Leading order down-stream asymptotics of non-symmetric stationary Navier-Stokes flows in two dimensions

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November 14, 2003

Abstract

We consider stationary solutions of the incompressible Navier-Stokes equations in two dimensions. We give a detailed description of the fluid flow in a half-plane through the construction of an inertial manifold for the dynamical system that one obtains when using the coordinate along the flow as a time.

1 Introduction

We consider, in $d = 2$ dimensions, the time independent incompressible Navier-Stokes equations

$$-(\mathbf{u} \cdot \nabla)\mathbf{u} + \Delta\mathbf{u} - \boldsymbol{\nabla} p = 0 \ , \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \ , \tag{2}$$

in the half-space $\Omega = \{(x, y) \in \mathbf{R}^2 \mid x > 1\}$. We are interested in modeling the situation where fluid enters Ω through the surface $\Sigma = \{(x, y) \in \mathbf{R}^2 \mid x = 1\}$ and where the fluid flows at infinity parallel to the x-axis at a nonzero constant speed $\mathbf{u}_\infty \equiv (1, 0)$. We therefore impose the boundary conditions

$$\lim_{\substack{x \geq 1 \\ x^2+y^2 \to \infty}} \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty \ , \tag{3}$$

$$\mathbf{u}|_\Sigma = \mathbf{u}_\infty + \mathbf{u}_* \ , \tag{4}$$

with $\mathbf{u}_* = (u_*, v_*)$ in a certain set of vector fields $\mathcal{S}$ to be defined later on, and satisfying $\lim_{|y|\to\infty} \mathbf{u}_*(y) = 0$.

The above problem has been studied in detail in [10], [11], for the special case where $u_*(y) = u_*(-y)$ and $v_*(y) = -v_*(-y)$. As a consequence the discussion could there be restricted to the case of symmetric vector fields $\mathbf{u} = \mathbf{u}_\infty + (u, v)$, *i.e.*, to functions u, v and p satisfying $u(x, y) = u(x, -y)$, $v(x, y) = -v(x, -y)$ and $p(x, y) = p(x, -y)$ for all $x \geq 1$, and this symmetry property was extensively used in the proofs. In order to get rid of this limitation one is forced to study the nonlinearity in (1) in much more detail than in [10], [11]. This makes the estimates somewhat lengthy, since many different terms have to be analyzed, but also simpler, since less information needs to be encoded in the function spaces.

The following theorem is our main result.

Theorem 1 *Let Σ and Ω be as defined above. Then, for each $\mathbf{u}_* = (u_*, v_*)$ in a certain set of vector fields $\mathcal{S}$ to be defined later on, there exist a vector field $\mathbf{u} = \mathbf{u}_\infty + (u, v)$ and a function p satisfying the Navier-Stokes equations (1) and (2) in Ω subject to the boundary conditions (3) and (4). Furthermore,*

$$\lim_{x\to\infty} x^{1/2} \left(\sup_{y\in\mathbf{R}} |(u - u_{as})(x, y)| \right) = 0 \ , \tag{5}$$

$$\lim_{x\to\infty} x \left(\sup_{y\in\mathbf{R}} |(v - v_{as})(x, y)| \right) = 0 \ , \tag{6}$$

*Supported in part by the Fonds National Suisse.

where

$$u_{as}(x,y) = \frac{c}{2\sqrt{\pi}} \frac{1}{\sqrt{x}} e^{-\frac{y^2}{4x}} + \frac{d}{\pi} \frac{x}{x^2+y^2} + \frac{b}{\pi} \frac{y}{x^2+y^2} \ , \tag{7}$$

$$v_{as}(x,y) = \frac{c}{4\sqrt{\pi}} \frac{y}{x^{3/2}} e^{-\frac{y^2}{4x}} + \frac{d}{\pi} \frac{y}{x^2+y^2} - \frac{b}{\pi} \frac{x}{x^2+y^2} \ , \tag{8}$$

with

$$b = \lim_{k \to 0^+} \int_{\mathbf{R}} \sin(ky) \ u_*(y) \ dy \ , \tag{9}$$

$$d = \lim_{k \to 0^+} \int_{\mathbf{R}} \sin(ky) \ v_*(y) \ dy \ , \tag{10}$$

$$c = \lim_{k \to 0} \int_{\mathbf{R}} \cos(ky) \ u_*(y) \ dy \ - d \ . \tag{11}$$

A proof of this theorem is given in Section 8.

Remark 2 *The integrals in (9), (10) and (11) have to be understood in the* (C,δ)*-sense, with* $0 < \delta \leq 1$ *(see e.g. [8], Theorem 15). Namely, let* $C(\delta, R) = (1 - |y|/R)^{\delta}$*. Then the exact version of (9) is*

$$b = \lim_{k \to 0_+} \left(\lim_{R \to \infty} \int_{-R}^{R} C(\delta, R) \sin(ky) u_*(y) \ dy \right) \ ,$$

and accordingly for the other cases.

The set $\mathcal{S}$ in Theorem 1 will be specified in Section 8, once appropriate function spaces have been introduced. For an interpretation of the results see [10], [11]. For related results see [2], [3] and [9]. For an application of the above results for an efficient numerical implementation of two-dimensional stationary exterior flow problems see [7].

The rest of this paper is organized as follows. In Section 2 and Section 3 we rewrite equation (1) and (2) as a dynamical system with the coordinate parallel to the flow playing the role of time. The discussion will be formal. At the end of the discussion we get a set of integral equations. In Sections 4 and 5 we then prove that these integral equations admit a solution. This solution is analyzed in some detail in Section 6 and Section 7. In Section 8 we finally prove Theorem 1 by using the results from Sections 4-7.

2 The dynamical system

Let $\mathbf{u} = \mathbf{u}_\infty + (u, v)$. Then, the equations (1), (2) are equivalent to

$$\begin{aligned} \omega &= \partial_x v - \partial_y u \ , \\ 0 &= -(\mathbf{u} \cdot \boldsymbol{\nabla})\omega + \boldsymbol{\Delta}\omega \ , \\ 0 &= \partial_x u + \partial_y v \ , \end{aligned} \tag{12}$$

since the pressure p is uniquely determined (modulo an additive constant) by solving

$$\Delta p = \boldsymbol{\nabla} \cdot ((\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\Delta}\mathbf{u}) \tag{13}$$

in Ω with the boundary condition

$$\partial_x p = -(\mathbf{u} \cdot \nabla)u + \Delta u \tag{14}$$

on Σ, once that (12) has been solved. The function ω is the vorticity of the fluid.

Provided $\partial_x u + \partial_y v = 0$, we have that

$$u\partial_x \omega + v\partial_y \omega = \partial_x(u\omega) + \partial_y(v\omega) \equiv q \ , \tag{15}$$

and - for reasons which are not obvious to us - it turns out to be important to discuss the nonlinearity q as represented by the expression on the *r.h.s.* of (15)[1].

The main idea underlying the tools developed in this paper is to consider the coordinate parallel to the flow as a time coordinate [1]. Let $\eta = \partial_x \omega$. Then, the equations (12) are equivalent to

$$\begin{aligned} \partial_x \omega &= \eta \ , \\ \partial_x \eta &= \eta - \partial_y^2 w + q \ , \\ \partial_x u &= -\partial_y v \ , \\ \partial_x v &= \partial_y u + \omega \ . \end{aligned} \tag{16}$$

Let

$$\omega(x,y) = \frac{1}{2\pi} \int_{\mathbf{R}} dk \ e^{-iky} \hat{\omega}(k,x) \ ,$$

and accordingly for the other functions. For (16) we then get (for simplicity we drop the hats and use in Fourier space t instead of x for the "time"-variable) the dynamical system

$$\begin{aligned} \dot{\omega} &= \eta \ , \\ \dot{\eta} &= \eta + k^2 \omega + q \ , \\ \dot{u} &= ikv \ , \\ \dot{v} &= -iku + \omega \ , \end{aligned} \tag{17}$$

where

$$q = \partial_t q_0 - ikq_1 \ , \tag{18}$$

with

$$q_0 = \frac{1}{2\pi} (u * \omega) \ , \tag{19}$$

$$q_1 = \frac{1}{2\pi} (v * \omega) \ . \tag{20}$$

The equations (17) are of the form $\dot{\mathbf{z}} = L\mathbf{z} + \mathbf{q}$, with $\mathbf{z} = (\omega, \eta, u, v)$, $\mathbf{q} = (0, q, 0, 0)$ and

$$L(k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & ik \\ 1 & 0 & -ik & 0 \end{pmatrix} \ .$$

The matrix $L(k)$ can be diagonalized. Namely, let $\sigma(k) \equiv \text{signum}(k)$, and define Λ_0, Λ_+ and Λ_- by

$$\begin{aligned} \Lambda_0(k) &= \sqrt{1 + 4k^2} \ , \\ \Lambda_+(k) &= \frac{1 + \Lambda_0(k)}{2} \ , \\ \Lambda_-(k) &= \frac{1 - \Lambda_0(k)}{2} \ . \end{aligned}$$

Let $\mathbf{z} = S\boldsymbol{\zeta}$ with

$$S(k) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \Lambda_+ & \Lambda_- & 0 & 0 \\ -\frac{i}{k}\Lambda_- & -\frac{i}{k}\Lambda_+ & 1 & 1 \\ 1 & 1 & -i\sigma & i\sigma \end{pmatrix} \ .$$

[1] In [5] part of the results of this paper were proved by using the expression on the *l.h.s.* of (15). That approach turned out to be much more complicated than the present one.

Then $\dot{\boldsymbol{\zeta}} = D\boldsymbol{\zeta} + S^{-1}\mathbf{q}$ with

$$S^{-1}(k) = \begin{pmatrix} -\frac{\Lambda_-}{\Lambda_0} & \frac{1}{\Lambda_0} & 0 & 0 \\ \frac{\Lambda_+}{\Lambda_0} & -\frac{1}{\Lambda_0} & 0 & 0 \\ -\frac{i}{2}(\sigma - \frac{1}{k}) & -\frac{1}{2}\frac{i}{k} & \frac{1}{2} & \frac{1}{2}i\sigma \\ \frac{i}{2}(\sigma + \frac{1}{k}) & -\frac{1}{2}\frac{i}{k} & \frac{1}{2} & -\frac{1}{2}i\sigma \end{pmatrix} ,$$

and $D = S^{-1}LS$ a diagonal matrix with diagonal entries Λ_+, Λ_-, $|k|$, and $-|k|$. Note that $\Lambda_+(k) \geq 1$ and $\Lambda_-(k) \leq 0$ and $\Lambda_-(k) \approx -k^2$ for small values of k. Let $\boldsymbol{\zeta} = (\omega_+, \omega_-, u_+, u_-)$. Using the definitions we find that (17) is equivalent to

$$\begin{aligned} \dot{\omega}_+ &= \Lambda_+\omega_+ + \frac{1}{\Lambda_0}q , \\ \dot{\omega}_- &= \Lambda_-\omega_- - \frac{1}{\Lambda_0}q , \\ \dot{u}_+ &= |k|\, u_+ - \frac{1}{2}\frac{i}{k}q , \\ \dot{u}_- &= -|k|\, u_- - \frac{1}{2}\frac{i}{k}q , \end{aligned} \tag{21}$$

with q as defined in (18). For convenience later on we also write $\mathbf{z} = S\boldsymbol{\zeta}$ in component form. Namely,

$$\begin{aligned} \omega &= \omega_+ + \omega_- , \\ \eta &= \Lambda_+\omega_+ + \Lambda_-\omega_- , \\ u &= -\frac{i}{k}\Lambda_-\omega_+ - \frac{i}{k}\Lambda_+\omega_- + u_+ + u_- , \\ v &= \omega_+ + \omega_- - i\sigma u_+ + i\sigma u_- . \end{aligned} \tag{22}$$

3 The integral equations

To solve (21) we convert it into an integral equation. The +-modes are unstable (remember that $\Lambda_+(k) \geq 1$) and we therefore have to integrate these modes backwards in time starting with $\omega_+(k,\infty) \equiv u_+(k,\infty) \equiv 0$ (see [4]). We get

$$\omega_+(k,t) = -\frac{1}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)}q(k,s)\; ds , \tag{23}$$

$$\omega_-(k,t) = \tilde{\omega}_-^*(k)e^{\Lambda_-(t-1)} - \frac{1}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)}q(k,s)\; ds , \tag{24}$$

$$u_+(k,t) = \frac{1}{2}\frac{i}{k}\int_t^\infty e^{|k|(t-s)}q(k,s)\; ds , \tag{25}$$

$$u_-(k,t) = \tilde{u}_-^*(k)e^{-|k|(t-1)} - \frac{1}{2}\frac{i}{k}\int_1^t e^{-|k|(t-s)}q(k,s)\; ds , \tag{26}$$

with $\tilde{\omega}_-^*$ and $\tilde{u}_-^*$ to be chosen later on. The integral equations (23)-(26) are identical to the ones discussed in [10], [11]. There, by restricting to symmetric flows, q was an odd function of k and therefore, in a space of continuously differentiable functions, the division by k that appears in (25) and (26) was compensated by a factor of k coming from q, so that u_+ and u_- were continuous functions of k. For non-symmetric functions q this strategy does not work anymore. As mentioned in the introduction, it is replaced in what follows by a more detailed analysis of the nonlinearity q, and by using the invariance properties of the equations which make that the singular terms compensate each other in the physically relevant functions ω, η, u and v as given by (22). In fact, after substitution of the integral equations (23)-(26) into the change of coordinates (22) we get, with (18), and after integrating by parts the time derivative acting on q_0, the following integral equations for ω, u and v:

$$
\begin{aligned}
\omega(k,t) &= \left(\tilde{\omega}_-^*(k) + \frac{1}{\Lambda_0} q_0(k,1)\right) e^{\Lambda_-(t-1)} \\
&\quad + \frac{1}{\Lambda_0} \int_1^t e^{\Lambda_-(t-s)} ik q_1(k,s)\, ds \\
&\quad + \frac{1}{\Lambda_0} \int_t^\infty e^{\Lambda_+(t-s)} ik q_1(k,s)\, ds \\
&\quad - \frac{\Lambda_-}{\Lambda_0} \int_1^t e^{\Lambda_-(t-s)} q_0(k,s)\, ds \\
&\quad - \frac{\Lambda_+}{\Lambda_0} \int_t^\infty e^{\Lambda_+(t-s)} q_0(k,s)\, ds\ , \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
u(k,t) &= -\frac{i}{k}\Lambda_+ \left(\tilde{\omega}_-^*(k) + \frac{1}{\Lambda_0} q_0(k,1)\right) e^{\Lambda_-(t-1)} \\
&\quad + \left(\tilde{u}_-^*(k) + \frac{1}{2}\frac{i}{k} q_0(k,1)\right) e^{-|k|(t-1)} \\
&\quad + \frac{\Lambda_+}{\Lambda_0} \int_1^t e^{\Lambda_-(t-s)} q_1(k,s)\, ds \\
&\quad - \frac{1}{2} \int_1^t e^{-|k|(t-s)} q_1(k,s)\, ds \\
&\quad + \frac{1}{2} \int_t^\infty e^{|k|(t-s)} q_1(k,s)\, ds \\
&\quad + \frac{\Lambda_-}{\Lambda_0} \int_t^\infty e^{\Lambda_+(t-s)} q_1(k,s)\, ds \\
&\quad + \frac{1}{2} i\sigma(k) \int_1^t e^{-|k|(t-s)} q_0(k,s)\, ds \\
&\quad + \frac{1}{2} i\sigma(k) \int_t^\infty e^{|k|(t-s)} q_0(k,s)\, ds \\
&\quad - \frac{ik}{\Lambda_0} \int_1^t e^{\Lambda_-(t-s)} q_0(k,s)\, ds \\
&\quad - \frac{ik}{\Lambda_0} \int_t^\infty e^{\Lambda_+(t-s)} q_0(k,s)\, ds\ , \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
v(k,t) &= \omega(k,t) \\
&\quad + i\sigma(k) \left(\tilde{u}_-^*(k) + \frac{1}{2}\frac{i}{k} q_0(k,1)\right) e^{-|k|(t-1)} \\
&\quad - \frac{1}{2} i\sigma(k) \int_1^t e^{-|k|(t-s)} q_1(k,s)\, ds \\
&\quad - \frac{1}{2} i\sigma(k) \int_t^\infty e^{|k|(t-s)} q_1(k,s)\, ds \\
&\quad - \frac{1}{2} \int_1^t e^{-|k|(t-s)} q_0(k,s)\, ds \\
&\quad + \frac{1}{2} \int_t^\infty e^{|k|(t-s)} q_0(k,s)\, ds\ , \qquad (29)
\end{aligned}
$$

with q_0 and q_1 given by (19) and (20), respectively. Note that the function η does not need to be constructed since it does not appear in the nonlinearities q_0 and q_1.

A closer look at (27)-(29) reveals, that the problem concerning the division by k in the equations (23)-(26) has not disappeared. However, in this new representation, the invariance properties of the equations have become manifest, and we see that the problem can be eliminated by a proper choice of initial conditions, *i.e.*, ω, u, and v are either regular or singular for all times. In particular, as we will see, if we set

$$\tilde{\omega}_{-}^{*}(k) = -ik\omega_{-}^{*}(k) - \frac{1}{\Lambda_0} q_0(k,1) \ , \tag{30}$$

$$\tilde{u}_{-}^{*}(k) = u_{-}^{*}(k) - \frac{1}{2}\frac{i}{k} q_0(k,1) \ , \tag{31}$$

with

$$u_{-}^{*}(k) = u_{-,1}^{*}(k) - i\sigma(k) u_{-,2}^{*}(k) \ , \tag{32}$$

and with ω_{-}^{*}, $u_{-,1}^{*}$, and $u_{-,2}^{*}$ smooth, then ω and q are smooth, and u and v are smooth modulo discontinuities at $k = 0$. This corresponds to choosing initial conditions exactly as singular as dictated by the nonlinearity. We expect this choice to be general enough to cover all cases of stationary exterior flows.

Below, we will prove existence of solutions to (27)-(29) for certain classes of continuous complex valued functions ω_{-}^{*}, $u_{-,1}^{*}$, and $u_{-,2}^{*}$. Once the existence of solutions has been established, we will restrict attention to even, real valued functions $u_{-,1}^{*}$, and $u_{-,2}^{*}$, and to complex valued functions ω_{-}^{*} of the form

$$\omega_{-}^{*}(k) = \omega_{-,1}^{*}(k) + i\omega_{-,2}^{*}(k) \ , \tag{33}$$

with $\omega_{-,1}^{*}$ and $\omega_{-,2}^{*}$ real valued, even and odd functions of k, respectively. This corresponds to the restriction to real valued solutions of (16).

It turns out that the decomposition of the nonlinearity q into q_0 and q_1 is not detailed enough to prove the existence of a solution to (27)-(29). To overcome this problem we split ω and u into a "dominant part" and a "remainder." Namely, we set

$$\omega(k,t) = \omega_0(k,t) + \omega_1(k,t) \ , \tag{34}$$

$$u(k,t) = u_0(k,t) + u_1(k,t) \ , \tag{35}$$

with ω and u given by (27) and (28), and with

$$u_0(k,t) = -\omega_{-}^{*}(k) e^{\Lambda_{-}(t-1)} + \frac{1}{\Lambda_0}\int_1^t e^{\Lambda_{-}(t-s)} q_1(k,s)\ ds \ , \tag{36}$$

$$\omega_0(k,t) = ik\ u_0(k,t) \ . \tag{37}$$

This decomposition allows us to split the function q_0 into a term which is "large", but zero at $k = 0$, and a remaining term which is "small" (see Section 4 for the definition of "large" and "small"). Namely, using (37) we find that

$$\begin{aligned} (u_0 * \omega_0)(k,t) &= (u_0 * (iku_0))(k,t) \\ &= \frac{1}{2} ik\ (u_0 * u_0)(k,t) \ , \end{aligned}$$

and therefore, using the definition (19) of q_0, we find that

$$q_0(k,t) = q_{0,0}(k,t) - ik\ q_{0,1}(k,t) \ , \tag{38}$$

with

$$q_{0,0}(k,t) = \frac{1}{2\pi}\left(u_0 * \omega_1 + u_1 * \omega_0 + u_1 * \omega_1\right)(k,t) \ , \tag{39}$$

$$q_{0,1}(k,t) = -\frac{1}{4\pi}(u_0 * u_0)(k,t) \ . \tag{40}$$

After some rearrangement, and using (30)-(32), we find for ω_1 and u_1 instead of the equations (34) and (35) the following explicit expressions, which we will use in the sequel:

$$\begin{aligned} \omega_1(k,t) = {} & \frac{1}{\Lambda_0}\int_t^{\infty} e^{\Lambda_{+}(t-s)} ik q_1(k,s)\ ds \\ & -\frac{\Lambda_{+}}{\Lambda_0}\int_t^{\infty} e^{\Lambda_{+}(t-s)} q_0(k,s)\ ds \\ & -\frac{\Lambda_{-}}{\Lambda_0}\int_1^{t} e^{\Lambda_{-}(t-s)} q_0(k,s)\ ds \ , \end{aligned} \tag{41}$$

and

$$
\begin{aligned}
u_1(k,t) &= \left(u^*_{-,1}(k) - i\sigma(k)u^*_{-,2}(k)\right) e^{-|k|(t-1)} \\
&+\omega^*_-(k)\Lambda_- e^{\Lambda_-(t-1)} \\
&-\frac{\Lambda_-}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)} q_1(k,s)\ ds \\
&-\frac{1}{2}\int_1^t e^{-|k|(t-s)} q_1(k,s)\ ds \\
&+\frac{1}{2}\int_t^\infty e^{|k|(t-s)} q_1(k,s)\ ds \\
&+\frac{\Lambda_-}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)} q_1(k,s)\ ds \\
&-\frac{ik}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)} q_0(k,s)\ ds \\
&+\frac{1}{2}i\sigma(k)\int_1^t e^{-|k|(t-s)} q_0(k,s)\ ds \\
&+\frac{1}{2}i\sigma(k)\int_t^\infty e^{|k|(t-s)} q_0(k,s)\ ds \\
&-\frac{ik}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)} q_0(k,s)\ ds\ . \qquad (42)
\end{aligned}
$$

4 Function spaces

In order to prove the existence of a solution for (27)-(29) we will apply, for fixed ω^*_- and fixed $u^*_{-,1}$, $u^*_{-,2}$, the contraction mapping principle to the map $(\tilde{q}_{0,0}, \tilde{q}_{0,1}, \tilde{q}_1) = \mathcal{N}(q_{0,0}, q_{0,1}, q_1)$ that is formally defined by computing first q_0 from $q_{0,0}$ and $q_{0,1}$ using (38), then ω_0, ω_1, ω, u_0, u_1, u and v using (37), (41), (34), (36), (42), (35), (29) and (31) and then $q_{0,0}$, $q_{0,1}$ and q_1 by using (39), (40) and (20). We now define the functions spaces that will be used below:

Let α, $p \geq 0$ and

$$
\mu^p_\alpha(k,t) = \frac{1}{1 + (|k|\, t^p)^\alpha}\ . \qquad (43)
$$

Let furthermore

$$
\begin{aligned}
\mu_\alpha(k,t) &= \mu^{1/2}_a(k,t)\ , \\
\bar{\mu}_\alpha(k,t) &= \mu^1_a(k,t)\ .
\end{aligned}
$$

We then consider, for fixed $\alpha \geq 0$, the Banach space $\mathcal{V}_\alpha$ of functions $f \in \mathcal{C}(\mathbf{R}, \mathbf{C})$ (continuous functions from $\mathbf{R}$ to $\mathbf{C}$), equipped with the norm

$$
\|f\|_\alpha = \sup_{k\in\mathbf{R}} \frac{|f(k)|}{\mu_\alpha(k,1)}\ ,
$$

and, for fixed α, $\beta \geq 0$, the Banach space $\mathcal{B}_{\alpha,\beta}$ of functions $f \in \mathcal{C}([1,\infty), \mathcal{V}_\alpha)$, equipped with the norm

$$
\|f\|_{\alpha,\beta} = \sup_{t\geq 1} t^\beta ||f(t^{-1/2}.\ ,t)||_\alpha\ .
$$

Finally, we define the Banach space $\mathcal{B}_\alpha$,

$$
\mathcal{B}_\alpha = \mathcal{B}_{\alpha,3/2} \oplus \mathcal{B}_{\alpha+1,1/2} \oplus \mathcal{B}_{\alpha,3/2}\ ,
$$

equipped with the norm

$$
\|(\rho_0, \rho_1, \rho_2)\|_\alpha = \|\rho_0\|_{\alpha,3/2} + \|\rho_1\|_{\alpha+1,1/2} + \|\rho_2\|_{\alpha,3/2}\ .
$$

Theorem 3 *Let $\alpha > 1$. Let $u^*_{-,1}$, $u^*_{-,2}$, $\omega^*_- \in \mathcal{V}_{\alpha+1}$, and let $\varepsilon_0 = \|u^*_{-,1}\|_{\alpha+1} + \|u^*_{-,2}\|_{\alpha+1} + \|\omega^*_-\|_{\alpha+1}$. Then, $\mathcal{N}$ is well defined as a map from $\mathcal{B}_\alpha$ to $\mathcal{B}_\alpha$ and contracts, for ε_0 sufficiently small, the ball $B_\alpha(\varepsilon_0) = \{\rho \in \mathcal{B}_\alpha \mid \|\rho\|_\alpha \leq \varepsilon_0\}$ into itself.*

Theorem 3 implies that for ε_0 small enough $\mathcal{N}$ has a unique fixed point in $B_\alpha(\varepsilon_0)$, *i.e.*, the integral equations (27)-(29) have a solution.

5 Proof of Theorem 3

The proof is organized as follows: we first prove that $\mathcal{N}$ is well defined and maps, for small enough initial conditions ω^*_- and u^*_-, a ball in $\mathcal{B}_\alpha$ into itself. Then, we show that $\mathcal{N}$ is a contraction on this ball.

Let ε_0 be as in Theorem 3. Throughout all proofs we then denote by ε a constant multiple of ε_0, *i.e.*, $\varepsilon = \text{const.}\ \varepsilon_0$ with a constant that may be different from instance to instance.

5.1 $\mathcal{N}$ is well defined

We first prove bounds on ω_0, ω_1, u_0, u_1 and v:

Proposition 4 *Let $\alpha > 0$. Let $u^*_{-,1}$, $u^*_{-,2}$, $\omega^*_- \in \mathcal{V}_{\alpha+1}$, with $\varepsilon_0 = \|u^*_{-,1}\|_{\alpha+1} + \|u^*_{-,2}\|_{\alpha+1} + \|\omega^*_-\|_{\alpha+1}$, and let $(q_{0,0}, q_{0,1}, q_1) \in B_\alpha(\varepsilon)$. Then, ω_0, ω_1 and u_0 as defined by (37), (41) and (36) are continuous functions from $\mathbf{R} \times [1,\infty)$ to $\mathbf{C}$, and u_1 and v as defined by (42) and (29) are of the form*

$$u_1(k,t) = u_{1,E}(k,t) + i\sigma(k) u_{1,O}(k,t) \ , \tag{44}$$

$$v(k,t) = v_E(k,t) + i\sigma(k) v_O(k,t) \tag{45}$$

with $u_{1,E}$, $u_{1,O}$, v_E and v_O continuous functions from $\mathbf{R} \times [1,\infty)$ to $\mathbf{C}$. Furthermore, we have the bounds

$$|\omega_0(k,t)| \leq \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t) \ , \tag{46}$$

$$|\omega_1(k,t)| \leq \frac{\varepsilon}{t} \mu_\alpha(k,t) \ , \tag{47}$$

$$|u_0(k,t)| \leq \varepsilon \mu_{\alpha+1}(k,t) \ , \tag{48}$$

$$|u_1(k,t)| \leq \varepsilon \bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t) \ , \tag{49}$$

$$|v(k,t)| \leq \varepsilon \bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t) \ . \tag{50}$$

uniformly in $k \in \mathbf{R}$ and $t \geq 1$.

See Appendix I for a proof.

Now we prove bounds on $q_{0,0}$, $q_{0,1}$ and q_1:

Proposition 5 *Let $\alpha > 1$. Let ω_0, ω_1 and u_0 be continuous functions from $\mathbf{R} \times [1,\infty)$ to $\mathbf{C}$ satisfying the bounds (46)-(48), and let u_1 and v be continuous functions from $\mathbf{R} \setminus \{0\} \times [1,\infty)$ to $\mathbf{C}$, satisfying the bounds (49) and (50), respectively. Then, $q_{0,0}$, $q_{0,1}$ and q_1 as defined by (39), (40) and (20) are continuous functions from $\mathbf{R} \times [0,\infty)$ to $\mathbf{C}$, and we have the bounds*

$$|q_{0,0}(k,t)| \leq \frac{\varepsilon^2}{t^{3/2}} \mu_a(k,t) \ , \tag{51}$$

$$|q_{0,1}(k,t)| \leq \frac{\varepsilon^2}{t^{1/2}} \mu_{a+1}(k,t) \ , \tag{52}$$

$$|q_1(k,t)| \leq \frac{\varepsilon^2}{t^{3/2}} \mu_a(k,t) \ , \tag{53}$$

uniformly in $k \in \mathbf{R}$ and $t \geq 1$, and therefore $\|(q_{0,0}, q_{0,1}, q_1)\|_\alpha \leq \varepsilon^2$.

See Appendix II for a proof.

Proposition 4 together with Proposition 5 imply that, for $\rho \in B_\alpha(\varepsilon)$, $\|\mathcal{N}(\rho)\|_\alpha \leq \varepsilon^2$. Therefore, $\mathcal{N}$ is well defined as a map from $\mathcal{B}_\alpha$ to $\mathcal{B}_\alpha$. Furthermore, since $\varepsilon^2 = \text{const.}\ \varepsilon_0^2$, it follows that $\mathcal{N}$ maps $B_\alpha(\varepsilon_0)$ into itself for ε_0 small enough.

5.2 $\mathcal{N}$ is Lipschitz

In order to complete the proof of Theorem 3 it remains to be shown that $\mathcal{N}$ is Lipschitz:

Proposition 6 *Let $\alpha > 1$. Let $u^*_{-,1}$, $u^*_{-,2}$, $\omega^*_- \in \mathcal{V}_{\alpha+1}$, with $\varepsilon_0 = \left\|u^*_{-,1}\right\|_{\alpha+1} + \left\|u^*_{-,2}\right\|_{\alpha+1} + \left\|\omega^*_-\right\|_{\alpha+1}$, and let ρ, $\tilde{\rho} \in B_\alpha(\varepsilon_0)$. Then*

$$\|\mathcal{N}(\rho) - \mathcal{N}(\tilde{\rho})\|_\alpha \leq \varepsilon \|\rho - \tilde{\rho}\|_\alpha \ . \tag{54}$$

See Appendix III for a proof.

Proposition 4 together with Proposition 5 show that, for $\alpha > 1$, $\mathcal{N}$ maps the ball $B_\alpha(\varepsilon_0)$ into itself for ε_0 small enough, and Proposition 6 therefore shows that $\mathcal{N}$ is a contraction of $B_\alpha(\varepsilon_0)$ into itself for ε_0 small enough. This completes the proof of Theorem 3.

6 Invariant quantities

We now restrict attention to real valued, even functions $u^*_{-,1}$, and $u^*_{-,2}$, and to complex valued functions ω^*_- of the form (33), with $\omega^*_{-,1}$ and $\omega^*_{-,2}$ real valued, even and odd functions of k, respectively.

Proposition 7 *In the limit $k \to 0_\pm$ the equations (35) and (29) reduce to*

$$\begin{aligned} u(0_\pm, t) &= -\omega^*_-(0) + \frac{1}{2}\int_1^\infty q_1(0,s)\ ds + u^*_{-,1}(0) \\ &\quad \pm i\left(-u^*_{-,2}(0) + \frac{1}{2}\int_1^\infty q_0(0,s)\ ds\right) \ , \end{aligned} \tag{55}$$

$$\begin{aligned} v(0_\pm, t) &= u^*_{-,2}(0) - \frac{1}{2}\int_1^\infty q_0(0,s)\ ds \\ &\quad + \int_t^\infty (1 - e^{t-s}) q_0(0,s)\ ds \\ &\quad \pm i\left(u^*_{-,1}(0) - \frac{1}{2}\int_1^\infty q_1(0,s)\ ds\right) \ , \end{aligned} \tag{56}$$

Proof. *This follows immediately using the fact that by Proposition 4 the functions u and v are continuous on $[0,\infty)$ and $(-\infty, 0]$, respectively, for all $t \geq 1$.* ∎

From (55) and (56) we see that the three (real) quantities b, c and d,

$$\begin{aligned} b &= \frac{u(0_+,t) - u(0_-,t)}{2i} \ , \\ d &= \frac{v(0_+,t) - v(0_-,t)}{2i} \ , \\ c &= \frac{u(0_+,t) + u(0_-,t)}{2} - d \ , \end{aligned}$$

are independent of $t \geq 1$. Explicitly, we have

$$b = -u^*_{-,2}(0) + \frac{1}{2}\int_1^\infty q_0(0,s)\ ds \ , \tag{57}$$

$$d = u^*_{-,1}(0) - \frac{1}{2}\int_1^\infty q_1(0,s)\ ds \ , \tag{58}$$

$$c = -\omega^*_-(0) + \int_1^\infty q_1(0,s)\ ds \ . \tag{59}$$

We also note that the quantity φ,

$$\varphi = 2d + c = 2u^*_{-,1}(0) - \omega^*_-(0) \ , \tag{60}$$

is directly given in terms of the initial conditions (see [10], [11] and [6] for the physical interpretation of φ).

7 Asymptotic behavior

The following theorem provides the leading order behavior of solutions whose existence has been shown in Theorem 3. We again restrict attention to even, real valued functions $u^*_{-,1}$, and $u^*_{-,2}$, and to complex valued functions ω^*_- of the form (33), with $\omega^*_{-,1}$ and $\omega^*_{-,2}$ real valued, even and odd functions of k, respectively.

Theorem 8 *Let $\alpha > 1$. Let $u^*_{-,1}$, $u^*_{-,2}$, $\omega^*_- \in \mathcal{V}_{\alpha+1}$, with $\varepsilon_0 = \|u^*_{-,1}\|_{\alpha+1} + \|u^*_{-,2}\|_{\alpha+1} + \|\omega^*_-\|_{\alpha+1}$ sufficiently small. Then, the equations (27)-(29) have a solution and*

$$\lim_{t\to\infty} t^{1/2} \int_{\mathbf{R}} |u(k,t) - u_{as}(k,t)|\ dk = 0\ , \tag{61}$$

$$\lim_{t\to\infty} t \int_{\mathbf{R}} |v(k,t) - v_{as}(k,t)|\ dk = 0\ , \tag{62}$$

where

$$\begin{aligned} u_{as}(k,t) &= c\ e^{-k^2 t} + d\ e^{-|k|t} + b\ i\sigma(k) e^{-|k|t}\ , \\ v_{as}(k,t) &= c\ ike^{-k^2 t} + d\ i\sigma(k) e^{-|k|t} - b\ e^{-|k|t}\ , \end{aligned}$$

with b, c and d as defined in (57), (58) and (59).

The existence of a solution follows from Theorem 3. A proof of (61) and (62) can be found in Appendix IV.

8 Proof of Theorem 1

We again restrict attention to even, real valued functions $u^*_{-,1}$, and $u^*_{-,2}$, and to complex valued functions ω^*_- of the form (33), with $\omega^*_{-,1}$ and $\omega^*_{-,2}$ real valued, even and odd functions of k, respectively. For $\alpha > 1$ we have proved in Section 5 the existence of a solution of the equations (27)-(29) satisfying (to avoid confusion we now write the hats for the Fourier transforms)

$$|\hat{u}(k,t)| \leq \varepsilon \mu_\alpha(k,t)\ , \tag{63}$$

$$|\hat{v}(k,t)| \leq \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t) + \varepsilon \bar{\mu}_{\alpha+1}(k,t)\ . \tag{64}$$

Since, for $\alpha > 1$, the real and imaginary parts of the functions $k \mapsto \hat{u}(k,t)$ and $k \mapsto \hat{v}(k,t)$ are respectively even and odd functions in $L^1(\mathbf{R}, dk)$ for all $t \geq 1$, their Fourier transforms

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi} \int_{\mathbf{R}} e^{-iky} \hat{u}(k,x)\ dk\ , \\ v(x,y) &= \frac{1}{2\pi} \int_{\mathbf{R}} e^{-iky} \hat{v}(k,x)\ dk\ , \end{aligned}$$

are by the Riemann-Lebesgue lemma real valued continuous functions of y and vanish as $|y| \to \infty$ for each $x \geq 1$. Moreover, using (63) and (64), we find that

$$\sup_{y\in\mathbf{R}} |u(x,y)| \leq \frac{\varepsilon}{|x|^{1/2}}\ , \tag{65}$$

$$\sup_{y\in\mathbf{R}} |v(x,y)| \leq \frac{\varepsilon}{|x|}\ . \tag{66}$$

As a consequence, u and v converge to zero whenever $|x|+|y| \to \infty$ in Ω (see Section 5 of [10] for details), and satisfy therefore not only (12) but also the boundary conditions (3), (4). The reconstruction of the pressure from u and v is standard. For $\alpha > 3$ second derivatives of u and v are continuous in direct space, and one easily verifies using the definitions that the triple (u, v, p) satisfies the Navier-Stokes equations (1). The set $\mathcal{S}$ in Theorem 1 is by definition the set of all vector fields (u, v) obtained this way, restricted to Σ. Finally, equations (5)-(11) are a direct consequence of Theorem 8. This completes the proof of Theorem 1.

9 Appendix I

In this appendix we give a proof of Proposition 4. We first prove the continuity, then the bounds.

The continuity of the functions u_0 and ω_0 is elementary. Similarly, using that $\Lambda_+(k) \geq 1$, the continuity of the function ω_1 is elementary, since the improper integrals in (41) converge uniformly in k. Next, we note that u_1 and v as given by (42) and (29) are explicitly of the form (44), (45). The continuity of the functions $u_{1,E}$, $u_{1,O}$, v_E and v_O is again elementary except for the improper integrals involving the function $e^{|k|(t-s)}$. There are two cases of such integrals; those involving q_1 and those involving q_0. Since $|q_1(k,s)| \leq \varepsilon/s^{3/2}$ and since $1/s^{3/2}$ is integrable at infinity, the continuity follows in these cases. Similarly, since $|q_{0,0}(k,s)| \leq \varepsilon/s^{3/2}$, the contribution of $q_{0,0}$ to the integrals involving q_0 defines continuous functions. This leaves us with the case of improper integrals involving $e^{|k|(t-s)}$ and $q_{0,1}$. For these cases we have the inequality

$$\begin{aligned}
\left|\frac{1}{2}i\sigma(k)\int_t^\infty e^{|k|(t-s)}\left(-ikq_{0,1}(k,s)\right)\,ds\right| &\leq \int_t^\infty e^{|k|(t-s)}\frac{\varepsilon}{s^{1/2}}\,|k|\,\mu_{a+1}(k,s)\,ds \\
&\leq \varepsilon\mu_{a+1}(k,t)\int_t^\infty e^{|k|(t-s)}\,|k|\,\frac{ds}{s^{1/2}} \\
&\leq \varepsilon\mu_{a+1}(k,t)\,|k|^{1/2}\,e^{|k|t}\int_{|k|t}^\infty e^{-\sigma}\frac{d\sigma}{\sigma^{1/2}} \\
&\leq \varepsilon\mu_{a+1}(k,t)\,|k|^{1/2}\,e^{|k|t}(1-\operatorname{erf}(|k|\,t)) \\
&\leq \varepsilon\,|k|^{1/2}\,\bar{\mu}_{1/2}(k,t)\mu_{a+1}(k,t)\ , \qquad (67)
\end{aligned}$$

with erf the error function. This shows continuity, since $\lim_{k\to 0}\varepsilon\,|k|^{1/2}\,\bar{\mu}_{1/2}(k,t)\mu_{a+1}(k,t) = 0$ for $t \geq 1$. This completes the proof of the continuity.

We next prove the bounds (46)-(50). We start by proving an inequality which will be routinely used below:

9.1 Main technical Lemma

Proposition 9 *Let $\alpha' \geq \beta' \geq \gamma' \geq 0$ and $\mu > 0$. Then, we have the bound*

$$\frac{1}{1+|k|^{\alpha'}}e^{\mu\Lambda_-(t-1)}\,|\Lambda_-|^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.}\ \frac{1}{t^{\beta'}}\frac{1}{1+\left(|k|\,t^{1/2}\right)^{\alpha'-\beta'+\gamma'}}\ , \qquad (68)$$

uniformly in $k \in \mathbf{R}$ and $t \geq 1$. Similarly, for positive α', β', γ' with $\alpha' - \beta' + \gamma' \geq 0$ and $\mu > 0$ we have the bound

$$\frac{1}{1+|k|^{\alpha'}}e^{-\mu|k|(t-1)}\,|k|^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.}\ \frac{1}{t^{\beta'}}\frac{1}{1+(|k|\,t)^{\alpha'-\beta'+\gamma'}}\ , \qquad (69)$$

uniformly in $k \in \mathbf{R}$ and $t \geq 1$.

Proof. We first prove (68). For $1 \leq t \leq 2$ we have that

$$\begin{aligned}
\frac{1}{1+|k|^{\alpha'}}e^{\mu\Lambda_-(t-1)}\,|\Lambda_-|^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'} &\leq \text{const.}\ \frac{1}{1+|k|^{\alpha'}}e^{\mu\Lambda_-(t-1)}\,|\Lambda_-\,(t-1)|^{\gamma'}\ \ |\Lambda_-|^{\beta'-\gamma'} \\
&\leq \text{const.}\ \frac{1}{1+|k|^{\alpha'}}\ \ |\Lambda_-|^{\beta'-\gamma'} \\
&\leq \text{const.}\ \frac{1}{1+|k|^{\alpha'-\beta'+\gamma'}} \\
&\leq \text{const.}\ \frac{1}{t^{\beta'}}\frac{1}{1+\left(|k|\,t^{1/2}\right)^{\alpha'-\beta'+\gamma'}}\ ,
\end{aligned}$$

as claimed, and for $t>2$ we use that

$$\begin{aligned}
&\left(1+\left(|k|\,t^{1/2}\right)^{\alpha'-\beta'+\gamma'}\right)e^{\mu\Lambda_-(t-1)}\,|\Lambda_- t|^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'}\\
&\leq \text{const.}\left(1+\left(|k|\,t^{1/2}\right)^{\alpha'}\right)e^{\frac{1}{2}\mu\Lambda_- t}\,|\Lambda_- t|^{\beta'}\\
&\leq \text{const.}\left(1+\frac{|k|^{\alpha'}}{|\Lambda_-|^{\alpha'/2}}\,|\Lambda_- t|^{\alpha'/2}\,|\Lambda_- t|^{\beta'}\,e^{\frac{1}{2}\mu\Lambda_- t}\right)\\
&\leq \text{const.}\left(1+\frac{|k|^{\alpha'}}{|\Lambda_-|^{\alpha'/2}}\right)\\
&\leq \text{const.}\left(1+|k|^{\alpha'/2}\right)\leq \text{const.}\left(1+|k|^{\alpha'}\right)\ ,
\end{aligned}$$

and (68) follows. We now prove (69). For $1\leq t\leq 2$ and $|k|\leq 1$ we have that

$$\begin{aligned}
\frac{1}{1+|k|^{\alpha'}}e^{-\mu|k|(t-1)}\,|k|^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'}&\leq \text{const.}\\
&\leq \text{const.}\,\frac{1}{t^{\beta'}}\frac{1}{1+(|k|\,t)^{\alpha'-\beta'+\gamma'}}\ ,
\end{aligned}$$

and for $1\leq t\leq 2$ and $|k|>1$ we have that

$$\begin{aligned}
\frac{1}{1+|k|^{\alpha'}}e^{-\mu|k|(t-1)}\,|k|^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'}&\leq \text{const.}\,\frac{1}{1+|k|^{\alpha'}}e^{-\mu|k|(t-1)}\left(|k|\,(t-1)\right)^{\gamma'}\ \ |k|^{\beta'-\gamma'}\\
&\leq \text{const.}\,\frac{1}{1+|k|^{\alpha'}}\ \ |k|^{\beta'-\gamma'}\\
&\leq \text{const.}\,\frac{1}{1+|k|^{\alpha'-\beta'+\gamma'}}\\
&\leq \text{const.}\,\frac{1}{t^{\beta'}}\frac{1}{1+(|k|\,t)^{\alpha'-\beta'+\gamma'}}\ .
\end{aligned}$$

Finally, for $t>2$ we use that

$$\begin{aligned}
&\left(1+(|k|\,t)^{\alpha'-\beta'+\gamma'}\right)e^{-\mu|k|(t-1)}\,(|k|\,t)^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'}\\
&\leq \text{const.}\left(1+(|k|\,t)^{\alpha'-\beta'+\gamma'}\right)e^{-\frac{1}{2}\mu|k|t}\,(|k|\,t)^{\beta'}\\
&\leq \text{const.}\leq \text{const.}\left(1+|k|^{\alpha'}\right)\ ,
\end{aligned}$$

and (69) follows. ■

We are now ready to prove (46)-(50).

9.2 Bound on ω_0

By definition (37) of ω_0 the inequality (46) follows from (48).

9.3 Bound on u_0

We write $u_0=\sum_{i=1}^{2}u_{0,i}$, with $u_{0,i}$ the i-th term in (36), and we bound each of the terms individually. The inequality (48) then follows using the triangle inequality.

Proposition 10 *For all $\alpha \geq 0$ we have the bounds*

$$|u_{0,1}(k,t)| \leq \varepsilon\mu_{a+1}(k,t) \ , \tag{70}$$
$$|u_{0,2}(k,t)| \leq \varepsilon\mu_{a+1}(k,t) \ , \tag{71}$$

uniformly in $t \geq 1$ and $k \in \mathbf{R}$.

For $u_{0,1}$ we have

$$\left|-\omega_{-}^{*}(k)e^{\Lambda_{-}(t-1)}\right| \leq \varepsilon\mu_{a+1}(k,1)e^{\Lambda_{-}(t-1)} \ ,$$

and (70) follows using Proposition 9. Next, splitting the integral defining of $u_{0,2}$ in two parts we find that

$$\begin{aligned}\left|\frac{1}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_{-}(t-s)}q_1(k,s)\ ds\right| &\leq \varepsilon\mu_{a+1}(k,1)e^{\Lambda_{-}\frac{t-1}{2}}\int_1^{\frac{t+1}{2}}\frac{1}{s^{3/2}}ds \\ &\leq \varepsilon\mu_{a+1}(k,1)e^{\Lambda_{-}\frac{t-1}{2}}\left(\frac{t-1}{t}\right) \\ &\leq \varepsilon\mu_{a+1}(k,t) \ ,\end{aligned}$$

and that

$$\begin{aligned}\left|\frac{1}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_{-}(t-s)}q_1(k,s)\ ds\right| &\leq \varepsilon\frac{1}{\Lambda_0}\mu_a(k,t)\int_{\frac{t+1}{2}}^{t}\frac{1}{s^{3/2}}ds \\ &\leq \frac{\varepsilon}{t^{1/2}}\frac{1}{\Lambda_0}\mu_a(k,t) \\ &\leq \varepsilon\mu_{a+1}(k,t) \ ,\end{aligned}$$

which proves (71) using the triangle inequality.

9.4 Bound on ω_1

We write $\omega_1 = \sum_{i=1}^{3}\omega_{1,i}$, with $\omega_{1,i}$ the i-th term in (41), and we bound each of the terms individually. The inequality (47) then follows using the triangle inequality.

Proposition 11 *For all $\alpha \geq 0$ we have the bounds*

$$|\omega_{1,1}(k,t)| \leq \frac{\varepsilon}{t^{3/2}}\mu_a(k,t) \ , \tag{72}$$
$$|\omega_{1,2}(k,t)| \leq \frac{\varepsilon}{t}\mu_a(k,t) \ , \tag{73}$$
$$|\omega_{1,3}(k,t)| \leq \frac{\varepsilon}{t}\mu_{a+1}(k,t) \ , \tag{74}$$

uniformly in $t \geq 1$ and $k \in \mathbf{R}$.

For $\omega_{1,1}$ we have that

$$\begin{aligned}\left|\frac{1}{\Lambda_0}\int_t^{\infty} e^{\Lambda_{+}(t-s)}ikq_1(k,s)\ ds\right| &\leq \frac{\varepsilon}{t^{3/2}}\frac{|k|}{\Lambda_0}\mu_a(k,t)\int_t^{\infty} e^{\Lambda_{+}(t-s)}ds \\ &\leq \frac{\varepsilon}{t^{3/2}}\frac{1}{\Lambda_{+}}\frac{|k|}{\Lambda_0}\mu_a(k,t) \\ &\leq \frac{\varepsilon}{t^{3/2}}\mu_a(k,t) \ ,\end{aligned}$$

which proves (72). Similarly, to prove the bound on $\omega_{1,2}$, we use that by definition (38) of q_0

$$|q_0(k,t)| \leq \frac{\varepsilon}{t^{3/2}}\mu_a(k,t) + \frac{\varepsilon}{t^{1/2}}|k|\,\mu_{a+1}(k,t) \tag{75}$$
$$\leq \frac{\varepsilon}{t}\mu_a(k,t) \ , \tag{76}$$

and therefore

$$\begin{aligned}\left|\frac{\Lambda_+}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)}q_0(k,s)\;ds\right| &\leq \frac{\varepsilon}{t}\frac{\Lambda_+}{\Lambda_0}\mu_a(k,t)\int_t^\infty e^{\Lambda_+(t-s)}ds\\ &\leq \frac{\varepsilon}{t}\frac{1}{\Lambda_0}\mu_a(k,t)\;,\end{aligned}$$

which proves (73). The integral defining $\omega_{1,3}$ we split in two parts. Using (75) and the identity $|k| = |\Lambda_-|^{1/2}|\Lambda_+|^{1/2}$, we find that

$$\begin{aligned}\left|\frac{\Lambda_-}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}q_0(k,s)\;ds\right| &\leq \varepsilon\mu_{a+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}|\Lambda_-|\int_1^{\frac{t+1}{2}}\frac{1}{s^{3/2}}ds\\ &\quad+\varepsilon\mu_{a+2}(k,1)e^{\Lambda_-\frac{t-1}{2}}|\Lambda_-|\,|k|\int_1^{\frac{t+1}{2}}\frac{1}{s^{1/2}}ds\\ &\leq \varepsilon\mu_{a+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}|\Lambda_-|\left(\frac{t-1}{t}\right)\\ &\quad+\varepsilon t^{1/2}\mu_{a+3/2}(k,1)e^{\Lambda_-\frac{t-1}{2}}|\Lambda_-|^{3/2}\left(\frac{t-1}{t}\right)\\ &\leq \frac{\varepsilon}{t}\mu_{a+1}(k,t)\;,\end{aligned}$$

and, using (76), we find that

$$\begin{aligned}\left|\frac{\Lambda_-}{\Lambda_0}\int_{\frac{t+1}{2}}^t e^{\Lambda_-(t-s)}q_0(k,s)\;ds\right| &\leq \frac{\varepsilon}{t}\frac{1}{\Lambda_0}\mu_a(k,t)\int_{\frac{t+1}{2}}^t e^{\Lambda_-(t-s)}|\Lambda_-|\,ds\\ &\leq \frac{\varepsilon}{t}\frac{1}{\Lambda_0}\mu_a(k,t)\;,\end{aligned}$$

and (74) follows using the triangle inequality.

9.5 Bound on u_1

We write $u_1 = \sum_{i=1}^{10} u_{1,i}$, with $u_{1,i}$ the i-th term in (42), and we bound each of the terms individually. The inequality (49) then follows using the triangle inequality.

Proposition 12 *For all $\alpha \geq 0$ we have the bounds*

$$|u_{1,1}(k,t)| \leq \varepsilon\bar{\mu}_{a+1}(k,t)\;, \tag{77}$$

$$|u_{1,2}(k,t)| \leq \frac{\varepsilon}{t}\mu_a(k,t)\;, \tag{78}$$

$$|u_{1,3}(k,t)| \leq \frac{\varepsilon}{t}\mu_{a+1}(k,t)\;, \tag{79}$$

$$|u_{1,4}(k,t)| \leq \varepsilon\bar{\mu}_{a+1}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_a(k,t)\;, \tag{80}$$

$$|u_{1,5}(k,t)| \leq \frac{\varepsilon}{t^{1/2}}\mu_a(k,t)\;, \tag{81}$$

$$|u_{1,6}(k,t)| \leq \frac{\varepsilon}{t}\mu_{a+1}(k,t)\;, \tag{82}$$

$$|u_{1,7}(k,t)| \leq \frac{\varepsilon}{t^{1/2}}\mu_{a+1/2}(k,t)\;, \tag{83}$$

$$|u_{1,8}(k,t)| \leq \varepsilon\bar{\mu}_{a+1}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_a(k,t)\;, \tag{84}$$

$$|u_{1,9}(k,t)| \leq \frac{\varepsilon}{t^{1/2}}\mu_a(k,t)\;, \tag{85}$$

$$|u_{1,10}(k,t)| \leq \frac{\varepsilon}{t}\mu_a(k,t)\;, \tag{86}$$

uniformly in $t \geq 1$ and $k \in \mathbf{R}$.

For $u_{1,1}$ we have that

$$\left|\left(u^*_{-,1}(k) - i\sigma(k)u^*_{-,2}(k)\right) e^{-|k|(t-1)}\right| \leq \varepsilon\mu_{\alpha+1}(k,1)e^{-|k|(t-1)} \ ,$$

and (77) follows using Proposition 9. Next, for $u_{1,2}$ we have that

$$\left|\omega^*_-(k)\Lambda_- e^{\Lambda_-(t-1)}\right| \leq \varepsilon\mu_{\alpha+1}(k,1)\left|\Lambda_-\right| e^{\Lambda_-(t-1)} \ ,$$

and (78) follows using Proposition 9. The integral defining $u_{1,3}$ we split in two parts. We have that

$$\begin{aligned}\left|\frac{\Lambda_-}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}q_1(k,s)\ ds\right| &\leq \varepsilon\mu_{\alpha+1}(k,1)\left|\Lambda_-\right| e^{\Lambda_-\frac{t-1}{2}}\int_1^{\frac{t+1}{2}}\frac{1}{s^{3/2}}ds \\ &\leq \varepsilon\mu_{\alpha+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}\left|\Lambda_-\right|\left(\frac{t-1}{t}\right) \\ &\leq \frac{\varepsilon}{t}\mu_{\alpha+1}(k,t)\ ,\end{aligned}$$

and that

$$\begin{aligned}\left|\frac{\Lambda_-}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}q_1(k,s)\ ds\right| &\leq \frac{\varepsilon}{t^{3/2}}\frac{1}{\Lambda_0}\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\left|\Lambda_-\right|\ ds \\ &\leq \frac{\varepsilon}{t^{3/2}}\frac{1}{\Lambda_0}\mu_\alpha(k,t) \\ &\leq \frac{\varepsilon}{t}\mu_{\alpha+1}(k,t)\ ,\end{aligned}$$

and (79) follows. The integral defining $u_{1,4}$ we also split in two parts. We have that

$$\begin{aligned}\left|\frac{1}{2}\int_1^{\frac{t+1}{2}} e^{-|k|(t-s)}q_1(k,s)\ ds\right| &\leq \varepsilon e^{-|k|\frac{t-1}{2}}\mu_\alpha(k,1)\int_1^{\frac{t+1}{2}}\frac{1}{s^{3/2}}\ ds \\ &\leq \varepsilon\mu_\alpha(k,1)e^{-|k|\frac{t-1}{2}}\left(\frac{t-1}{t}\right) \\ &\leq \varepsilon\bar{\mu}_{\alpha+1}(k,t)\ ,\end{aligned}$$

and that

$$\begin{aligned}\left|\frac{1}{2}\int_{\frac{t+1}{2}}^{t} e^{-|k|(t-s)}q_1(k,s)\ ds\right| &\leq \varepsilon\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^{t}\frac{1}{s^{3/2}}\ ds \\ &\leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t)\ ,\end{aligned}$$

and (80) follows. Next, to bound $u_{1,5}$, we use that

$$\begin{aligned}\left|\frac{1}{2}\int_t^{\infty} e^{|k|(t-s)}q_1(k,s)\ ds\right| &\leq \varepsilon\mu_\alpha(k,t)\int_t^{\infty}\frac{1}{s^{3/2}}\ ds \\ &\leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t)\ ,\end{aligned}$$

and (81) follows. Next, for $u_{1,6}$ we have that

$$\begin{aligned}\left|\frac{\Lambda_-}{\Lambda_0}\int_t^{\infty} e^{\Lambda_+(t-s)}q_1(k,s)\ ds\right| &\leq \frac{\varepsilon}{t^{3/2}}\frac{\left|\Lambda_-\right|}{\Lambda_0}\mu_\alpha(k,t)\int_t^{\infty} e^{\Lambda_+(t-s)}\ ds \\ &\leq \frac{\varepsilon}{t^{3/2}}\frac{\left|\Lambda_-\right|}{\Lambda_0\Lambda_+}\mu_\alpha(k,t) \\ &\leq \frac{\varepsilon}{t}\mu_{\alpha+1}(k,t)\ ,\end{aligned}$$

and (82) follows. To bound $u_{1,7}$ we split the integral into two parts and estimate the contributions from $q_{0,0}$ and $q_{0,1}$ separately. For the contribution to $u_{1,7}$ coming from $q_{0,0}$ we have that

$$\begin{aligned}\left|\frac{-ik}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}q_{0,0}(k,s)\;ds\right| &\leq \frac{|k|}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}\frac{\varepsilon}{s^{3/2}}\mu_a(k,s)\;ds \\ &\leq \varepsilon\frac{|k|}{\Lambda_0}e^{\Lambda_-\frac{t-1}{2}}\mu_a(k,1)\int_1^{\frac{t+1}{2}}\frac{1}{s^{3/2}}\;ds \\ &\leq \frac{\varepsilon}{t^{1/2}}\mu_{a+1/2}(k,t)\;,\end{aligned}$$

and that

$$\begin{aligned}\left|\frac{-ik}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}q_{0,0}(k,s)\;ds\right| &\leq \frac{|k|}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\frac{\varepsilon}{s^{3/2}}\mu_a(k,s)\;ds \\ &\leq \frac{\varepsilon}{t^{3/2}}\mu_a(k,t)\frac{\Lambda_+^{1/2}}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\left|\Lambda_-\right|^{1/2}\;ds \\ &\leq \frac{\varepsilon}{t^{3/2}}\frac{1}{\Lambda_0^{1/2}}\mu_a(k,t)\int_{\frac{t+1}{2}}^{t}\frac{1}{\sqrt{t-s}}\;ds \\ &\leq \frac{\varepsilon}{t}\frac{1}{\Lambda_0^{1/2}}\mu_a(k,t)\;,\end{aligned}$$

and for the contribution to $u_{1,7}$ coming from $q_{0,1}$ we have that

$$\begin{aligned}\left|\frac{-ik}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}q_{0,1}(k,s)\;ds\right| &\leq \frac{|k|}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}\frac{\varepsilon}{s^{1/2}}\left|k\right|\mu_{a+1}(k,s)\;ds \\ &\leq \varepsilon\mu_{a+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}\left|\Lambda_-\right|\int_1^{\frac{t+1}{2}}\frac{\varepsilon}{s^{1/2}}\;ds \\ &\leq \frac{\varepsilon}{t^{1/2}}\mu_{a+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}\left|\Lambda_-\right|(t-1) \\ &\leq \frac{\varepsilon}{t^{1/2}}\mu_{a+1}(k,t)\;,\end{aligned}$$

and that

$$\begin{aligned}\left|\frac{-ik}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}q_{0,1}(k,s)\;ds\right| &\leq \frac{|k|}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\frac{\varepsilon}{s^{1/2}}\left|k\right|\mu_{a+1}(k,s)\;ds \\ &\leq \frac{\varepsilon}{t^{1/2}}\mu_{a+1}(k,t)\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\left|\Lambda_-\right|\;ds \\ &\leq \frac{\varepsilon}{t^{1/2}}\mu_{a+1}(k,t)\;,\end{aligned}$$

and (83) follows using the triangle inequality. The proof of the estimate for the $q_{0,0}$-contribution to $u_{1,8}$ is identical to the proof of (80), and we therefore have that

$$\left|\frac{1}{2}i\sigma(k)\int_1^t e^{-|k|(t-s)}q_{0,0}(k,s)\;ds\right| \leq \varepsilon\bar{\mu}_{\alpha+1}(k,t)+\frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t)\;.$$

To estimate the contribution of $q_{0,1}$ to $u_{1,8}$ we split the integral into two and we get

$$\begin{aligned}\left|\frac{1}{2}i\sigma(k)\int_1^{\frac{t+1}{2}} e^{-|k|(t-s)}q_{0,1}(k,s)\;ds\right| &\leq \int_1^{\frac{t+1}{2}} e^{-|k|(t-s)}\frac{\varepsilon}{s^{1/2}}\left|k\right|\mu_{a+1}(k,s)\;ds \\ &\leq \varepsilon\mu_{a+1}(k,1)\left|k\right|e^{-|k|\frac{t-1}{2}}\int_1^{\frac{t+1}{2}}\frac{1}{s^{1/2}}ds \\ &\leq \varepsilon t^{1/2}\mu_{a+1}(k,1)e^{-|k|\frac{t-1}{2}}\left|k\right|\left(\frac{t-1}{t}\right) \\ &\leq \frac{\varepsilon}{t^{1/2}}\bar{\mu}_{a+1}(k,t)\;,\end{aligned}$$

and

$$\left|\frac{1}{2}i\sigma(k)\int_{\frac{t+1}{2}}^{t} e^{-|k|(t-s)}q_{0,1}(k,s)\; ds\right| \;\leq\; \int_{\frac{t+1}{2}}^{t} e^{-|k|(t-s)}\frac{\varepsilon}{s^{1/2}}\,|k|\,\mu_{a+1}(k,s)\; ds$$
$$\leq\; \frac{\varepsilon}{t^{1/2}}\mu_{a+1}(k,t)\int_{\frac{t+1}{2}}^{t} e^{-|k|(t-s)}\,|k|\; ds$$
$$\leq\; \frac{\varepsilon}{t^{1/2}}\mu_{a+1}(k,t)\ ,$$

and (84) follows using the triangle inequality. The proof of the estimate for the $q_{0,0}$-contribution to $u_{1,9}$ is identical to the proof of (81), so that

$$\left|\frac{1}{2}i\sigma(k)\int_{t}^{\infty} e^{|k|(t-s)}q_{0,0}(k,s)\; ds\right| \leq \frac{\varepsilon}{t^{1/2}}\mu_{\alpha}(k,t)\ .$$

For the contribution of $q_{0,1}$ to $u_{1,9}$ we get (this is a more elementary but less precise bound than (67))

$$\left|\frac{1}{2}i\sigma(k)\int_{t}^{\infty} e^{|k|(t-s)}\left(-ikq_{0,1}(k,s)\right)\; ds\right| \;\leq\; \int_{t}^{\infty} e^{|k|(t-s)}\frac{\varepsilon}{s^{1/2}}\,|k|\,\mu_{a+1}(k,s)\; ds$$
$$\leq\; \frac{\varepsilon}{t^{1/2}}\mu_{a+1}(k,t)\int_{t}^{\infty} e^{|k|(t-s)}\,|k|\; ds$$
$$\leq\; \frac{\varepsilon}{t^{1/2}}\mu_{a+1}(k,t)\ ,$$

and (85) follows using the triangle inequality. Finally, since $u_{1,10} = ik/\Lambda_{+}\ \omega_{1,2}$, and since $|ik/\Lambda_{+}| \leq$ const., the proof of the estimate (86) is identical to the proof of (73).

9.6 Bound on v

We write $v = \sum_{i=1}^{6} v_i$, with v_i the i-th term in (29), and we bound each of the terms individually. The inequality (50) then follows using the triangle inequality.

Proposition 13 *For all $\alpha \geq 0$ we have the bounds*

$$|v_1(k,t)| \;\leq\; \frac{\varepsilon}{t^{1/2}}\mu_a(k,t)\ , \tag{87}$$
$$|v_2(k,t)| \;\leq\; \varepsilon\bar{\mu}_{\alpha+1}(k,t)\ , \tag{88}$$
$$|v_3(k,t)| \;\leq\; \varepsilon\bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_{\alpha}(k,t)\ , \tag{89}$$
$$|v_4(k,t)| \;\leq\; \frac{\varepsilon}{t^{1/2}}\mu_a(k,t)\ , \tag{90}$$
$$|v_5(k,t)| \;\leq\; \varepsilon\bar{\mu}_{a+1}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_a(k,t)\ , \tag{91}$$
$$|v_6(k,t)| \;\leq\; \frac{\varepsilon}{t^{1/2}}\mu_a(k,t)\ , \tag{92}$$

uniformly in $k \in \mathbf{R}$ and $t \geq 1$.

Inequality (87) follows from (46) and (47) using the triangle inequality. Next, after substitution of (31) and (32) into v_2 we find that $v_2(k,t) = i\sigma(k)u_{1,1}(k,t)$. Therefore,

$$|v_2(k,t)| \leq |u_{1,1}(k,t)|\ ,$$

and (88) therefore follows from (77). Similarly, (89), (90), (91) and (92) follow from (80), (81), (84) and (85), since

$$v_3(k,t) \;=\; i\sigma(k)\ u_{1,4}(k,t)\ ,$$
$$v_4(k,t) \;=\; -i\sigma(k)\ u_{1,5}(k,t)\ ,$$
$$v_5(k,t) \;=\; -i\sigma(k)\ u_{1,8}(k,t)\ ,$$
$$v_6(k,t) \;=\; i\sigma(k)\ u_{1,9}(k,t)\ .$$

This completes the proof of Proposition 4.

10 Appendix II

In this appendix we give a proof of Proposition 5.

10.1 Bounds on convolutions

Proposition 14 *Let α, $\beta > 1$, $p \geq q \geq 0$ and let a be a piecewise continuous, and b be a continuous function from $\mathbf{R}\times[1,\infty)$ to $\mathbf{C}$ satisfying the bounds (see (43) for the definition of μ_α^p and μ_β^q, respectively),*

$$\begin{aligned}|a(k,t)| &\leq \mu_\alpha^p(k,t) \ , \\ |b(k,t)| &\leq \mu_\beta^q(k,t) \ .\end{aligned}$$

Then, the convolution $a * b$ is a continuous function from $\mathbf{R}\times[1,\infty)$ to $\mathbf{C}$ and we have the bounds

$$|(a*b)(k,t)| \leq \text{const.}\left(\frac{1}{t^p}\mu_\beta^q(k,t) + |k|\,\mu_\alpha^p(k,t)\right) \ , \tag{93}$$

$$|(a*b)(k,t)| \leq \text{const.}\ \frac{1}{t^p}\mu^q_{\min\{\alpha-1,\beta\}}(k,t) \ , \tag{94}$$

$$|(a*b)(k,t)| \leq \text{const.}\left(\frac{1}{t^p}\mu_\beta^q(k,t) + \frac{1}{t^q}\mu_\alpha^p(k,t)\right) \ , \tag{95}$$

$$|(a*b)(k,t)| \leq \text{const.}\ \frac{1}{t^{\min\{p,q\}}}\mu^{\min\{p,q\}}_{\min\{\alpha,\beta\}}(k,t) \ , \tag{96}$$

uniformly in $t \geq 1$, $k \in \mathbf{R}$.

Continuity is elementary. The bound (94) follows immediately from (93) using that $|k|\,\mu_\alpha^p(k,t) \leq$ const. $\mu_{\alpha-1}^p(k,t)/t^p$, and (96) follows from (95). We now prove (93). Let $k \geq 0$. Then, we have for $a * b$

$$\begin{aligned}|(a*b)(k,t)| &\leq \int_{-\infty}^{\infty}\mu_\alpha^p(k',t)\mu_\beta^q(k-k',t)\ dk' \\ &\leq \mu_\beta^q(k/2,t)\int_{-\infty}^{k/2}\mu_\alpha^p(k',t)\ dk' \\ &\quad + \int_{k/2}^{3k/2}\mu_\alpha^p(k',t)\ dk' + \mu_\beta^q(k/2,t)\int_{3k/2}^{\infty}\mu_\alpha^p(k',t)\ dk' \\ &\leq \text{const.}\left(\frac{1}{t^p}\mu_\beta^q(k,t) + |k|\,\mu_\alpha^p(k/2,t)\right) \ ,\end{aligned} \tag{97}$$

and (93) follows for $k \geq 0$. Similarly we have for $k < 0$,

$$\begin{aligned}|(a*b)(k,t)| &\leq \int_{-\infty}^{\infty}\mu_\alpha^p(k',t)\mu_\beta^q(k-k',t)\ dk' \\ &\leq \mu_\beta^q(k/2,t)\int_{-\infty}^{3k/2}\mu_\alpha^p(k',t)\ dk' \\ &\quad + \int_{3k/2}^{k/2}\mu_\alpha^p(k',t)\ dk' + \mu_\beta^q(k/2,t)\int_{k/2}^{\infty}\mu_\alpha^p(k',t)\ dk' \\ &\leq \text{const.}\left(\frac{1}{t^p}\mu_\beta^q(k,t) + |k|\,\mu_\alpha^p(k/2,t)\right) \ ,\end{aligned}$$

and (93) follows for $k < 0$. We now prove (95). Let $k \geq 0$. Then we have, cutting the integral into three parts as in (97),

$$\begin{aligned}|(a*b)(k,t)| &\leq \mu_\beta^q(k/2,t)\int_{-\infty}^{k/2}\mu_\alpha^p(k',t)\ dk' \\ &\quad + \mu_\alpha^p(k/2,t)\int_{k/2}^{3k/2}\mu_\beta^q(k-k',t)\ dk' + \mu_\beta^q(k/2,t)\int_{3k/2}^{\infty}\mu_\alpha^p(k',t)\ dk' \\ &\leq \text{const.}\left(\frac{1}{t^p}\mu_\beta^q(k,t) + \frac{1}{t^q}\mu_\alpha^p(k,t)\right) \ ,\end{aligned}$$

and (95) follows for $k \geq 0$. Similarly we have for $k < 0$,

$$\begin{aligned}
|(a*b)(k,t)| &\leq \mu_\beta^q(k/2,t)\int_{-\infty}^{3k/2}\mu_\alpha^p(k',t)\ dk' \\
&\quad +\mu_\alpha^p(k/2,t)\int_{3k/2}^{k/2}\mu_\beta^q(k-k',t)\ dk' + \mu_\beta^q(k/2,t)\int_{k/2}^{\infty}\mu_\alpha^p(k',t)\ dk' \\
&\leq \text{const.}\left(\frac{1}{t^p}\mu_\beta^q(k,t)+\frac{1}{t^q}\mu_\alpha^p(k,t)\right) ,
\end{aligned}$$

and (95) follows for $k < 0$. This completes the proof of Proposition 14.

10.2 Proof of Proposition 5

We first prove the bound on $q_{0,0}$. Namely, using Proposition 14, we find from (46), (47), (48) and (49) that

$$\begin{aligned}
|(u_0*\omega_1)(k,t)| &\leq \frac{\varepsilon^2}{t^{3/2}}\mu_a(k,t) , \\
|(u_1*\omega_0)(k,t)| &\leq \frac{\varepsilon^2}{t^{3/2}}\mu_a(k,t) , \\
|(u_1*\omega_1)(k,t)| &\leq \frac{\varepsilon^2}{t^2}\mu_a(k,t) ,
\end{aligned}$$

and (51) follows using the triangle inequality. Next we prove the bound on $q_{0,1}$. Namely, using (96) we find from (48) that

$$|(u_0*u_0)(k,t)| \leq \frac{\varepsilon^2}{t^{1/2}}\mu_{a+1}(k,t) ,$$

which proves (52). Finally, we prove the bound on q_1. First, the bounds (46) and (47) imply using the triangle inequality, that

$$|\omega(k,t)| \leq \frac{\varepsilon}{t^{1/2}}\mu_a(k,t) ,$$

which, together with (50) and using Proposition 14 implies that

$$|(v*\omega)(k,t)| \leq \frac{\varepsilon^2}{t^{3/2}}\mu_\alpha(k,t) ,$$

which proves (53). This completes the proof of Proposition 5.

11 Appendix III

In this appendix we prove Proposition 6. Let $\rho^1 \equiv (\rho_1^1,\rho_2^1,\rho_3^1)$, $\rho^2 \equiv (\rho_1^2,\rho_2^2,\rho_3^2) \in B_\alpha(\varepsilon_0)$. Then, by Proposition 4 and Proposition 5, $\rho \equiv \mathcal{N}(\rho^1) - \mathcal{N}(\rho^2)$ is well defined and $\rho \in \mathcal{B}_\alpha$. Let $\rho \equiv (\rho_1,\rho_2,\rho_3)$, and let ω_0^i, ω_1^i, ω^i, u_0^i, u_0^i, v_0^i, $i = 1,2$, be the quantities (37), (41), (34), (36), (42) and (29), computed from ρ^1 and ρ^2, respectively. Using the identity $ab - \tilde{a}\tilde{b} = (a-\tilde{a})b + \tilde{a}(b-\tilde{b})$ (distributive law) we find that

$$\begin{aligned}
\rho_1 &= \frac{1}{2\pi}\left(u_0^1*\omega_1^1+u_1^1*\omega_0^1+u_1^1*\omega_1^1\right) \\
&\quad -\frac{1}{2\pi}\left(u_0^2*\omega_1^2+u_1^2*\omega_0^2+u_1^2*\omega_1^2\right) \\
&= \frac{1}{2\pi}\left[\left(u_0^1-u_0^2\right)*\omega_1^1+u_0^2*\left(\omega_1^1-\omega_1^2\right)\right] \\
&\quad +\frac{1}{2\pi}\left[\left(u_1^1-u_0^2\right)*\omega_0^1+u_1^2*\left(\omega_0^1-\omega_0^2\right)\right] \\
&\quad +\frac{1}{2\pi}\left[\left(u_1^1-u_0^2\right)*\omega_1^1+u_1^2*\left(\omega_1^1-\omega_1^2\right)\right] ,
\end{aligned}$$

and similarly that

$$\rho_2 = -\frac{1}{4\pi}\left[\left(u_0^1 - u_0^2\right) * u_0^1 + u_0^2 * \left(u_0^1 - u_0^2\right)\right] ,$$
$$\rho_3 = \frac{1}{2\pi}\left[\left(v^1 - v^2\right) * \omega^1 + v^2 * \left(\omega^1 - \omega^2\right)\right] .$$

Therefore, and since the quantities ω_0^i, ω_1^i, ω^i, u_0^i, u_0^i, v_0^i, $i = 1, 2$ are linear (respectively affine) in ρ^1 and ρ^2, the bound (54) now follows *mutatis mutandis* as in the proof of Proposition 4 and Proposition 5.

12 Appendix IV

In this appendix we prove (61) and (62).

12.1 Asymptotic behavior of u

Let

$$U(k,t) = \left(-\omega_-^*(k) + \frac{1}{\Lambda_0}\int_1^t q_1(k,s)\ ds\right) e^{\Lambda_-(t-1)} .$$

Using the triangle inequality we get that

$$\begin{aligned} |u(k,t) - u_{as}(k,t)| \leq\ & \left|U(k,t) - c\ e^{-k^2 t}\right| \\ & + \left|u_{as}(k,t) - c\ e^{-k^2 t}\right| \\ & + |u_0(k,t) - U(k,t)| \\ & + |u_1(k,t)| . \end{aligned} \tag{98}$$

We bound each term in (98) separately. First, we have that

$$\lim_{t\to\infty} U(\frac{k}{t^{1/2}}, t) = c\ e^{-k^2} , \tag{99}$$

and furthermore that

$$\begin{aligned} |U(k,t)| &\leq \left(\varepsilon\mu_{\alpha+1}(k,1) + \varepsilon\mu_{\alpha+1}(k,1)\int_1^\infty \frac{1}{s^{3/2}} ds\right) e^{\Lambda_-(t-1)} \\ &\leq \varepsilon\mu_{\alpha+1}(k,t) , \end{aligned}$$

so that

$$\left|U(\frac{k}{t^{1/2}}, t)\right| \leq \varepsilon\mu_{\alpha+1}(k,1) . \tag{100}$$

From (99) and (100) it follows by the Lebesgue dominated convergence theorem that

$$\begin{aligned} & \lim_{t\to\infty} t^{1/2}\int_{\mathbf{R}} \left|U(k,t) - c\ e^{-k^2 t}\right|\ dk \\ =\ & \lim_{t\to\infty}\int_{\mathbf{R}} \left|U(\frac{k}{t^{1/2}}, t) - c\ e^{-k^2}\right|\ dk \\ =\ & 0 , \end{aligned}$$

as required. Next

$$\begin{aligned} \lim_{t\to\infty} t^{1/2}\int_{\mathbf{R}} \left|u_{as}(k,t) - c\ e^{-k^2 t}\right|\ dk &\leq \lim_{t\to\infty} t^{1/2}\int_{\mathbf{R}} (|d| + |b|)\, e^{-|k|t}\ dk \\ &= \lim_{t\to\infty} t^{1/2} 2\frac{|d| + |b|}{t} = 0 , \end{aligned}$$

as required. Next, using that $1-e^x \le -x$ for all $x \le 0$, we find that

$$
\begin{aligned}
|u_0(k,t)-U(k,t)| &\le \frac{1}{\Lambda_0}\int_1^t \left(e^{\Lambda_-(t-s)}-e^{\Lambda_-(t-1)}\right)|q_1(k,s)|\ ds \\
&\le \frac{1}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)}\left(1-e^{\Lambda_-(s-1)}\right)\frac{\varepsilon}{s^{3/2}}\mu_\alpha(k,s)\ ds \\
&\le -\frac{1}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)}\Lambda_-(s-1)\frac{\varepsilon}{s^{3/2}}\mu_\alpha(k,s)\ ds\ ,
\end{aligned}
$$

and therefore, since

$$
\begin{aligned}
-\frac{1}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}\Lambda_-(s-1)\frac{\varepsilon}{s^{3/2}}\mu_\alpha(k,s)\ ds &\le \varepsilon\mu_{\alpha+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}\Lambda_-\left(\frac{t-1}{t}\right)^2 t^{1/2} \\
&\le \frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t)\ ,
\end{aligned}
$$

and since

$$
\begin{aligned}
-\frac{1}{\Lambda_0}\int_{\frac{t+1}{2}}^t e^{\Lambda_-(t-s)}\Lambda_-(s-1)\frac{\varepsilon}{s^{3/2}}\mu_\alpha(k,s)\ ds &\le -\frac{\varepsilon}{t^{1/2}}\frac{1}{\Lambda_0}\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^t e^{\Lambda_-(t-s)}\Lambda_-\ ds \\
&\le \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t)\ ,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\lim_{t\to\infty} t^{1/2}\int_{\mathbf{R}} |u_0(k,t)-U(k,t)|\ dk &\le \lim_{t\to\infty} t^{1/2}\int_{\mathbf{R}} \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t)\ dk \\
&\le \lim_{t\to\infty} t^{1/2}\frac{\varepsilon}{t} = 0\ ,
\end{aligned}
$$

as required. Finally, we have that

$$
\begin{aligned}
\lim_{t\to\infty} t^{1/2}\int_{\mathbf{R}} |u_1(k,t)|\ dk &\le \lim_{t\to\infty} t^{1/2}\int_{\mathbf{R}} \left(\varepsilon\bar{\mu}_\alpha(k,t)+\frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t)\right)\ dk \\
&\le \lim_{t\to\infty} t^{1/2}\frac{\varepsilon}{t} = 0\ ,
\end{aligned}
$$

as required. This completes the proof of (61).

12.2 Asymptotic behavior of v

Let

$$
V(k,t) = V_0(k,t) + V_1(k,t)\ ,
$$

where

$$
\begin{aligned}
V_0(k,t) &= \left(\omega_-^*(k) - \frac{1}{\Lambda_0}\int_1^t q_1(k,s)\ ds\right)(-ik)e^{\Lambda_-(t-1)}\ , \\
V_1(k,t) &= \left(i\sigma(k)u_-^*(k) - \frac{1}{2}i\sigma(k)\int_1^t q_1(k,s)\ ds - \frac{1}{2}\int_1^t q_{0,0}(k,s)\ ds\right)e^{-|k|(t-1)}\ ,
\end{aligned}
$$

with u_-^* given by (32). Using the triangle inequality we get that

$$
\begin{aligned}
|v(k,t)-v_{as}(k,t)| \le\ & \left|V_0(k,t) - c\ ike^{-k^2t}\right| \\
&+ \left|V_1(k,t) - \left(d\ i\sigma(k)e^{-|k|t} - b\ e^{-|k|t}\right)\right| \\
&+ |v(k,t)-V(k,t)|\ . \qquad (101)
\end{aligned}
$$

We bound each term in (101) separately. First, we have that

$$
\lim_{t\to\infty} t^{1/2}V_0(\frac{k}{t^{1/2}},t) = c\ (-ik)e^{-k^2}\ , \qquad (102)
$$

and furthermore that

$$
\begin{aligned}
|V_0(k,t)| &\leq \left(\varepsilon\mu_{\alpha+1}(k,1) + \varepsilon\mu_{\alpha+1}(k,1)\int_1^\infty \frac{1}{s^{3/2}}ds\right)|k|\, e^{\Lambda_-(t-1)} \\
&\leq \varepsilon\,|k|\,\mu_{\alpha+1}(k,t)\ ,
\end{aligned}
$$

so that

$$
\left|t^{1/2}V_0(\frac{k}{t^{1/2}},t)\right| \leq \varepsilon\,|k|\,\mu_{\alpha+1}(k,1)\ . \tag{103}
$$

From (102) and (103) it follows by the Lebesgue dominated convergence theorem that

$$
\begin{aligned}
&\lim_{t\to\infty} t\int_{\mathbf{R}} \left|V_0(k,t) - c\,(-ik)e^{-k^2 t}\right|\ dk \\
=&\ \lim_{t\to\infty}\int_{\mathbf{R}}\left|t^{1/2}V_0(\frac{k}{t^{1/2}},t) - c\,(-ik)e^{-k^2}\right|\ dk \\
=&\ 0\ ,
\end{aligned}
$$

as required. Similarly, since $q_0(0,t) = q_{0,0}(0,t)$, for $t \geq 1$, we find that

$$
\lim_{t\to\infty} V_1(\frac{k}{t},t) = d\ i\sigma(k)e^{-|k|} - b\ e^{-|k|}\ , \tag{104}
$$

and furthermore that

$$
\begin{aligned}
|V_1(k,t)| &\leq \left(\varepsilon\mu_{\alpha+1}(k,1) + \varepsilon\mu_\alpha(k,1)\int_1^\infty \frac{1}{s^{3/2}}ds + \mu_{\alpha+1}(k,1)\,|k|\int_1^t \frac{1}{s^{1/2}}ds\right)e^{-|k|(t-1)}\ , \\
&\leq \varepsilon\mu_\alpha(k,1)e^{-|k|(t-1)} + t^{1/2}\mu_{\alpha+1}(k,1)e^{-|k|(t-1)}\,|k|\left(\frac{t-1}{t}\right) \\
&\leq \varepsilon\bar{\mu}_\alpha(k,t)\ ,
\end{aligned}
$$

so that

$$
\left|V_1(\frac{k}{t},t)\right| \leq \varepsilon\mu_\alpha(k,1)\ , \tag{105}
$$

for $t \geq 1$. From (104) and (105) it follows by the Lebesgue dominated convergence theorem that

$$
\begin{aligned}
&\lim_{t\to\infty} t\int_{\mathbf{R}}\left|V_1(k,t) - \left(d\ i\sigma(k)e^{-|k|t} - b\ e^{-|k|t}\right)\right|\ dk \\
=&\ \lim_{t\to\infty}\int_{\mathbf{R}}\left|V_1(\frac{k}{t},t) - \left(d\ i\sigma(k)e^{-|k|} - b\ e^{-|k|}\right)\right|\ dk \\
=&\ 0\ ,
\end{aligned}
$$

as required. Finally, for the last term in (101) we have the following Proposition:

Proposition 15 *Let v and V be as defined above. Then,*

$$
\begin{aligned}
|v(k,t) - V(k,t)| \leq&\ \frac{\varepsilon}{t^{2/3}}\mu_\alpha(k,t) + \frac{\varepsilon}{t^{1/3}}\mu_\alpha^{5/6}(k,t) \\
&+\frac{\varepsilon}{t^{1/2}}\mu_\alpha^{3/4}(k,t) + \frac{\varepsilon}{t^{1/2}}\bar{\mu}_\alpha(k,t)\ .
\end{aligned} \tag{106}
$$

See Appendix V for a proof.

From Proposition 15 it follows that

$$
\begin{aligned}
&\lim_{t\to\infty} t\int_{\mathbf{R}} |v(k,t) - V(k,t)|\ dk \\
\leq&\ \lim_{t\to\infty} t\int_{\mathbf{R}}\left(\frac{\varepsilon}{t^{2/3}}\mu_\alpha(k,t) + \frac{\varepsilon}{t^{1/3}}\mu_\alpha^{5/6}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_\alpha^{3/4}(k,t) + \frac{\varepsilon}{t^{1/2}}\bar{\mu}_\alpha(k,t)\right)\ dk \\
\leq&\ \lim_{t\to\infty} t\frac{\varepsilon}{t^{7/6}} = 0\ ,
\end{aligned}
$$

as required. This completes the proof of (62).

13 Appendix V

In this appendix we prove Proposition 15. The proof is rather lengthy and we therefore split it in several pieces. We start by proving some general bound.

13.1 Two inequalities

Proposition 16 *Let $\alpha \geq 0$. Then,*

$$\int_1^t \left(e^{-|k|(t-s)} - e^{-|k|(t-1)}\right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \leq \text{const.} \left(\frac{1}{t^{2/3}}\mu_\alpha(k,t) + \frac{1}{t^{1/3}}\mu_\alpha^{5/6}(k,t)\right) , \quad (107)$$

$$\int_t^\infty e^{|k|(t-s)} \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \leq \text{const.} \left(\frac{1}{t^{2/3}}\mu_\alpha(k,t) + \frac{1}{t^{1/2}}\mu_\alpha^{3/4}(k,t)\right) , \quad (108)$$

uniformly in $k \in \mathbf{R}$ and $t \geq 1$.

We first prove (107) for $1 \leq t \leq 2$. We have

$$\begin{aligned}
\int_1^t \left(e^{-|k|(t-s)} - e^{-|k|(t-1)}\right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds &\leq \varepsilon\mu_\alpha(k,1) \int_1^t \frac{ds}{s^{3/2}} \\
&\leq \varepsilon\mu_\alpha(k,1) \leq \frac{\varepsilon}{t^{2/3}}\mu_\alpha(k,t) ,
\end{aligned}$$

as required. For $t > 2$ we split the integral in (107) into two. For the first part we have

$$\begin{aligned}
&\int_1^{t-(t-1)^{5/6}} \left(e^{-|k|(t-s)} - e^{-|k|(t-1)}\right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
\leq\ & \varepsilon\mu_\alpha(k,1) \int_1^{t-(t-1)^{5/6}} \left(e^{-|k|(t-s)} - e^{-|k|(t-1)}\right) \frac{1}{s^{3/2}}\ ds \\
\leq\ & \varepsilon\mu_\alpha(k,1) \int_1^{t-(t-1)^{5/6}} e^{-|k|(t-s)} \left(1 - e^{|k|(s-1)}\right) \frac{1}{s^{3/2}}\ ds \\
\leq\ & \varepsilon\mu_\alpha(k,1) e^{-|k|(t-1)^{5/6}} \int_1^{t-(t-1)^{5/6}} (s-1)\,|k|\, \frac{1}{s^{3/2}}\ ds \\
\leq\ & \varepsilon t^{1/2} \mu_\alpha(k,1) e^{-|k|(t-1)^{5/6}}\, |k| \left(\frac{t-1}{t}\right)^2 \\
\leq\ & \frac{\varepsilon}{t^{1/3}} \mu_\alpha^{5/6}(k,t) ,
\end{aligned}$$

as required, and for the other part we get,

$$\begin{aligned}
&\int_{t-(t-1)^{5/6}}^t \left(e^{-|k|(t-s)} - e^{-|k|(t-1)}\right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
\leq\ & \frac{\varepsilon}{t^{3/2}} \mu_\alpha(k,t) \int_{t-(t-1)^{5/6}}^t ds \\
\leq\ & \frac{\varepsilon}{t^{3/2}} \mu_\alpha(k,t) t^{5/6} \leq \frac{\varepsilon}{t^{2/3}} \mu_\alpha(k,t) ,
\end{aligned}$$

as required. We now prove (108). Namely,

$$
\begin{aligned}
& \int_t^\infty e^{|k|(t-s)} \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
\leq \ & \varepsilon \mu_\alpha(k,t) \left(\int_t^{t+t^{3/4}} e^{|k|(t-s)} \frac{1}{s^{3/2}}\ ds + \int_{t+t^{3/4}}^\infty e^{|k|(t-s)} \frac{1}{s^{3/2}}\ ds \right) \\
\leq \ & \varepsilon \mu_\alpha(k,t) \left(\int_t^{t+t^{3/4}} \frac{1}{s^{3/2}}\ ds + e^{-|k|t^{3/4}} \int_{t+t^{3/4}}^\infty \frac{1}{s^{3/2}}\ ds \right) \\
\leq \ & \varepsilon \mu_\alpha(k,t) \left(\frac{1}{t^{3/4}} + \frac{1}{t^{1/2}} e^{-|k|t^{3/4}} \right) \\
\leq \ & \frac{\varepsilon}{t^{3/4}} \mu_\alpha(k,t) + \frac{\varepsilon}{t^{1/2}} \mu_\alpha^{3/4}(k,t) \\
\leq \ & \frac{\varepsilon}{t^{2/3}} \mu_\alpha(k,t) + \frac{\varepsilon}{t^{1/2}} \mu_\alpha^{3/4}(k,t)\ ,
\end{aligned}
$$

as required. This completes the proof of Proposition 16.

13.2 Proof of Proposition 15

Let $v_D = v - V$. Using the definitions we find that

$$
\begin{aligned}
v_D(k,t) \ = \ & \omega_1(k,t) \\
& + \frac{1}{\Lambda_0} \int_1^t \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)} \right) ikq_1(k,s)\ ds \\
& - \frac{1}{2} \int_1^t \left(e^{-|k|(t-s)} - e^{-|k|(t-1)} \right) (q_{0,0}(k,s) + i\sigma(k) q_1(k,s))\ ds \\
& + \frac{1}{2} \int_t^\infty e^{|k|(t-s)} (q_{0,0}(k,s) - i\sigma(k) q_1(k,s))\ ds \\
& - \frac{1}{2} \left(\int_1^t e^{-|k|(t-s)} ikq_{0,1}(k,s)\ ds - \int_t^\infty e^{|k|(t-s)} ikq_{0,1}(k,s)\ ds \right)\ .
\end{aligned}
\tag{109}
$$

We write $v_D = \sum_{i=1}^5 v_{D,i}$, with $v_{D,i}$ the i-th term in (109), and we bound each of the terms individually. The inequality (106) then follows using the triangle inequality.

Proposition 17 *For all $\alpha \geq 0$ we have the bounds*

$$
|v_{D,1}(k,t)| \leq \frac{\varepsilon}{t} \mu_\alpha(k,t)\ , \tag{110}
$$

$$
|v_{D,2}(k,t)| \leq \frac{\varepsilon}{t} \mu_\alpha(k,t)\ , \tag{111}
$$

$$
|v_{D,3}(k,t)| \leq \frac{\varepsilon}{t^{2/3}} \mu_\alpha(k,t) + \frac{\varepsilon}{t^{1/3}} \mu_\alpha^{5/6}(k,t)\ , \tag{112}
$$

$$
|v_{D,4}(k,t)| \leq \frac{\varepsilon}{t^{2/3}} \mu_\alpha(k,t) + \frac{\varepsilon}{t^{1/2}} \mu_\alpha^{3/4}(k,t)\ , \tag{113}
$$

$$
\begin{aligned}
|v_{D,5}(k,t)| \leq \ & \frac{\varepsilon}{t^{1/2}} \bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{2/3}} \mu_\alpha(k,t) \\
& + \frac{\varepsilon}{t^{1/3}} \mu_\alpha^{5/6}(k,t) + \frac{\varepsilon}{t^{1/2}} \mu_\alpha^{3/4}(k,t)\ ,
\end{aligned}
\tag{114}
$$

uniformly in $k \in \mathbf{R}$ and $t \geq 1$.

The bound on $v_{D,1}$ has already been established in (47). Next, to bound $v_{D,2}$, we split the integral into two parts. We have

$$\begin{aligned}
& \frac{1}{\Lambda_0}\int_1^{\frac{t+1}{2}} \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right)|k|\,|q_1(k,s)|\ ds \\
\leq\ & \varepsilon\mu_{\alpha+1/2}(k,1)\int_1^{\frac{t+1}{2}} \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right)|\Lambda_-|^{1/2}\frac{1}{s^{3/2}}\ ds \\
\leq\ & \varepsilon\mu_{\alpha+1/2}(k,1)\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}\left(1 - e^{\Lambda_-(s-1)}\right)|\Lambda_-|^{1/2}\ \frac{1}{s^{3/2}}\ ds \\
\leq\ & \varepsilon\mu_{\alpha+1/2}(k,1)e^{\Lambda_-\frac{t-1}{2}}\int_1^{\frac{t+1}{2}} (s-1)\,|\Lambda_-|^{3/2}\frac{1}{s^{3/2}}\ ds \\
\leq\ & \varepsilon t^{1/2}\mu_{\alpha+1/2}(k,1)e^{\Lambda_-\frac{t-1}{2}}\,|\Lambda_-|^{3/2}\left(\frac{t-1}{t}\right)^2 \\
\leq\ & \frac{\varepsilon}{t}\mu_{\alpha+1/2}(k,t)\ ,
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right)|k|\,|q_1(k,s)|\ ds \\
\leq\ & \varepsilon\frac{\Lambda_+^{1/2}}{\Lambda_0}\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\,|\Lambda_-|^{1/2}\,\frac{1}{s^{3/2}}\ ds \\
& +\varepsilon\frac{\Lambda_+^{1/2}}{\Lambda_0}\mu_\alpha(k,t)e^{\Lambda_-(t-1)}\,|\Lambda_-|^{1/2}\int_{\frac{t+1}{2}}^{t}\frac{1}{s^{3/2}}\ ds \\
\leq\ & \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\,|\Lambda_-|^{1/2}\ ds \\
& +\frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t)e^{\Lambda_-(t-1)}\,|\Lambda_-|^{1/2}\left(\frac{t-1}{t}\right) \\
\leq\ & \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^{t}\frac{1}{\sqrt{t-s}}\ ds + \frac{\varepsilon}{t}\mu_\alpha(k,t) \\
\leq\ & \frac{\varepsilon}{t}\mu_\alpha(k,t)\ ,
\end{aligned}$$

and (111) now follows using the triangle inequality. The bounds (112) and (113) on $v_{D,3}$ and $v_{D,4}$ are a consequence of Proposition 16, using that

$$|q_{0,0}(k,s) + i\sigma(k)q_1(k,s)| \leq \frac{\varepsilon}{s^{3/2}}\mu_\alpha(k,s)\ .$$

To complete the proof of Proposition 17 we still need to prove the bound (114) on $v_{D,5}$. This bound is somewhat tricky, since the dominant contributions to the two integrals defining $v_{D,5}$ compensate one another. A proof of this bound is the content of the final subsections of this paper.

13.3 Proof of inequality (114)

Using the definition of $v_{D,5}$ we find after integration by parts, that

$$\begin{aligned}
-i\sigma(k)v_{D,5}(k,t) &= \int_1^t e^{-|k|(t-s)}\,|k|\,q_{0,1}(k,s)\ ds - \int_t^\infty e^{|k|(t-s)}\,|k|\,q_{0,1}(k,s)\ ds \\
&= \left[e^{-|k|(t-s)}q_{0,1}(k,s)\right]_{s=1}^{s=t} - \int_1^t e^{-|k|(t-s)}\partial_s q_{0,1}(k,s)\ ds \\
&\quad + \left[e^{|k|(t-s)}q_{0,1}(k,s)\right]_{s=t}^{s=\infty} - \int_t^\infty e^{|k|(t-s)}\partial_s q_{0,1}(k,s)\ ds\ .
\end{aligned}$$

Therefore,

$$
\begin{aligned}
-i\sigma(k)v_{D,5}(k,t) &= q_{0,1}(k,t) - e^{-|k|(t-1)}q_{0,1}(k,1) - q_{0,1}(k,t) \\
&\quad -e^{-|k|(t-1)}\int_1^t \partial_s q_{0,1}(k,s)\ ds \\
&\quad -\int_1^t \left(e^{-|k|(t-s)} - e^{-|k|(t-1)}\right)\partial_s q_{0,1}(k,s)\ ds \\
&\quad -\int_t^\infty e^{|k|(t-s)}\partial_s q_{0,1}(k,s)\ ds\ ,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
v_{D,5}(k,t) &= -i\sigma(k)e^{-|k|(t-1)}q_{0,1}(k,t) \\
&\quad -i\sigma(k)\int_1^t \left(e^{-|k|(t-s)} - e^{-|k|(t-1)}\right)\partial_s q_{0,1}(k,s)\ ds \\
&\quad -i\sigma(k)\int_t^\infty e^{|k|(t-s)}\partial_s q_{0,1}(k,s)\ ds\ . \qquad (115)
\end{aligned}
$$

From the representation (115) of $v_{D,5}$ we get the inequality

$$
\begin{aligned}
|v_{D,5}(k,t)| &\le e^{-|k|(t-1)}\,|q_{0,1}(k,t)| \\
&\quad +\int_1^t \left(e^{-|k|(t-s)} - e^{-|k|(t-1)}\right)|\partial_s q_{0,1}(k,s)|\ ds \\
&\quad +\int_t^\infty e^{|k|(t-s)}\,|\partial_s q_{0,1}(k,s)|\ ds\ . \qquad (116)
\end{aligned}
$$

We now prove a bound for each of the terms in (116) separately. The bound (114) on $v_{D,5}$ then follows by the triangle inequality. For the first term in (116) we have that

$$
\begin{aligned}
e^{-|k|(t-1)}\,|q_{0,1}(k,t)| &\le e^{-|k|(t-1)}\frac{\varepsilon}{t^{1/2}}\mu_{\alpha+1}(k,t) \\
&\le \frac{\varepsilon}{t^{1/2}}\bar{\mu}_{\alpha+1}(k,t)\ ,
\end{aligned}
$$

as required, and the bounds on the second and the third term in (116) follow using Proposition 16, together with:

Proposition 18 *For all $\alpha \ge 0$ we have that*

$$
|\partial_t q_{0,1}(k,t)| \le \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)\ , \qquad (117)
$$

uniformly in $k \in \mathbf{R}$ and $t \ge 1$.

See the next subsection for a proof.

This completes the proof of inequality (114).

13.4 Proof of Proposition 18

By definition (40) of $q_{0,1}$ we have that

$$
\partial_t q_{0,1}(k,t) = -\frac{1}{2\pi}\left(u_0 * \partial_t u_0\right)(k,t)\ ,
$$

and for $\partial_t u_0$ we have:

Proposition 19 *For all $\alpha \ge 0$ we have*

$$
|\partial_t u_0(k,t)| \le \frac{\varepsilon}{t}\mu_\alpha(k,t)\ , \qquad (118)
$$

uniformly in $k \in \mathbf{R}$ and $t \ge 1$.

See the next subsection for a proof.

Using (96) we find from (48) and (118) that

$$|(u_0 * \partial_t u_0)(k,t)| \leq \frac{\varepsilon^2}{t^{3/2}} \mu_a(k,t) \ ,$$

and (117) follows. This completes the proof of Proposition 18.

13.5 Proof of Proposition 19

By definition (36) of u_0 we have that

$$\begin{aligned} \partial_t u_0(k,t) &= -\omega_-^*(k)\Lambda_- e^{\Lambda_-(t-1)} \\ &\quad + \frac{1}{\Lambda_0} q_1(k,t) \\ &\quad + \frac{1}{\Lambda_0} \int_1^t e^{\Lambda_-(t-s)} \Lambda_- q_1(k,s) \, ds \ . \end{aligned} \tag{119}$$

We write $\partial_t u_0 = \sum_{i=1}^3 \partial_t u_{0,i}$, with $\partial_t u_{0,i}$ the i-th term in (119), and we bound each of the terms separately. The inequality (118) then follows using the triangle inequality.

Proposition 20 *For all $\alpha \geq 0$ we have the bounds*

$$|\partial_t u_{0,1}(k,t)| \leq \frac{\varepsilon}{t} \mu_a(k,t) \ , \tag{120}$$

$$|\partial_t u_{0,2}(k,t)| \leq \frac{\varepsilon}{t^{3/2}} \mu_a(k,t) \ , \tag{121}$$

$$|\partial_t u_{0,3}(k,t)| \leq \frac{\varepsilon}{t} \mu_{a+1}(k,t) \ , \tag{122}$$

uniformly in $t \geq 1$ and $k \in \mathbf{R}$.

For $\partial_t u_{0,1}$ we have

$$\left| -\omega_-^*(k)\Lambda_- e^{\Lambda_-(t-1)} \right| \leq \varepsilon \mu_{a+1}(k,1) e^{\Lambda_-(t-1)} |\Lambda_-| \ ,$$

and (120) follows using Proposition 9. Next,

$$\left| \frac{1}{\Lambda_0} q_1(k,t) \right| \leq \frac{1}{\Lambda_0} \frac{\varepsilon}{t^{3/2}} \mu_a(k,t) \ ,$$

and (121) follows. Finally, splitting the integral defining $\partial_t u_{0,3}$ in two parts we find that

$$\begin{aligned} \left| \frac{1}{\Lambda_0} \int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)} \Lambda_- q_1(k,s) \, ds \right| &\leq \varepsilon \mu_{a+1}(k,1) e^{\Lambda_- \frac{t-1}{2}} |\Lambda_-| \int_1^{\frac{t+1}{2}} \frac{1}{s^{3/2}} ds \\ &\leq \varepsilon \mu_{a+1}(k,1) e^{\Lambda_- \frac{t-1}{2}} |\Lambda_-| \left(\frac{t-1}{t} \right) \\ &\leq \frac{\varepsilon}{t} \mu_{a+1}(k,t) \ , \end{aligned}$$

and that

$$\begin{aligned} \left| \frac{1}{\Lambda_0} \int_{\frac{t+1}{2}}^t e^{\Lambda_-(t-s)} |\Lambda_-| \, q_1(k,s) \, ds \right| &\leq \frac{\varepsilon}{t^{3/2}} \frac{1}{\Lambda_0} \mu_a(k,t) \int_{\frac{t+1}{2}}^t e^{\Lambda_-(t-s)} |\Lambda_-| \, ds \\ &\leq \frac{\varepsilon}{t^{3/2}} \frac{1}{\Lambda_0} \mu_a(k,t) \\ &\leq \frac{\varepsilon}{t} \mu_{a+1}(k,t) \ , \end{aligned}$$

and (122) follows using the triangle inequality. This completes the proof of Proposition 20 and Proposition 19.

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