-| (t-s) \ \frac{ds}{s^{3/2}} \\
&\leq \varepsilon \frac{1}{\Delta} \mu_{\alpha+1}(k,t) \ , \\
\frac{1}{\Delta} \int_1^t e^{\Lambda_-(t-s)} |\partial_k q(k,s)| \ ds &\leq \varepsilon \mu_{\alpha+1}(k,t) \ , \qquad (51)
\end{aligned}
$$

and (47) follows. We now prove (48). We multiply the inequality (49) with $\Lambda_-$and again bound the terms on the right hand side individually. Namely,

$$
\begin{aligned}
\left|\partial_k \omega_-^*(k)\right| |\Lambda_-| e^{\Lambda_-(t-1)} &\leq \varepsilon \mu_{\alpha+1}(k,1) |\Lambda_-| e^{\Lambda_-(t-1)} \leq \frac{\varepsilon}{t} \mu_\alpha(k,t) \ , \\
\left|\omega_-^*(k)\right| \frac{|k|}{\Delta} |\Lambda_-| (t-1) e^{\Lambda_-(t-1)} &\leq \varepsilon \mu_{\alpha+2}(k,1) |\Lambda_-|^2 (t-1) e^{\Lambda_-(t-1)} \leq \frac{\varepsilon}{t} \mu_\alpha(k,t) \ , \\
\frac{|\Lambda_-| |k|}{\Delta^2} \int_1^t e^{\Lambda_-(t-s)} |q(k,s)| \ ds &\leq \varepsilon \mu_\alpha(k,t) \ ,
\end{aligned}
$$

and furthermore

$$
\begin{aligned}
&\frac{|k|}{\Delta^2} \int_1^t e^{\Lambda_-(t-s)} |\Lambda_-| |t-s| \ \ |q(k,s)| \ ds \\
\leq \ & \varepsilon \frac{|k|^2}{\Delta^2} \mu_{\alpha+1}(k,1) |\Lambda_-| (t-1) e^{\Lambda_- \frac{t-1}{2}} + \frac{|k|^2}{\Delta^2} \frac{\varepsilon}{t^{3/2}} \mu_{\alpha+1}(k,\frac{t}{2}) \int_{t/2}^t e^{\Lambda_-(t-s)} |\Lambda_-| (t-s) \ ds \\
\leq \ & \varepsilon \mu_{\alpha+2}(k,1) |\Lambda_-|^2 (t-1) e^{\Lambda_- \frac{t-1}{2}} + \frac{\Lambda_+}{\Delta^2} \frac{\varepsilon}{t^{3/2}} \mu_{\alpha+1}(k,\frac{t}{2}) \left[\int_{t/2}^t e^{-|\Lambda_-|(t-s)} (|\Lambda_-| (t-s)) \ \ |\Lambda_-| \, ds \right] \\
\leq \ & \frac{\varepsilon}{t} \mu_\alpha(k,t) \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{|\Lambda_-|}{\Delta} \int_1^t e^{\Lambda_-(t-s)} |\partial_k q(k,s)| \ ds \\
\leq \ & \frac{|\Lambda_-|}{\Delta} \int_1^{(t+1)/2} e^{\Lambda_-(t-s)} |\partial_k q(k,s)| \ ds + \frac{|\Lambda_-|}{\Delta} \int_{t/2}^t e^{\Lambda_-(t-s)} |\partial_k q(k,s)| \ ds \\
\leq \ & \varepsilon \mu_{\alpha+1}(k,1) |\Lambda_-| e^{\Lambda_-(t-1)/2} \int_1^t \frac{ds}{s^{3/2}} + \frac{|\Lambda_-|}{\Delta} \int_{t/2}^t e^{\Lambda_-(t-s)} |\partial_k q(k,s)| \ ds \\
\leq \ & \frac{\varepsilon}{t} \mu_\alpha(k,1) + \frac{1}{\Delta} \frac{\varepsilon}{t^{3/2}} \mu_\alpha(k,t/2) \int_{t/2}^t e^{\Lambda_-(t-s)} |\Lambda_-| \ ds \leq \frac{\varepsilon}{t} \mu_\alpha(k,t) \ ,
\end{aligned}
$$

and (48) follows. ■

From the bounds on $\partial_k\omega_+$ and $\partial_k\omega_-$ we get the following bound on $ik\partial_k\omega$ and $\partial_k\eta$:

Proposition 11 *Let u_-^*, ω_-^* and q as defined above. Then,*

$$\begin{aligned} |ik\partial_k\omega(k,t)| &\leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t) \ , & (52)\\ |\partial_k\eta(k,t)| &\leq \frac{\varepsilon}{t}\mu_\alpha(k,t) \ . & (53)\end{aligned}$$

Proof. We apply the triangle inequality to the derivatives of ω and η in (15), and get

$$\begin{aligned} |ik\partial_k\omega(k,t)| &\leq |k|\left[\frac{1}{\Delta}\frac{1}{\Lambda_+}\frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)+\varepsilon\mu_{\alpha+1}(k,t)\right] \leq \varepsilon\,|k|\,\mu_{\alpha+1}(k,t) \leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t) \ ,\\ |\partial_k\eta(k,t)| &\leq \frac{|k|}{\Delta}\,|\omega_+(k,t)| + \frac{|k|}{\Delta}\,|\omega_-(k,t)| + \Lambda_+\,|\partial_k\omega_+(k,t)| + |\Lambda_-|\ \ |\partial_k\omega_-(k,t)| \ ,\\ &\leq \frac{|k|}{\Delta}\,|\omega(k,t)| + \frac{1}{\Delta}\frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t) + \frac{\varepsilon}{t}\mu_\alpha(k,t) \ ,\\ &\leq \frac{|k|}{\Delta}\frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t) + \frac{\varepsilon}{t}\mu_\alpha(k,t) \leq \frac{\varepsilon}{t}\mu_\alpha(k,t) \ ,\end{aligned}$$

as claimed. ■

We can now estimate the convolution products in (45):

Proposition 12 *Let u_-^*, ω_-^* and q as defined above. Then,*

$$\begin{aligned} |(u*\partial_k\eta)(k,t)| &\leq \frac{\varepsilon^2}{t^{3/2}}\mu_\alpha(k,t) \ , & (54)\\ |(v*(-i\omega))(k,t)| &\leq \frac{\varepsilon^2}{t^{3/2}}\mu_\alpha(k,t) \ , & (55)\\ |(v*(-ik\partial_k\omega))(k,t)| &\leq \frac{\varepsilon^2}{t^{3/2}}\mu_\alpha(k,t) \ . & (56)\end{aligned}$$

Proof. We use the bounds (36), (39), (40) and (52), (53). The bounds on (54)-(56) then follow immediately since all the resulting convolutions have already been bounded in the proof of Proposition 8. ■

Note that the bounds (54)-(56) show that $|\partial_k\mathcal{N}(q)(k,t)| \leq \varepsilon^2\mu_\alpha(k,t)/t^{3/2}$ as required.

3.3 Bound on $\|\mathcal{N}(q)-\mathcal{N}(\tilde{q})\|_{\alpha,2}$

In order to proof Theorem 2 it remains to be shown that $\mathcal{N}$ is Lipschitz.

Proposition 13 *Let ω_-^* and u_-^* as above and let q, $\tilde{q} \in B_\alpha(\varepsilon_0)$. Then*

$$\|\mathcal{N}(q)-\mathcal{N}(\tilde{q})\|_{\alpha,2} \leq \text{const.}\ \varepsilon_0\,\|q-\tilde{q}\|_{\alpha,2} \ . \qquad (57)$$

Proof. We have, using the identity $ab-\tilde{a}\tilde{b} = (a-\tilde{a})b+\tilde{a}(b-\tilde{b})$, that

$$\begin{aligned} \mathcal{N}(q)-\mathcal{N}(\tilde{q}) &= \frac{1}{2\pi}\left[(u*\eta+v*(-ik\omega))-(\tilde{u}*\tilde{\eta}+\tilde{v}*(-ik\tilde{\omega}))\right]\\ &= \frac{1}{2\pi}\left[(u-\tilde{u})\eta+\tilde{u}(\eta-\tilde{\eta})+(v-\tilde{v})(-ik\omega)+\tilde{v}((-ik\omega)-(-ik\tilde{\omega}))\right] \ .\end{aligned}$$

Furthermore, since $q-\tilde{q}\in\mathcal{B}_{\alpha,2}$, we have as in (18)-(20) that

$$\begin{aligned} |q(k,t)-\tilde{q}(k,t)| &\leq \text{const.}\ \|q-\tilde{q}\|_{\alpha,2}\frac{1}{t^2}\mu_\alpha(k,t) \ ,\\ |\partial_k q(k,t)-\partial_k\tilde{q}(k,t)| &\leq \text{const.}\ \|q-\tilde{q}\|_{\alpha,2}\frac{1}{t^{3/2}}\mu_\alpha(k,t) \ ,\\ |q(k,t)-\tilde{q}(k,t)| &\leq \text{const.}\ \|q-\tilde{q}\|_{\alpha,2}\frac{|k|}{t^{3/2}}\ \mu_{\alpha+1}(k,t) \ .\end{aligned}$$

Finally, since ω, η, u, v and $\tilde{\omega}$, $\tilde{\eta}$, $\tilde{u}$, $\tilde{v}$ are linear (respectively affine) in q and $\tilde{q}$, the bound (57) follows *mutatis mutandis* from the proof of the bound on $\mathcal{N}$ and $\partial_k \mathcal{N}$. ■

In Section 3.1 and 3.2 we have shown that $\mathcal{N}$ maps the ball $B_\alpha(\varepsilon_0)$ into itself, and Proposition 13 therefore shows that $\mathcal{N}$ is a contraction of $B_\alpha(\varepsilon_0)$ into itself for ε_0 small enough. This completes the proof of Theorem 2.

4 Choice of initial conditions

We briefly discuss the choice of initial conditions. Instead of ω_-^* it would be more natural to be able to prescribe the vorticity on Σ. From (16) we see that

$$\begin{aligned} \omega_+(k,1) &= -\frac{1}{\Delta}\int_1^\infty e^{\Lambda_+(1-s)} q(k,s)\, ds \,, \\ \omega_-(k,1) &= \omega_-^*(k) \,. \end{aligned} \tag{58}$$

Since $\omega = \omega_+ + \omega_-$ this means that we have to choose

$$\omega_-^*(k) = \omega(k,1) - \omega_+(k,1) \tag{59}$$

to construct a solution with given vorticity $\omega(k,1)$. If we evaluate the bounds (27) and (46) at $t=1$, we immediately see that $|\omega_+(k,1)| \leq \varepsilon\mu_{\alpha+1}(k,1)$ and $|\partial_k\omega_+(k,1)| \leq \varepsilon\mu_{\alpha+1}(k,1)$ and therefore $k \mapsto \omega_+(k,1) \in \mathcal{V}^1_{\alpha+1}$. As a consequence, if we choose $k \mapsto \omega(k,1) \in \mathcal{V}^1_{\alpha+1}$ and replace ω_-^* in (16) by the right hand side of the equation (59), we get instead of the map $\mathcal{N}$ a map $\mathcal{N}_1$, where the equation for ω_- in (16) is replaced by

$$\omega_-(k,t) = \omega(k,1)e^{\Lambda_-(t-1)} - \omega_+(k,1)e^{\Lambda_-(t-1)} - \frac{1}{\Delta}\int_1^t e^{\Lambda_-(t-s)} q(k,s)\, ds \,,$$

with $\omega_+(k,1)$ given by (58). However, since $k \mapsto \omega_+(k,1) \in \mathcal{V}^1_{\alpha+1}$, all the bounds in the proof of Theorem 1 remain unchanged, and therefore $\mathcal{N}_1$ is well defined on $\mathcal{B}_{\alpha,2}$ and a contraction on $B_\alpha(\varepsilon_0)$ provided $k \mapsto \omega(k,1) \in \mathcal{V}^1_{\alpha+1}$ and $\left\|u_-^*\right\|_\alpha + \|\omega(\ .\ ,1)\|^1_\alpha \leq \varepsilon_0$ with ε_0 small enough.

We now discuss the role of the initial condition u_-^*. From (15) we find that u and v are of the form $u(k,t) = \cdots + u_E(k,t)$, $v(k,t) = \cdots + v_E(k,t)$, where

$$\begin{aligned} u_E(k,t) &= u_-^*(k)e^{-|k|(t-1)} \,, \\ v_E(k,t) &= i\sigma(k)u_-^*(k)e^{-|k|(t-1)} \,. \end{aligned}$$

The vector field (u_E, v_E) is a potential flow. Namely, let $\psi(k,t) = \frac{1}{-ik}u_-^*(k,1)e^{-|k|(t-1)}$, then $u_E(k,t) = -ik\psi(k,t)$ and $v_E(k,t) = -\partial_t\psi(k,t)$ and moreover $\left(\partial_t^2 - k^2\right)\psi(k,t) \equiv 0$, which implies that u_E and v_E are harmonic functions in direct space, provided $\alpha > 2$.

5 Proof of Theorem 1

In Section 3 we have proved that (to avoid confusion we now write the hats for the Fourier transforms)

$$\begin{aligned} |\hat{u}(k,t)| &\leq \varepsilon\mu_{\alpha+1}(k,t) \,, \\ |\hat{v}(k,t)| &\leq \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t) + \varepsilon\bar{\mu}_{\alpha+1}(k,t) \,. \end{aligned} \tag{60}$$

Since we have assumed that $\alpha > 0$ it follows in particular that the functions $k \mapsto \hat{u}(k,t)$ and $k \mapsto \hat{v}(k,t)$ are in $L^1(\mathbf{R})$ for all $t \geq 1$, and therefore, by the Riemann-Lebesgue lemma, their Fourier transforms,

$$\begin{aligned} u(x,y) &= \int_{\mathbf{R}} e^{-iky}\hat{u}(k,x)\, dk \,, \\ v(x,y) &= \int_{\mathbf{R}} e^{-iky}\hat{v}(k,x)\, dk \,, \end{aligned}$$

are continuous functions that vanish as $|y| \to \infty$ for each $x \geq 1$. Moreover using the bounds (60) we find that

$$\sup_{y\in\mathbf{R}} |u(x,y)| \leq \frac{\varepsilon}{|x|^{1/2}} , \tag{61}$$

$$\sup_{y\in\mathbf{R}} |v(x,y)| \leq \frac{\varepsilon}{|x|} , \tag{62}$$

and therefore u and v converge to zero whenever $|x| + |y| \to \infty$ in Ω and hence satisfy the boundary condition (3). The reconstruction of the pressure from u and v is standard. For $\alpha > 2$ second derivatives of u and v are continuous in direct space, and one easily verifies using the definitions that the triple (u, v, p) satisfies the Navier-Stokes equations (1). The details are left to the reader. The set $\mathcal{S}$ in Theorem (1) is by definition the set of all vector fields (u, v) obtained this way, restricted to Σ.

Note that (62) proves (5). Finally, to prove (6) we use the integral equations (16) to get more detailed information on the asymptotic behavior of u. We first prove an identity for (7) (we again drop the hats):

Proposition 14 *Let ω_+, ω_-, u_+, u_- be a solution of the integral equation (16) and let u, v be as defined in (15), and c as defined in (7). Then,*

$$c \equiv u(0,1) + iv(0_+,1) = -i\partial_k\omega_-^*(0) + i\int_1^\infty \partial_k q(0,s)\ ds . \tag{63}$$

Proof. Note that $c \equiv u(0,1) + iv(0_+,1)$ by definition (7). From (16) we find that

$$\begin{aligned}
\partial_k\omega_-(0,t) &= \partial_k\omega_-^*(0) - \int_1^t \partial_k q(0,s)\ ds ,\\
u_+(0,t) &= \frac{i}{2}\int_t^\infty \partial_k q(0,s)\ ds ,\\
u_-(0,t) &= u_-^*(0) - \frac{i}{2}\int_1^t \partial_k q(0,s)\ ds ,
\end{aligned}$$

and from (15) we find that

$$\begin{aligned}
u(0,t) &= -i\partial_k\omega_-(0,t) + u_+(0,t) + u_-(0,t)\\
&= -i\partial_k\omega_-^*(0) + \frac{i}{2}\int_1^\infty \partial_k q(0,s)\ ds + u_-^*(0) = u(0,1) ,\\
v(0_+,t) &= -iu_+(0,t) + iu_-(0,t)\\
&= iu_-^*(0) + \frac{1}{2}\int_1^\infty \partial_k q(0,s)\ ds = v(0_+,1) ,
\end{aligned}$$

and (63) follows. ■

To prove (6) we proceed in several steps:

Proposition 15 *Let u_-^*, ω_-^* and q as defined above. Then,*

$$\left|u(k,t) + \frac{i}{k}\Lambda_+\omega_-\right| \leq \varepsilon\left(\bar{\mu}_{\alpha+1}(k,t) + \frac{1}{t^{1/2}}\mu_{\alpha+1}(k,t)\right) . \tag{64}$$

Proof. This is an immediate consequence of (27)-(32). ■

Note that (64) implies in direct space that all contributions to u with the exception of the ω_--term are bounded by $\mathcal{O}(1/x)$. It is therefore sufficient to prove a more detailed bound on $\frac{i}{k}\Lambda_+\omega_-$ in order to prove (6). We have

Proposition 16 *Let u_-^*, ω_-^* and q as defined above, and let*

$$W_-(k,t) = -\frac{i}{k}\Lambda_+\left(\omega_-^*(k)e^{\Lambda_-(t-1)} - \frac{1}{\Delta}e^{\Lambda_-(t-1)}\int_1^t q(k,s)\ ds\right) . \tag{65}$$

Then,

$$\left|-\frac{i}{k}\Lambda_+\omega_-(k,t) - W_-(k,t)\right| \leq \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t) . \tag{66}$$

Proof. We first note that, for $1 \le s \le t$,

$$0 \le e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)} \le \text{const. } e^{\Lambda_-(t-s)} \frac{|\Lambda_-|\,(s-1)}{1+|\Lambda_-|\,(s-1)} \ .$$

Furthermore, proceeding as in the proof of Proposition 7 we find that

$$\mu_{\alpha+1}(k,1) e^{\Lambda_-(t-1)/2} |\Lambda_-| \min\left\{(t-1), t^{1/2}\right\} \le \frac{\text{const.}}{t^{1/2}} \mu_{\alpha+1}(k,t) \ ,$$

and therefore

$$\begin{aligned}
& \frac{\Lambda_+}{\Delta} \frac{1}{|k|} \int_1^t \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right) |q(k,s)| \ ds \\
\le \ & \text{const.} \left(\int_1^{(t+1)/2} e^{\Lambda_-(t-s)} \frac{|\Lambda_-|\,(s-1)}{1+|\Lambda_-|\,(s-1)} \frac{|q(k,s)|}{|k|} ds + \int_{t/2}^t e^{\Lambda_-(t-s)} \frac{|\Lambda_-|\,(s-1)}{1+|\Lambda_-|\,(s-1)} \frac{|q(k,s)|}{|k|} ds \right) \\
\le \ & \varepsilon \mu_{\alpha+1}(k,1) |\Lambda_-| e^{\Lambda_-(t-1)/2} \int_1^t \frac{(s-1)}{s^{3/2}} ds + \varepsilon \mu_{\alpha+1}(k,t/2) \int_{t/2}^t \frac{ds}{s^{3/2}} \\
\le \ & \varepsilon \mu_{\alpha+1}(k,1) e^{\Lambda_-(t-1)/2} |\Lambda_-| \min\left\{(t-1)^2, t^{1/2}\right\} + \varepsilon \mu_{\alpha+1}(k,t/2) \int_{t/2}^t \frac{ds}{s^{3/2}} \ ,
\end{aligned}$$

and (66) follows. ■

Note that (66) implies in direct space that, with the exception of the W_- term, all contributions from $\frac{i}{k}\Lambda_+\omega_-$ to u are bounded by $\mathcal{O}(1/x)$. It is therefore sufficient to prove a more detailed bound on W_- in order to prove (6). We have

Proposition 17 *Let u_-^*, ω_-^*, q and c as defined above. Then,*

$$\lim_{t\to\infty} W_-\left(\frac{k}{t^{1/2}}, t\right) = c\ e^{-k^2} \ . \tag{67}$$

Proof. The bound (67) is immediate using the definition of W_- and (63). ■

Note that (66) implies, provided $\alpha > 0$, that $k \mapsto W_-(k,t)$ is in $L^1(\mathbf{R})$ for all $t \ge 1$. Therefore (67) implies (6). This completes the proof of Theorem 1.

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