On the vorticity of the Oseen problem in a half plane

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Abstract

We derive the equation for the vorticity of the incompressible Oseen problem in a half plane with homogeneous (no slip) boundary conditions. The resulting equation is a scalar Oseen equation with certain Dirichlet boundary conditions which are determined by the incompressibility condition and the boundary conditions of the original problem. We prove existence and uniqueness of solutions for this equation in function spaces that provide detailed information on the asymptotic behavior of the solution. We show that, in contrast to the Oseen problem in the whole space where the vorticity decays exponentially fast outside the wake region, the vorticity only decays algebraically in the present case. This algebraic decay is however faster than what one would expect for a generic problem, since the dominant volume and boundary contributions cancel each other as a consequence of the incompressibility and the no slip boundary conditions of the original problem.

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1 Introduction and main results

This is the first of a series of papers in which we develop the mathematical framework which is necessary for the precise computation of the hydrodynamic forces that act on a body that moves at constant speed parallel to a wall in an otherwise unbounded space filled with a fluid.

A very important practical application of such a situation is the description of the motion of bubbles rising in a liquid parallel to a nearby wall. Interesting recent experimental work is described in [12] and in [16]. Numerical studies can be found in [2], [5], [11], and [14]. The computation of hydrodynamic forces is reviewed in [10].

In what follows we consider the situation of a single bubble of fixed shape which rises with constant velocity in a regime of Reynolds numbers less than about fifty. The resulting fluid flow is then typically laminar. The Stokes equations provide a good quantitative description (forces determined within an error of one percent, say) only for Reynolds numbers less than one. For the larger Reynolds numbers under consideration the Navier-Stokes equations need to be solved. The vertical speed of the bubble depends on the drag, and the distance from the wall at which the bubble rises requires one to find the position relative to the wall where the transverse force is zero. Since at low Reynolds numbers the transverse forces are orders of magnitude smaller than the forces along the flow, this turns out to be a very delicate problem which needs to be solved numerically with the help of high precision computations. But, if done by brute force, such computations are excessively costly even with today's computers. In [3] we have developed techniques that lead for similar problems to an overall gain of computational efficiency of typically several orders of magnitude. See also [4] and [10]. These techniques use as an input a precise asymptotic description of the flow. In all cases considered so far the basis of such an analysis is a detailed description of an appropriate linear problem. The goal of the work that we start here is to extend this technique to the case of motions close to a wall.

In what follows we consider the two dimensional case. In a frame attached to the body the Navier-Stokes equations are

$$-\mathbf{u}\cdot\nabla\mathbf{u} - \partial_x\mathbf{u} + \Delta\mathbf{u} - \nabla p = 0 \;, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \;, \tag{1.2}$$

which have to be solved in the domain $\Omega = \mathbb{R}^2_+ \setminus \mathbf{B}$, subject to the boundary conditions

$$\mathbf{u}(x,0) = 0 \;, \qquad x \in \mathbb{R} \;, \tag{1.3}$$

$$\lim_{\mathbf{x}\to\infty} \mathbf{u}(\mathbf{x}) = 0 \;, \tag{1.4}$$

$$\mathbf{u}|_{\partial\mathbf{B}} = -\mathbf{e}_1 \;. \tag{1.5}$$

Here, $\mathbf{x} = (x,y)$, $\mathbb{R}^2_+ = \{(x,y)\in\mathbb{R}^2 \mid y > 0\}$, $\mathbf{B}\subset\mathbb{R}^2_+$ is a compact set with smooth boundary $\partial\mathbf{B}$, and $\mathbf{e}_1 = (1,0)$.

Based on preliminary numerical studies we expect that the relevant linear problem for the asymptotic analysis of (1.1)-(1.5) is given by the inhomogeneous stationary Oseen equation,

$$-\partial_x\mathbf{u} + \Delta\mathbf{u} - \nabla p = \mathbf{f} \;, \tag{1.6}$$

in the domain $\mathbb{R}^2_+$, subject to the incompressibility conditions (1.2), the boundary conditions (1.3) and (1.4), and with $\mathbf{f} = (f_1, f_2)$ a smooth vector field with compact support. Note that for $\mathbf{f}$ in appropriate function spaces the solution of equation (1.6) can also be used as a starting point for a proof of the existence of solutions of the Navier-Stokes equations (1.1) based on the contraction mapping principle. Solving the Oseen equation (1.6) turns out to be surprisingly complicated, since there exists no reflection principle like in the case of the Stokes equation (see for example [7]). We therefore discuss here in a first step the vorticity of the vector field $\mathbf{u}$, which is crucial for a detailed understanding of the asymptotic behavior of solutions of (1.6). See [1] for an early publication that stresses the importance of analyzing the vorticity. The reconstruction of the velocity field and the pressure, as well as the analysis of the original Navier-Stokes problem (1.1), is the content of upcoming publications.

Let $\mathbf{u} = (u, v)$ and let

$$\omega = -\partial_y u + \partial_x v \tag{1.7}$$

be the vorticity. If we take the curl of equation (1.6) we see that ω has to satisfy the equation

$$-\partial_x \omega + \Delta\omega = \varphi \ , \tag{1.8}$$

with

$$\varphi = -\partial_y f_1 + \partial_x f_2 \ . \tag{1.9}$$

As we will show in Section 2, it follows from the incompressibility condition (1.2) and the boundary condition (1.3), that we have to impose at $y = 0$ the Dirichlet boundary condition

$$\omega(x, 0) = \omega_0(x) \ , \tag{1.10}$$

with ω_0 a certain function which depends on the data φ. Let

$$\kappa = \sqrt{k^2 - ik} \ , \tag{1.11}$$

and

$$m_0(k, \eta) = -\frac{e^{-\kappa\eta} - e^{-|k|\eta}}{\kappa - |k|} \ . \tag{1.12}$$

Then, we find (see Section 2), that

$$\omega_0 = \mathcal{F}_{m_0}[\varphi] \ , \tag{1.13}$$

where $\mathcal{F}_m$ is the linear operator which, for a given function $m \colon \mathbb{R}^2_+ \to \mathbb{C}$, is formally defined by

$$\mathcal{F}_m[\varphi](x) = \int_0^\infty \left[\mathcal{F}^{-1}\left(m\ \mathcal{F}\varphi\right)\right](x, \eta)\ d\eta \ . \tag{1.14}$$

Here, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are, respectively, the Fourier transform with respect to the first variable, and its inverse. Explicitly, we use the following sign and normalization conventions for $\mathcal{F}$,

$$(\mathcal{F}\varphi)(k, y) = \int_{\mathbb{R}} e^{ikx} \varphi(x, y)\ dx \ , \tag{1.15}$$

and

$$\left[\mathcal{F}^{-1}\left(m\ \mathcal{F}\varphi\right)\right](x, \eta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} m(k, \eta)(\mathcal{F}\varphi)(k, \eta)\ dk \ . \tag{1.16}$$

The following two theorems summarize our main results.

Theorem 1 *For all functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2_+)$ the equation (1.8) subject to the boundary condition (1.10) has a solution $\omega \in \mathcal{C}^\infty(\mathbb{R}^2_+) \cap \mathcal{C}(\overline{\mathbb{R}^2_+})$. This solution is unique among functions decaying sufficiently rapidly at infinity.*

Theorem 2 *Let $f \in \mathcal{C}_0^\infty(\mathbb{R}_+^2)$, and let, for $i = 1, 2$, ω_i be the solution of equation (1.8) with boundary condition (1.10) and data $\varphi = \varphi_i$, where $\varphi_1 = \partial_x f$ and $\varphi_2 = \partial_y f$. Then, the functions $\tilde{\omega}_i$ defined by*

$$\widetilde{\omega}_i(x, y) = (1 + y)^{d_i - \varepsilon} \omega_i(x, y) \ ,$$

with $d_1 = 4$ and $d_2 = 3$, are in $L^p(\mathbb{R}_+^2)$, for all $0 < \varepsilon < 2$ and all $3/\varepsilon < p < \infty$.

The remainder of the paper is organized as follows. In Section 2 we discuss the equation for the vorticity and compute the function ω_0. In Section 3 we construct a solution for data with compact support and discuss the asymptotic behavior of solutions on a heuristic level, in order to motivate the choice of function spaces. Since, as has been mentioned earlier, the ultimate goal of the work that we start here is to solve and analyze in detail the Navier-Stokes equations in the above setting, we are interested in solving the Oseen problem and to study the properties of the mapping $\varphi \mapsto \omega[\varphi]$ in a well-designed functional framework. With this in mind we introduce in Section 4 appropriate function spaces, give precise formulations of the above theorems, formulate a third theorem which describes the behavior of solutions close to the boundary $y = 0$, and prove a uniqueness theorem. Section 5 contains the main technical Lemmas on which the proofs of the theorems in Section 6 to Section 8 are based. As we will see at the end of Section 4.3 and at the end of Section 4.2, respectively, Theorem 1 is a consequence of Lemma 4 (existence) and Theorem 15 (uniqueness), and Theorem 1 is a consequence of Theorem 12, Theorem 13, and Theorem 14. Appendices A and B, finally, contain various technical details.

Notation

Throughout the paper we use the following conventions. First, the subsets $\mathbb{R}_+^2$ and $\mathbb{R}_{++}^2$ are defined by $\mathbb{R}_+^2 = \mathbb{R} \times (0, \infty)$, and $\mathbb{R}_{++}^2 = \mathbb{R} \times (1, \infty)$. For most function spaces we use the names that are classical. Notations that are particular to the present work are introduced in Section 4. Furthermore, given $\Omega \subset \mathbb{R}^2$, we define $\mathcal{C}_0^\infty(\Omega)$ to be the the set of smooth functions with compact support. Partial differentiations with respect to the first and second variable are denoted by ∂_x and ∂_y, respectively, and $L^p(\Omega)$ and $L^p_{loc}(\Omega)$ stand for the usual Lebesgue spaces. Also, to simplify notation, we use the same expressions for vector and scalar quantities. The norm on $L^p(\Omega)$ is denoted by $|\cdot|_p$, and whenever necessary in order to avoid confusions, we indicate Ω either with a subscript (i.e., $|\cdot|_{p,\Omega}$), or if more details on the space are needed by $|\cdot|_{L^p(\Omega)}$. Throughout the paper, $\mathcal{F}$ denotes the Fourier transform with respect to the variable x. Finally, we will often write $\hat{\varphi}$ instead of $\mathcal{F}\varphi$ for the Fourier transform of a function φ.

2 The vorticity boundary condition

In this section we derive the expression (1.13) for ω_0 on a formal level. Taking partial derivatives of equation (1.7) with respect to x and y, and using the incompressibility condition (1.2) and the boundary condition (1.3), one finds that the components of the vector field $\mathbf{u} = (u, v)$ have to satisfy the equations

$$\begin{aligned} \Delta u &= -\partial_y \omega \ , \\ u(x, 0) &= 0 \ , \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \Delta v &= \partial_x \omega \ , \\ v(x, 0) &= 0 \ . \end{aligned} \tag{2.2}$$

Once the equation (1.8) for the vorticity is solved we can therefore *a priori* reconstruct the vector field $\mathbf{u} = (u, v)$ using (2.1) and (2.2). But if we solve, for a given function ω, the equations (2.1) and (2.2) for u and v, then, in general, the corresponding vector field $\mathbf{u} = (u, v)$ will not be divergence free. Indeed, unless the vorticity ω is chosen in a special way, the equations (2.1), and (2.2) only imply that

$$\Delta(\partial_x u + \partial_y v)(x, y) = 0 \ , \tag{2.3}$$

for $(x, y) \in \mathbb{R}_+^2$, and not necessarily (1.2). However, as we now show, for functions ω that are solution of (1.8), we can enforce the divergence freeness (1.2) by an appropriate choice of the boundary condition in

(1.10). First, since the Laplace equation in the half plane is well-posed (see for example [6]), (1.2) follows from (2.3) provided that

$$\partial_x u(x,0) + \partial_y v(x,0) = 0 \ . \tag{2.4}$$

Next, since $u(x,0) = 0$ by (2.1), it follows that $\partial_x u(x,0) = 0$, and (2.4) therefore reduces to $\partial_y v(x,0) = 0$. Taking the Fourier transform with respect to the variable x we get from (1.8) and (1.10) the equation

$$\partial_y^2 \hat{\omega}(k,y) - \kappa^2 \hat{\omega}(k,y) = \hat{\varphi}(k,y) \ , \tag{2.5}$$

$$\hat{\omega}(k,0) = \hat{\omega}_0(k) \ , \tag{2.6}$$

with, $\hat{\omega} = \mathcal{F}\omega$, $\hat{\omega}_0 = \mathcal{F}\omega_0$, and $\hat{\varphi} = \mathcal{F}\varphi$, and where κ is defined in (1.11). Similarly, we get from (2.2) after Fourier transform with respect to x the equation

$$\partial_y^2 \hat{v}(k,y) - k^2 \hat{v}(k,y) = -ik\hat{\omega}(k,y) \ , \tag{2.7}$$

$$\hat{v}(k,0) = 0 \ . \tag{2.8}$$

The solution of (2.7,2.8) is

$$\hat{v}(k,y) = -\frac{1}{2\,|k|} \int_0^\infty \left(e^{-|k||y-\eta|} - e^{-|k|(y+\eta)} \right) ik\hat{\omega}(k,\eta)\ d\eta \ , \tag{2.9}$$

from which we get that $\partial_y v(x,0) = 0$ if and only if

$$\partial_y \hat{v}(k,0) = -ik \int_0^\infty e^{-|k|\eta} \hat{\omega}(k,\eta)\ d\eta = 0 \ . \tag{2.10}$$

Therefore, the vector field $\mathbf{u} = (u,v)$ is divergence free if and only if for all $k \in \mathbb{R} \setminus \{0\}$,

$$\int_0^\infty e^{-|k|\eta} \hat{\omega}(k,\eta)\ d\eta = 0 \ . \tag{2.11}$$

We now show that (2.11) implies (1.13). Equation (2.11) follows from

$$\int_0^\infty \left(e^{-\kappa\eta} - e^{-|k|\eta} \right) \hat{\omega}(k,\eta)\ d\eta = \int_0^\infty e^{-\kappa\eta} \hat{\omega}(k,\eta)\ d\eta \ . \tag{2.12}$$

Multiplying (2.12) with κ^2 and using (2.5) we find that (2.12) is equivalent to

$$\int_0^\infty \left(e^{-\kappa\eta} - e^{-|k|\eta} \right) \left(\partial_\eta^2 \hat{\omega}(k,\eta) - \hat{\varphi}(k,\eta) \right)\ d\eta = \int_0^\infty e^{-\kappa\eta} \kappa^2 \hat{\omega}(k,\eta)\ d\eta \ . \tag{2.13}$$

We now integrate in (2.13) the term containing $\partial_\eta^2 \hat{\omega}$ twice by parts. The boundary term of the first integration by parts is zero. The integral that one obtains after the second integration by parts contains a term which is equal to zero if (2.11) is satisfied. The other term simplifies with the right hand side in (2.13), and one obtains from the remaining boundary term the following equation for $\hat{\omega}_0$,

$$\hat{\omega}_0(k) = \int_0^\infty m_0(k,\eta) \hat{\varphi}(k,\eta)\ d\eta \ , \tag{2.14}$$

with m_0 as defined in (1.12), and therefore

$$\begin{aligned} (\mathcal{F}^{-1}\hat{\omega}_0)(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} \hat{\omega}_0(k)\ dk \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} dk\ e^{-ikx} \int_0^\infty dy\ m_0(k,y) \hat{\varphi}(k,y) \ . \end{aligned} \tag{2.15}$$

For functions φ of compact support we can exchange the integrals in (2.15) and we get (1.13). This completes the formal construction of ω_0.

3 Solution for data with compact support

For the case where the domain is $\mathbb{R}^2$ instead of $\mathbb{R}^2_+$ the unique solution $\tilde{\omega}$ of the scalar Oseen equation (1.8) decaying sufficiently rapidly at infinity is well known. It is given in terms of the Green's function $\mathcal{K}$, which is defined for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ in terms of the modified Bessel function K_0 of the second kind of order zero. Namely,

$$\mathcal{K}(x,y) = -\frac{e^{\frac{x}{2}}}{2\pi} K_0(\frac{r}{2}) \ , \tag{3.1}$$

with

$$r = \sqrt{x^2 + y^2} \ , \tag{3.2}$$

and

$$\tilde{\omega}(x,y) = (\mathcal{K} * \varphi)(x,y) = \int_{\mathbb{R}^2} \mathcal{K}(x - x_0, y - y_0) \varphi(x_0, y_0) \ dx_0 dy_0 \ . \tag{3.3}$$

See for example [8], [9]. Properties of the Green's function $\mathcal{K}$ that are needed in subsequent sections are summarized in Appendix A, which also contains a proof of the following proposition.

Proposition 3 *The function $\mathcal{K}$ defined in (3.1) has the following properties:*

i) $\mathcal{K} \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$,

ii) $\mathcal{K}$ *is even as a function of* y ,

iii) $(\mathcal{K}, \nabla\mathcal{K}, \Delta\mathcal{K}) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

3.1 Construction of a solution ω

The solution ω of the Oseen problem (1.8) in the domain $\mathbb{R}^2_+$ can also be written in terms of the Green's function $\mathcal{K}$ in a standard way. It is the sum of two terms, a "volume term" obtained by the reflection principle which solves the inhomogeneous problem but is equal to zero at $y = 0$, and a "boundary term" which adds a solution of the homogeneous problem satisfying the boundary condition (1.10). Namely,

$$\omega = \omega_V + \omega_B \ , \tag{3.4}$$

with $\omega_V = T_V\{\mathcal{K}\}[\varphi]$ and with $\omega_B = T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]]$, where

$$T_V\{\mathcal{K}\}[\varphi](x,y) = \int_{\mathbb{R}^2_+} (\mathcal{K}(x - x_0, y - y_0) - \mathcal{K}(x - x_0, y + y_0)) \ \varphi(x_0, y_0) \ dx_0 dy_0 \ , \tag{3.5}$$

$$T_B\{\mathcal{K}\}[g](x,y) = \int_{\mathbb{R}} \mathcal{K}_1(x_0, y) \, g(x - x_0) \ dx_0 \ , \tag{3.6}$$

where $\mathcal{K}_1 = 2\partial_y \mathcal{K}$. From (3.1) one gets for $\mathcal{K}_1$ for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ the following explicit expression,

$$\mathcal{K}_1(x,y) = \frac{y e^{\frac{x}{2}}}{2\pi r} K_1(\frac{r}{2}) \ , \tag{3.7}$$

with r as defined in (3.2) and with K_1 the modified Bessel function of the second kind of order one.

The following Lemma shows that for smooth data with compact support the function ω is well defined and solves the Oseen equation (1.8) with boundary conditions (1.10).

Lemma 4 *Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2_+)$. Let ω_0 be as defined in (1.13) and let ω be as defined in (3.4). Then we have:*

(i) $\omega_0 \in \mathcal{C}^\infty(\mathbb{R})$, *and ω_0 and all its derivatives converge to zero at infinity.*

(ii) $\omega \in \mathcal{C}^\infty(\mathbb{R}^2_+) \cap \mathcal{C}(\overline{\mathbb{R}^2_+})$.

(iii) ω *is a solution of (1.8), (1.10).*

Proof. The proof of this lemma follows standard ideas. We first show *i*). Since $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2_+)$, $\varphi(x,y)$ is in particular in $\mathcal{C}_0^\infty(\mathbb{R})$ as a function of x and of compact support as a function of y. Its Fourier transform with respect to the first variable, $\hat{\varphi}(k,y)$, decreases therefore faster than any polynomial in k, uniformly in y, and so does $m(k,y)\hat{\varphi}(k,y)$. Therefore, by the Riemann-Lebesgue lemma, the function $\omega_0 = \mathcal{F}_{m_0}[\varphi]$ as well as all its derivatives are continuous and converge to zero at infinity. This completes the proof of *i*). We now prove *ii*), and *iii*) with appropriate boundary conditions, separately for ω_V and ω_B. The result then follows by the linearity of the equations. We first discuss the volume term ω_V. Instead of (3.5), we can equivalently write ω_V as a convolution product over $\mathbb{R}^2$. Namely, if we extend the function φ from $\mathbb{R}^2_+$ to $\mathbb{R}^2$ by anti-symmetry, *i.e.*, if we define $\widetilde{\varphi}$ by

$$\widetilde{\varphi}(x,y) = \begin{cases} \varphi(x,y) & \text{if } y > 0 \ , \\ -\varphi(x,y) & \text{if } y < 0 \ , \end{cases} \tag{3.8}$$

then we have, for $(x,y) \in \mathbb{R}^2_+$, that $\omega_V = \mathcal{K} * \widetilde{\varphi}$. Now, since by Proposition 3 $\mathcal{K} \in L^1_{\text{loc}}(\mathbb{R}^2)$, and since $\widetilde{\varphi}$ is smooth, and moreover $\mathcal{K}$ is an even function of y and $\widetilde{\varphi}$ is by definition an odd function of y it follows that ω_V extends to an odd function in $\mathcal{C}^\infty(\mathbb{R}^2)$. This in turn implies in particular that $\omega_V \in \mathcal{C}^\infty(\overline{\mathbb{R}^2_+})$, and that $\omega_V(x,0) = 0$. Similarly, since by Proposition 3 $\partial_x \mathcal{K} \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $\Delta\mathcal{K} \in L^1_{\text{loc}}(\mathbb{R}^2)$, it follows that $(-\partial_x + \Delta)\omega_V = ((-\partial_x + \Delta)\mathcal{K}) * \varphi$, which implies that $(-\partial_x + \Delta)\omega_V(x,y) = \widetilde{\varphi}(x,y)$ for all $(x,y) \in \mathbb{R}^2$, and in particular that $(-\partial_x + \Delta)\omega_V(x,y) = \varphi(x,y)$ for $(x,y) \in \mathbb{R}^2_+$. This completes the proof of *ii*) and of *iii*) with homogeneous Dirichlet boundary conditions for the volume term. We now discuss the boundary term. Standard results on parameter dependent convolutions imply that $\omega_B \in \mathcal{C}^\infty(\mathbb{R}^2_+)$ (see Proposition 39 for the uniform domination argument), and furthermore that for $(x,y) \in \mathbb{R}^2_+$,

$$(-\partial_x + \Delta)\,\omega_B(x,y) = \left(\partial_y \left[(-\partial_x + \Delta)\,\mathcal{K}\right] *_x \omega_0\right)(x,y) \ ,$$

where $*_x$ means convolution with respect to the first variable. Since $(-\partial_x + \Delta)\,\mathcal{K}(x,y) = 0$ for (x,y) in $\mathbb{R}^2 \setminus \{(0,0)\}$, we get in particular that $(-\partial_x + \Delta)\,\omega_B = 0$ for (x,y) in $\mathbb{R}^2_+$. We still need to prove that ω_B is continuous at the boundary, *i.e.*, that $\omega_B \in \mathcal{C}^\infty(\overline{\mathbb{R}^2_+})$ and that $\lim_{y\to 0} \omega_B(x,y) = \omega_0(x)$ for all $x \in \mathbb{R}$. Since $K_1(z) = 1/z + o(1)$, asymptotically as $z \to 0$, and $K_1(z) = O(e^{-z}/\sqrt{z})$ as $z \to +\infty$, we find that there is a constant C_1, such that for $r < 1$ and $y \in (0,1)$,

$$|\mathcal{K}_1(x,y) - \frac{y}{\pi r^2}| \leq C_1 \frac{y}{r} \leq C_1 \frac{y}{r^{3/2}} \ , \tag{3.9}$$

and that there is a non-negative decreasing function C_2, such that, for arbitrary $\varepsilon > 0$, $r > \varepsilon$, and $y \in (0,1)$,

$$|\mathcal{K}_1(x,y)| \leq y \frac{C_2(\varepsilon)}{r^{\frac{3}{2}}} \ . \tag{3.10}$$

From (3.9) and (3.10) it follows in particular that for arbitrary $r > 0$ and $y \in (0,1)$,

$$|\mathcal{K}_1(x,y) - \frac{y}{\pi r^2}| \leq \max\{C_1, C_2(1) + 1/\pi\} \frac{y}{r^{3/2}} \ . \tag{3.11}$$

The function $\mathcal{K}_1(\cdot,y)$ is positive and infinitely differentiable for all $y \in (0,1)$. Moreover, (3.11) implies that

$$\lim_{y\to 0} \int_{\mathbb{R}} \mathcal{K}_1(x,y)\ dx = 1 \ , \tag{3.12}$$

since

$$\lim_{y\to 0} \max\{C_1, C_2(1) + 1/\pi\} \int_{\mathbb{R}} \frac{y}{r^{3/2}}\ dx = 0 \ , \tag{3.13}$$

and (3.10) implies that, for arbitrary $\varepsilon > 0$,

$$\lim_{y\to 0} \int_{|x|>\varepsilon} |\mathcal{K}_1(x,y)|\ dx \leq C_2(\varepsilon) \lim_{y\to 0} \int_{\mathbb{R}} \frac{y}{r^{3/2}}\ dx = 0 \ . \tag{3.14}$$

Consequently, $(\mathcal{K}_1(\cdot,y))_{y\in(0,1)}$ is a regularizing sequence, and therefore, since $\omega_0 = \mathcal{F}_{m_0}[\varphi]$ is bounded together with its first derivative we have that $\lim_{y\to 0} \omega_B[\varphi](\cdot,y) = \mathcal{F}_{m_0}[\varphi]$ in $(\mathcal{C}(\mathbb{R}), \|\cdot\|_\infty)$. This completes the proof of Lemma 4. ■

We close this section with a proof that the boundary term $T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]]$ can be written alternatively in terms of Fourier transforms.

Lemma 5 *Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+^2)$ and let κ be as defined in (1.11). Define $E\colon \mathbb{R}_+^2 \to \mathbb{C}$ by*

$$E(k,y) = e^{-\kappa y} \ , \tag{3.15}$$

and let $T_B\{\mathcal{K}\}$ be as defined in (3.6). Then,

$$E \cdot \mathcal{F}[\mathcal{F}_{m_0}[\varphi]] \in L^1(\mathbb{R}) \ , \tag{3.16}$$

and

$$T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]] = \mathcal{F}^{-1}\left[E \cdot \mathcal{F}[\mathcal{F}_{m_0}[\varphi]]\ \right] \ . \tag{3.17}$$

Proof. By Proposition 39 $\partial_y\mathcal{K}(\cdot,y) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for all $y > 0$. Moreover, $\mathcal{F}_{m_0}[\varphi] \in L^2(\mathbb{R})$ by arguments similar to the ones given at the beginning of the proof of Lemma 4. Therefore, $T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]]$ is the convolution with respect to the variable x of $\partial_y\mathcal{K}(\cdot,y) \in L^1(\mathbb{R})$ with $\mathcal{F}_{m_0}[\varphi] \in L^2(\mathbb{R})$, and therefore $T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]](\cdot,y) \in L^2(\mathbb{R})$, for all $y > 0$. We can therefore take the Fourier transform of equation (3.6) and we get that

$$\mathcal{F}[T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]](\cdot,y)] = \mathcal{F}[\partial_y\mathcal{K}(\cdot,y)] \cdot \mathcal{F}[\mathcal{F}_{m_0}[\varphi]] \ . \tag{3.18}$$

From (3.18) it follows that $\mathcal{F}[T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]](\cdot,y)] \in L^1(\mathbb{R})$. Finally, an explicit computation shows that

$$\mathcal{F}[\partial_y\mathcal{K}(\cdot,y)](k) = E(k,y) \ . \tag{3.19}$$

This completes the proof of Lemma 5. ■

Explicitly we have for (3.17) the following expression,

$$T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]](x,y) = \mathcal{F}^{-1}\left[\int_0^\infty m_B(\cdot,\eta,y)\ \mathcal{F}[\varphi](\cdot,\eta)\ d\eta\right](x) \ . \tag{3.20}$$

where, for all $(k,\eta,y) \in \mathbb{R} \times (0,\infty)^2$, m_B is defined by

$$m_B(k,\eta,y) = e^{-\kappa y} m_0(k,\eta) \ , \tag{3.21}$$

with m_0 as defined in (1.12). For functions φ of compact support we can exchange the integrals in (3.20) with the integral of the Fourier transform, so that (3.20) can also be written as,

$$T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]](x,y) = \int_0^\infty \mathcal{F}^{-1}\left[m_B(\cdot,\eta,y)\ \mathcal{F}[\varphi](\cdot,\eta)\right](x)\ d\eta \ . \tag{3.22}$$

3.2 Expected asymptotic behavior

In order to motivate the choice of function spaces in the next section we briefly discuss the expected asymptotic behavior of solutions on a heuristic level. For the Navier-Stokes equation as well as for the Oseen equation we expect the formation of a parabolic wake in the region downstream of the support of f. Far enough downstream, this wake will interact with the border at $y = 0$ which will change its behavior when compared to the case without boundary. The horizontal component u of the vector field still satisfies asymptotically the heat equation $-\partial_x u + \partial_y^2 u = 0$ (see for example [10]), but with y in $\mathbb{R}_+$ rather than in $\mathbb{R}$, and with Dirichlet boundary conditions at $y = 0$. One therefore expects that asymptotically, as $x \to \infty$ for fixed $z = y/x^{1/2}$

$$u(x,y) \approx \partial_y\mathcal{K}(x,y) \approx \text{const}\ .\frac{1}{x} z e^{-\frac{z^2}{4}} = \text{const}\ .\frac{1}{y^2} z^3 e^{-\frac{z^2}{4}} \ . \tag{3.23}$$

From (3.23) one concludes using the incompressibility condition, that the vertical component v is negligible asymptotically when compared with u, and concludes that the vorticity ω should behave asymptotically, as $x \to \infty$ for fixed $z = y/x^{1/2}$, like $\partial_y u$, *i.e.*,

$$\omega(x,y) \approx \partial_y^2\mathcal{K}(x,y) \approx \text{const}\ .\frac{1}{x^{3/2}} e^{-\frac{z^2}{4}} = \text{const}\ .\frac{1}{y^3} e^{-\frac{z^2}{4}} \ . \tag{3.24}$$

The asymptotic term in (3.24) corresponds to a "volume term" in the terminology of Section 3. Outside the wake region such a term decays exponentially fast. As the following formal asymptotic expansion shows there is however a second order term (a "boundary term" in the terminology of Section 3), with a completely different behavior, and it is this term which dominates the large distance behavior of the vorticity outside the wake region, and in particular as $y \to \infty$ for fixed values of x. Namely, if we plug the Ansatz

$$\omega(x, y) = \frac{1}{y^3} f(x/y^2) + \frac{1}{y^4} g(x/y^2) + \dots \ , \tag{3.25}$$

into equation (1.8) and expand in inverse powers of y at fixed values of $\zeta = x/y^2$, we get for f and g the ordinary differential equations

$$4\zeta^2 f''(\zeta) + (18\zeta - 1) f'(\zeta) + 12 f(\zeta) = 0 \ , \tag{3.26}$$

$$4z^2 g''(\zeta) + (22\zeta - 1) g'(\zeta) + 20 g(\zeta) = 0 \ . \tag{3.27}$$

The equations (3.26) and (3.27) have two parameter families of solutions, but, as we will see later on, the relevant solution of (3.26) is

$$f(\zeta) = \begin{cases} c_1 \frac{1-2\zeta}{\zeta^{5/2}} e^{-\frac{1}{4\zeta}} & \zeta > 0 \\ 0 & \zeta \leq 0 \end{cases} \tag{3.28}$$

with $c_1 \neq 0$. The second solution of (3.26) is absent from the asymptotics of ω due to the no slip boundary conditions and the incompressibility condition of the original problem. The relevant solution for (3.27) is

$$g(\zeta) = \begin{cases} c_2 \dfrac{1}{96} \left(\pi^{1/2} \dfrac{1 - 6\zeta}{\zeta^{7/2}} e^{-\frac{1}{4\zeta}} \operatorname{erfi}(\dfrac{1}{2\zeta^{1/2}}) + \dfrac{8}{\zeta^2} - \dfrac{2}{4\zeta^3} \right) + c_3 \dfrac{1 - 6\zeta}{\zeta^{7/2}} e^{-\frac{1}{4\zeta}} & \zeta > 0 \\ c_2 & \zeta = 0 \\ c_2 \dfrac{1}{96} \left(\pi^{1/2} \dfrac{1 - 6\zeta}{(-\zeta)^{7/2}} e^{-\frac{1}{4\zeta}} \operatorname{erfc}(\dfrac{1}{2(-\zeta)^{1/2}}) + \dfrac{8}{\zeta^2} - \dfrac{2}{\zeta^3} \right) & \zeta < 0 \end{cases} \tag{3.29}$$

with $c_2 \neq 0$, with $\operatorname{erfi}(x) = \int_0^x e^{t^2} dt$, and with $\operatorname{erfc}(x) = \int_x^\infty e^{-t^2} dt$. Note that the functions f and g are continuous at $\zeta = 0$, that $f(\zeta) \approx \zeta^{-3/2}$ as $\zeta \to +\infty$ and that $g(\zeta) \approx \zeta^{-2}$ as $\zeta \to \pm\infty$.

From (3.28) and (3.29) it is now easy to see that ω behaves within the wake, *i.e.*, for $x/y^2 = \zeta =$ const., asymptotically like $f(\zeta)/y^3$, whereas outside the wake, and in particular for x fixed and y going to infinity, ω behaves asymptotically like $g(0)/y^4 = c_2/y^4$. This implies in particular also that our results in Theorem 2 are optimal for the norms under consideration.

4 Formulation of results in function spaces

The formal discussion at the end of Section 3 motivates the introduction of the following weighted Sobolev spaces.

Definition 6 *Let $p \geq 1$, $\sigma \geq 0$, and let $\Omega = \mathbb{R}^2$, $\mathbb{R}^2_+$, or $\mathbb{R}^2_{++}$. Then, we define the sets $L^p_\sigma(\Omega)$ and $\dot{W}^{1,p}_\sigma(\Omega)$ by*

$$L^p_\sigma(\Omega) = \left\{ f \in L^p(\Omega) \mid |f|_{L^p_\sigma(\Omega)} = \int_\Omega [(1 + |y|)^\sigma |f(x, y)|]^p \, dxdy < \infty \right\} \ , \tag{4.1}$$

where $L^p(\Omega)$ is the usual Lebesgue space, and where $\dot{W}^{1,p}_\sigma(\Omega)$ is the completion of $\mathcal{C}^\infty_0(\overline{\Omega})$ with respect to the norm

$$|f|_{\dot{W}^{1,p}_\sigma(\Omega)} = |\nabla f|_{L^p_\sigma(\Omega)} \ . \tag{4.2}$$

The following remarks are elementary:

Remark 7 *(i) Given $f \in L^p(\Omega)$, then $f \in L^p_\sigma(\Omega)$ if and only if there exists $\widetilde{f} \in L^p(\Omega)$ for which $f(x, y) = (1 + |y|)^{-\sigma} \widetilde{f}(x, y)$ for all $(x, y) \in \Omega$.*

(ii) For all $p \geq 1$ and $\sigma \geq 0$ the sets $L^p_\sigma(\Omega)$ and $\dot{W}^{1,p}_\sigma(\Omega)$ are Banach spaces when endowed with their respective norms $|\cdot|_{L^p_\sigma(\Omega)}$ and $|\cdot|_{\dot{W}^{1,p}_\sigma(\Omega)}$.

(iii) *The set* $\mathcal{C}_0^\infty(\overline{\Omega})$ *is dense in* $L^p_\sigma(\Omega)$ *and* $\dot{W}^{1,p}_\sigma(\Omega)$.

(iv) *For all* $0 \le \sigma < \sigma'$ *the imbeddings* $L^p_{\sigma'}(\Omega) \subset L^p_\sigma(\Omega)$ *and* $\dot{W}^{1,p}_{\sigma'}(\Omega) \subset \dot{W}^{1,p}_\sigma(\Omega)$ *are continuous.*

As has already been mentioned in the introduction we use the notation $|\cdot|_{p,\sigma}$ as a shorthand for $|\cdot|_{L^p_\sigma(\Omega)}$, whenever no confusion concerning the domain Ω under consideration is possible. We should emphasize that in the notation $|\cdot|_{p,\sigma}$ the first subscript stands for the exponent of a Lebesgue space $L^p(\Omega)$ and the second one for the exponent of the weight function $(1+|y|)^\sigma$. We note that no confusion arises with the standard notation for Sobolev spaces, since we use here only the homogeneous Sobolev spaces $\dot{W}^{1,p}_0$, and for these spaces we always denote the norm of a function f by $|\nabla f|_p$.

4.1 The space $\dot{W}^{1,p}_\sigma(\mathbb{R}^2_+)$

In this section we analyze the basic properties of $\dot{W}^{1,p}_\sigma(\mathbb{R}^2_+)$. The goal here is not necessarily to prove optimal results, concerning the imbeddings in particular, but rather to establish with as little effort as possible those results which are needed in later sections. For the case $\Omega = \mathbb{R}^2_+$, a first important aspect of the above spaces is that integration with respect to y is well behaved in the following sense:

Lemma 8 *Let* $p \in [1,\infty)$. *Then, the following imbeddings are continuous:*

(i) $\dot{W}^{1,p}_\sigma(\mathbb{R}^2_+) \subset L^p(\mathbb{R}\times(0,1))$, *for all* $\sigma > 1 - 1/p$.

(ii) $\dot{W}^{1,p}_\sigma(\mathbb{R}^2_+) \subset L^p_{\sigma'}(\mathbb{R}^2_+)$, *for all* $\sigma > 1$ *and* $0 \le \sigma' < \sigma - 1$.

Proof. We first prove (i). Given $f \in \mathcal{C}_0^\infty(\overline{\mathbb{R}^2_+})$, we set $\varphi = -\partial_y f$ so that

$$f(x,y) = \int_y^\infty \varphi(x,z)\, dz \ , \tag{4.3}$$

since φ is also a function with compact support. Denoting by $q = (1-1/p)^{-1}$ the conjugate exponent of p, we get, for all $\alpha > 0$, that

$$|f(x,y)|^p \le \left(\int_0^\infty \frac{1}{(1+z)^{\alpha q}}\right)^{\frac{p}{q}} \int_y^\infty |(1+z)^\alpha \varphi(x,z)|^p\, dz \ . \tag{4.4}$$

The first integral in the right-hand side is finite provided $\alpha > 1/q$. For $\sigma > 1 - 1/p$ we set $\alpha = \sigma$ and we get that

$$|f|_{p,\mathbb{R}\times(0,1)} \le C(\sigma)|\partial_y f|_{p,\sigma} \ , \tag{4.5}$$

which completes the proof of (i). We now prove (ii). Let $\alpha = \sigma - \sigma'$. Using that $\sigma > 1$ and $\sigma' < \sigma - 1$, we get that

$$\begin{aligned} &\int_{\mathbb{R}^2_+} \Big[(1+y)^{\sigma'}|f(x,y)|\Big]^p\, dxdy \\ &\le \int_0^\infty \left(\left[\int_y^\infty \frac{dz}{(1+z)^{\alpha q}}\right]^{\frac{p}{q}} \int_{\mathbb{R}} \left(\int_y^\infty \Big[(1+y)^{\sigma'}(1+z)^\alpha|\varphi(x,z)|\Big]^p\, dz\right) dx\right) dy \ . \end{aligned} \tag{4.6}$$

Because $\alpha > 1/q$, and $y < z$ in the z-integral, we get that

$$\begin{aligned} &\int_{\mathbb{R}^2_+} \Big[(1+y)^{\sigma'}|f(x,y)|\Big]^p\, dxdy \\ &\le \left(\int_0^\infty \frac{dy}{(1+y)^{p(\alpha q-1)/q}}\right)\left(\int_{\mathbb{R}^2_+} [(1+z)^\sigma|\varphi(x,z)|]^p\, dxdz\right) \ . \end{aligned} \tag{4.7}$$

This completes the proof of (ii), since

$$\int_0^\infty \frac{1}{(1+y)^{p(\alpha q-1)/q}}\, dy < \infty \ , \tag{4.8}$$

for $\alpha > 1 \geq 1/q$. ■

Applying the above lemma we find that $\dot{W}^{1,p}_\sigma(\mathbb{R}^2_+) \subset W^{1,p}(\mathbb{R} \times (0,1))$ for $\sigma > 1 - 1/p$. Consequently, provided $p > 1$ and $\sigma > 1 - 1/p$, the elements of $\dot{W}^{1,p}_\sigma(\mathbb{R}^2_+)$ have a well defined trace at $y = 0$. We will need this property when we discuss the boundary conditions of our problem.

Definition 9 *Let $(p, \sigma) \in \mathbb{R}^2_+$ with $p > 1$ and $\sigma > 1 - 1/p$. Then, we define $D^{1,p}_\sigma(\mathbb{R}^2_+)$ by*

$$D^{1,p}_\sigma(\mathbb{R}^2_+) = \left\{ u \in \dot{W}^{1,p}_\sigma(\mathbb{R}^2_+) \mid u(x,0) = 0 \text{ for all } x \in \mathbb{R} \right\} . \tag{4.9}$$

Note that the set $\mathcal{C}^\infty_0(\mathbb{R}^2_+)$ is dense in $D^{1,p}_\sigma(\mathbb{R}^2_+)$ for all $p \geq 1$ and $\sigma \geq 0$. The spaces $D^{1,p}_\sigma(\mathbb{R}^2_+)$ are inspired by the spaces $D^{1,p}_0(\mathbb{R}^2_+)$ in [8]. In particular, the spaces $D^{1,p}_\sigma(\mathbb{R}^2_+)$ imbed continuously in $D^{1,p}_0(\mathbb{R}^2_+)$ for arbitrary values of $\sigma > 1 - 1/p$. We have:

Lemma 10 *Let $1 < p < 2$, $\sigma > 1 - 1/p$ and let $p^* = 2p/(2-p)$. Then, $D^{1,p}_\sigma(\mathbb{R}^2_+) \subset L^{p^*}(\mathbb{R}^2_+)$.*

Proof. Since $D^{1,p}_\sigma(\mathbb{R}^2_+) \subset D^{1,p}_0(\mathbb{R}^2_+)$, Lemma 10 is an immediate consequence of [8, Theorem 5.2], for the case of functions that have zero trace at $y = 0$. By definition, this condition is satisfied by elements of $D^{1,p}_\sigma(\mathbb{R}^2_+)$. ■

Similarly, we have:

Lemma 11 *Let $p \geq 1$ and $\sigma \geq 1$. Then, $D^{1,p}_\sigma(\mathbb{R}^2_+) \subset L^p(\mathbb{R}^2_+)$.*

Proof. For $u \in \mathcal{C}^\infty_0(\mathbb{R}^2_+)$ and $p \geq 1$ we have that

$$\int_{\mathbb{R}^2_+} \partial_y((1+y)|u|^p(x,y)) \, dxdy = 0 ,$$

and therefore

$$\int_{\mathbb{R}^2_+} |u|^p(x,y) \, dxdy = -\int_{\mathbb{R}^2_+} (1+y)\partial_y|u|^p(x,y) \, dxdy ,$$

Applying Hölder's inequality we see that for some constant $C < \infty$,

$$\int_{\mathbb{R}^2_+} |u(x,y)|^p \, dxdy \leq C \int_{\mathbb{R}^2_+} |(1+y)\partial_y u(x,y)|^p \, dxdy + \frac{1}{2} \int_{\mathbb{R}^2_+} |u(x,y)|^p \, dxdy .$$

This completes the proof for $\sigma = 1$. Since $D^{1,p}_\sigma(\mathbb{R}^2_+) \subset D^{1,p}_1(\mathbb{R}^2_+)$ for $\sigma > 1$, the general result follows. ■

4.2 Exact formulation of main theorems

Now that the function spaces are introduced, we can formulate our results in detail. Concerning the behavior of the solution close to the boundary we prove:

Theorem 12 *Let $f \in \mathcal{C}^\infty_0(\mathbb{R}^2_+)$ and $\mathbf{a} \in \mathbb{R}^2$, and let $\varphi = \mathbf{a} \cdot \nabla f$. Then, for all $p \in (3/2, \infty)$ the solution ω of Theorem 1 is in $L^p(\mathbb{R} \times (0,1))$, and satisfies, for all $q < 3p/(3+p)$ and all $\sigma > 2 - 1/p$ the bound*

$$|\omega|_{p,\mathbb{R}\times(0,1)} \leq C(\mathbf{a}, p, q, \sigma)(|f|_q + |\varphi|_{p,\sigma}) , \tag{4.10}$$

for a constant $C(\mathbf{a}, p, q, \sigma)$ depending only on $\mathbf{a}$ and the choice of p, q and σ. Similarly, $\partial_x\omega$ and $\partial_y\omega$ are in $L^p(\mathbb{R} \times (0,1))$ for all $p \in (1, \infty)$ and satisfy, for all $q < p$, all $\sigma_x > 1 - 1/p$, and all $\sigma_y > 2 - 1/p$ the bounds

$$|\partial_x\omega|_{p,\mathbb{R}\times(0,1)} \leq C(\mathbf{a}, p, q, \sigma_x)(|f|_q + |\varphi|_{p,\sigma_x} + |\partial_y\varphi|_{p,\mathbb{R}\times(0,1)}) , \tag{4.11}$$

$$|\partial_y\omega|_{p,\mathbb{R}\times(0,1)} \leq C(\mathbf{a}, p, q, \sigma_y)(|f|_q + |\varphi|_{p,\sigma_y} + |\partial_y\varphi|_{p,\mathbb{R}\times(0,1)}) , \tag{4.12}$$

for constants $C(\mathbf{a}, p, q, \sigma_x)$ and $C(\mathbf{a}, p, q, \sigma_y)$ depending only on $\mathbf{a}$, and the choice of p, q and σ_x or σ_y, respectively.

We now formulate our results concerning the asymptotic behavior of ω. We recall that (1.8), (1.10) is obtained by taking the curl of (1.1). We can therefore restrict the study of the asymptotic behavior of ω to the cases $\varphi = \partial_x f$ and $\varphi = \partial_y f$. Since, as mentioned already in Theorem 2, the results are considerably different in the two cases, we state the two results separately:

Theorem 13 *Let f be a smooth function with compact support and let $\varphi = \partial_x f$. Let $(m_x, m_y) \in \{0,1\}^2$ such that $m_x + m_y \leq 1$, let $(p,\sigma) \in [1,\infty)^2$, and let $m_{xy} = 2m_x + m_y + 4$. Then,*

$$\partial_x^{m_x} \partial_y^{m_y} \omega \in L^p_\sigma(\mathbb{R}^2_{++}) \ , \tag{4.13}$$

provided $\sigma < m_{xy}$ and $p > 3/(m_{xy} - \sigma)$. Moreover we have, for all q with $1 \leq q < \min(q^\sigma_{m_x,m_y}, \widetilde{q}^\sigma_{m_x,m_y})$, with $q^\sigma_{m_x,m_y}$ and $\widetilde{q}^\sigma_{m_x,m_y}$ as defined in (7.7) and (7.9) respectively, and for $\sigma' > \max(1+\sigma, 2-1/q)$, that

$$|\partial_x^{m_x} \partial_y^{m_y} \omega|_{p,\sigma,\mathbb{R}^2_{++}} \leq C(p,q,\sigma) \Big(|\varphi|_{p,\sigma} + |f|_{q,\sigma'} \Big) \ , \tag{4.14}$$

where the constant $C(p,q,\sigma)$ depends only on p, σ and the choice of q.

Theorem 14 *Let f be a smooth function with compact support and let $\varphi = \partial_y f$. Let $(m_x, m_y) \in \{0,1\}^2$ such that $m_x + m_y \leq 1$, let $(p,\sigma) \in [1,\infty)$, and let $m_{xy} = 2m_x + m_y + 3$. Then,*

$$\partial_x^{m_x} \partial_y^{m_y} \omega \in L^p_\sigma(\mathbb{R}^2_{++}) \ , \tag{4.15}$$

provided $\sigma < m_{xy}$ and $p > 3/(m_{xy} - \sigma)$. Moreover we have, for all q with $1 \leq q < \min(q^\sigma_{m_x,m_y}, \widetilde{q}^\sigma_{m_x,m_y})$, with $q^\sigma_{m_x,m_y}$ and $\widetilde{q}^\sigma_{m_x,m_y}$ as defined in (7.7) and (7.9) respectively, and all r such that $1 < r < 2\widetilde{q}^\sigma_{m_x,m_y}/(\widetilde{q}^\sigma_{m_x,m_y} + 2)$, and for all $\sigma' > 2 + \sigma$, that

$$|\partial_x^{m_x} \partial_y^{m_y} \omega|_{p,\sigma,\mathbb{R}^2_{++}} \leq C(p,q,r,\sigma,\sigma') \Big(|\varphi|_{p,\sigma} + |f|_{q,\sigma'} + |\varphi|_r + |f|_{r,1} \Big) \ , \tag{4.16}$$

where the constant $C(p,q,r,\sigma,\sigma')$ depends only on p, σ and the choice for q, r and σ'.

Proofs of the preceding three theorems are the content of the remaining sections of this paper. Note that Theorem 2 follows from the above two theorems with $m_x = m_y = 0$, using that a function f is in $L^p_\sigma(\Omega)$ if and only if there exists a function $\widetilde{f}$ in $L^p(\Omega)$ such that $f(x,y) = (1+y)^{-\sigma} \widetilde{f}(x,y)$ for all $(x,y) \in \Omega$. See point (i) of Remark 7.

4.3 Uniqueness of solutions

Before going any further we now show that there is at most one solution of our problem in $\dot{W}^{1,p}_\sigma(\mathbb{R}^2_+)$, for all values of $(p,\sigma) \in [4/3, 2] \times [1,\infty)$. As a consequence, since for the cases $\varphi = \partial_x f$ and $\varphi = \partial_y f$ Theorem 13 and Theorem 14 imply that the solution ω constructed in Section 3 is in each of the function spaces under consideration, it follows that there is only one solution in the union of all these spaces.

Theorem 15 *Given $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^2_+)$ and $(p,\sigma) \in [4/3, 2) \times [1,\infty)$, there exists at most one solution to (1.8) and (1.13) in $\dot{W}^{1,p}_\sigma(\mathbb{R}^2_+)$.*

Proof. Let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^2_+)$, $p \geq 4/3$ and $\sigma \geq 1$, and assume ω_1 and ω_2 are two different solutions of (1.8) and (1.13) in $\dot{W}^{1,p}_\sigma(\mathbb{R}^2_+)$. By linearity the difference $\omega = \omega_1 - \omega_2$ is a solution to (1.8), (1.13) with $\varphi = 0$. Therefore, $\omega \in D^{1,p}_\sigma(\mathbb{R}^2_+)$, and $\Delta\omega - \partial_x \omega = 0$ in $\mathbb{R}^2_+$. Applying Lemma 10, we find that $\omega \in L^p(\mathbb{R}^2_+) \cap L^{p^*}(\mathbb{R}^2_+)$, and standard bootstrap techniques then imply that $\omega \in \mathcal{C}^\infty(\mathbb{R}^2_+)$. Let $r = \sqrt{x^2+y^2}$, and let χ be a cut-off function in $\mathcal{C}^\infty(\mathbb{R}^2)$, *i.e.*, $\chi(x,y) = 1$ for $r \leq 1/2$, and $\chi(x,y) = 0$ for $r > 1$, say, and let $\chi_R = \chi((x,y)/R)$. We now multiply the equation satisfied by ω with $\chi_R \omega$ and integrate over the half space. We get that (to simplify the notation we drop in what follows the arguments x and y)

$$\int_{\mathbb{R}^2_+} \Delta\omega \cdot \chi_R \omega \ dxdy - \int_{\mathbb{R}^2_+} \partial_x \omega \cdot \chi_R \omega \ dxdy = 0 \ .$$

We note that

$$\int_{\mathbb{R}^2_+} \Delta\omega \cdot \chi_R \omega \; dxdy = -\int_{\mathbb{R}^2_+} \chi_R |\nabla\omega|^2 \; dxdy - \int_{\mathbb{R}^2_+} \omega \nabla\chi_R \cdot \nabla\omega \; dxdy \; ,$$

and that

$$\int_{\mathbb{R}^2_+} \partial_x\omega \cdot \chi_R \omega \; dxdy = -\int_{\mathbb{R}^2_+} \omega^2 \partial_x \chi_R \; dxdy \; .$$

Since $2 \in [p, p^*]$ we have that $\omega \in L^2(\mathbb{R}^2_+)$, and therefore, since $|\nabla\chi_R|_\infty = |\nabla\chi|_\infty / R$, we find that

$$\lim_{R\to\infty} \int_{\mathbb{R}^2_+} \partial_x\omega \cdot \chi_R \omega \; dxdy = 0 \; .$$

Next, since $p \geq 4/3$, we have that $1/p + 1/p^* \leq 1$. Therefore, there exists $q \in [1, \infty)$ such that $1/q = 1 - (1/p + 1/p^*)$. Using Hölder's inequality we therefore get that

$$\left| \int_{\mathbb{R}^2_+} \omega \nabla\chi_R \cdot \nabla\omega \right| \; dxdy \leq |\omega|_{p^*} \; |\nabla\omega|_p \; |\nabla\chi_R|_q \; ,$$

and therefore, since $|\chi_R|_q = R^{1/q-1}|\chi|$, we find that

$$\lim_{R\to\infty} \int_{\mathbb{R}^2_+} \omega \nabla\chi_R \cdot \nabla\omega \; dxdy = 0 \; .$$

Finally, let $\mathcal{B}^R = \mathbb{R}^2 \setminus \mathcal{B}_R$ be the complement of a ball $\mathcal{B}_R$ of radius R centered at the origin. Since $\chi_R(\mathbb{R}^2) \subset [0, 1]$, we have that

$$\int_{\mathcal{B}^R \cap \mathbb{R}^2_+} |\nabla\omega|^2 \; dxdy \leq \int_{\mathbb{R}^2_+} \chi_R |\nabla\omega|^2 \; dxdy \; ,$$

and therefore

$$\limsup_{R\to\infty} \int_{\mathcal{B}^R \cap \mathbb{R}^2_+} |\nabla\omega|^2 \; dxdy = 0 \; .$$

It follows that $\nabla\omega$ is identically zero in $\mathbb{R}^2_+$ and therefore ω is also identically zero in $\mathbb{R}^2_+$, since ω is zero at the boundary $y = 0$. ■

5 Technical lemmas

In the following two subsections we prove technical lemmas which will allow us to bound the volume term (3.5) and the boundary term (3.6), respectively. In order to emphasize that the lemmas do not use the detailed properties of the Green's function $\mathcal{K}$, we state the results for general (Green's) functions $\mathcal{G}$. See Appendix A for a proof that the Green's function $\mathcal{K}$ satisfies all the stated properties.

5.1 Volume terms

As has been explained in Section 3.1 $T_V\{\mathcal{G}\}$ can be rewritten as a convolution product. Namely, let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2_+)$, and let $\widetilde{\varphi}(x, y) = \varphi(x, y)$ for $x \in \mathbb{R}$, and $y > 0$ and $\widetilde{\varphi}(x, y) = -\varphi(x, -y)$ for $x \in \mathbb{R}$, and $y < 0$. Then, for $(x, y) \in \mathbb{R}^2$ the convolution $\mathcal{G} * \widetilde{\varphi}$ is well defined and we can write $T_V\{\mathcal{G}\}[\varphi]$ as the restriction of this convolution product to $\mathbb{R}^2_+$. We have:

Lemma 16 *Let $(p, q, r) \in [1, \infty)^3$ such that $1/r + 1/q = 1 + 1/p$. Let furthermore $\mathcal{G} \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0, 0)\}) \cap L^q(\mathbb{R}^2)$. Then, $T_V\{\mathcal{G}\}[\varphi] \in L^p(\mathbb{R}^2_+)$, for all $\varphi \in L^r(\mathbb{R}^2_+)$, and we have the bound*

$$|T_V\{\mathcal{G}\}[\varphi]|_p \leq C(p, q, r, \mathcal{G}) \; |\varphi|_r \; . \tag{5.1}$$

Proof. Lemma 16 is an immediate consequence of the Young inequality for convolutions, using that $|\widetilde{\varphi}|_{p,\mathbb{R}^2} = 2|\varphi|_{p,\mathbb{R}^2_+}$ for all $p < \infty$. ■

The next Lemma makes the compensation at large values of y that is inherent in the definition of $T_V\{\mathcal{G}\}$ explicit:

Lemma 17 *Let $\sigma \in (0,\infty)$, and let $(p,q,r) \in [1,\infty)^3$ such that $1/r + 1/q = 1 + 1/p$. Let furthermore $\mathcal{G} \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{0\})$, with $\partial_y \mathcal{G} \in L^q_\sigma(\mathbb{R}^2)$. Then $T_V\{\mathcal{G}\}[\varphi] \in L^p_\sigma(\mathbb{R}^2_+)$ for all $\varphi \in L^r_{\sigma+1}(\mathbb{R}^2_+)$, and we have the bound*

$$|T_V\{\mathcal{G}\}[\varphi]|_{p,\sigma} \leq C(p,q,r,\sigma,\mathcal{G})\ |\varphi|_{r,\sigma+1}\ . \tag{5.2}$$

Proof. The proof relies on the following identity:

Proposition 18 *For any smooth function φ with compact support in $\overline{\mathbb{R}^2_+}$, we have*

$$T_V\{\mathcal{G}\}[\varphi](x,y) = -\int_0^1 \frac{dt}{(1-2t)^2} \int_{\mathbb{R}^2_+} 2\partial_y \mathcal{G}(x-x_0, y-y_0)\ y_0\ \varphi(x_0, \frac{y_0}{1-2t})\ dx_0 dy_0\ . \tag{5.3}$$

Proof. Since

$$\mathcal{G}(x-x_0, y-y_0) - \mathcal{G}(x-x_0, y+y_0) = -2y_0 \int_0^1 \partial_y \mathcal{G}(x-x_0, y-(1-2t)y_0)\ dt\ , \tag{5.4}$$

we get for $T_V\{\mathcal{G}\}$,

$$T_V\{\mathcal{G}\}[\varphi](x,y) = -\int_0^1 dt \int_{\mathbb{R}^2_+} 2\partial_y \mathcal{G}(x-x_0, y-(1-2t)y_0) y_0 \varphi(x_0,y_0)\ dx_0 dy_0\ , \tag{5.5}$$

and (5.3) follows by a change of variables. ■

Using the identity (5.3) we can now prove Lemma 17. We have that

$$|(1+y)^\sigma T_V\{\mathcal{G}\}[\varphi](x,y)| \leq C(\sigma)(T_1(x,y) + T_2(x,y))\ , \tag{5.6}$$

with T_1 and T_2 defined by

$$T_1(x,y) = \int_0^1 \frac{dt}{(1-2t)^2} \int_{\mathbb{R}^2_+} |(1+|y-y_0|)^\sigma \partial_y \mathcal{G}(x-x_0, y-y_0)| \cdot |y_0 \varphi(x_0, \frac{y_0}{1-2t})|\ dx_0 dy_0\ , \tag{5.7}$$

and

$$T_2(x,y) = \int_0^1 \frac{dt}{(1-2t)^2} \int_{\mathbb{R}^2_+} |\partial_y \mathcal{G}(x-x_0, y-y_0)| \cdot |y_0|^{1+\sigma} \varphi(x_0, \frac{y_0}{1-2t})|\ dx_0 dy_0\ . \tag{5.8}$$

We first bound T_1. Let $0 < 2\alpha < 1 - 1/p$, and let $q \geq 1$ such that $1/p + 1/q = 1$. Then, we have that

$$|T_1(x,y)|^p \leq \left(\int_0^1 \frac{dt}{|1-2t|^{2\alpha q}}\right)^{\frac{p}{q}} \int_0^1 \frac{|\Phi_t(x,y)|^p}{|1-2t|^{2(1-\alpha)p}}\ dt\ , \tag{5.9}$$

where, for any $(x,y) \in \mathbb{R}^2_+$ and $t \in [0,1]$, Φ_t is defined by

$$\Phi_t(x,y) = \int_{\mathbb{R}^2_+} |(1+|y-y_0|)^\sigma \partial_y \mathcal{G}(x-x_0, y-y_0)| \cdot |y_0 \varphi(x_0, \frac{y_0}{1-2t})|\ dx_0 dy_0\ . \tag{5.10}$$

Note that the first integral in (5.9) is finite since $2\alpha < 1/q$. Hence, by Tonelli's theorem,

$$|T_1|_p \leq C\left(\int_0^1 \frac{|\Phi_t|_p^p}{|1-2t|^{2(1-\alpha)p}}\ dt\right)^{\frac{1}{p}}\ . \tag{5.11}$$

Now, since $1 + 1/p = 1/q + 1/r$, we can apply Young's inequality for convolutions. We get

$$|\Phi_t|_p \leq |\partial_y \mathcal{G}|_{q,\sigma} \left(\int_{\mathbb{R}_2^+} \left[y_0 \varphi(x_0, \frac{y_0}{1-2t})\right]^r\ dx_0 dy_0\right)^{\frac{1}{r}}\ , \tag{5.12}$$

and therefore, using a scaling argument,

$$\int_{\mathbb{R}_2^+} \left[y_0 \varphi(x_0, \frac{y_0}{1-2t})\right]^r\ dx_0 dy_0 = |1-2t|^{r+1}\ |\varphi|_{r,1}^r\ . \tag{5.13}$$

Therefore,

$$|T_1|_p \leq |\partial_y \mathcal{G}|_{q,\sigma} \ \ |\varphi|_{p,1} \left(\int_0^1 |1-2t|^{p(1/r+1-2(1-\alpha))} \ dt \right)^{\frac{1}{p}} . \tag{5.14}$$

Let $p' = p(1/r + 1 - 2(1-\alpha))$. Then, the integral in (5.14) is finite, provided $p' > -1$. Since $1/r = 1 + 1/p - 1/q$, this is equivalent to $2/p > 1/q - 2\alpha$. Therefore, the integral in (5.14) is finite, provided 2α is chosen sufficiently close to $1/q$.

We now bound T_2.Following exactly the same strategy as for the case of T_1 we get that

$$|T_2|_p \leq |\partial_y \mathcal{G}|_q \ |\varphi|_{p,1+\sigma} \left(\int_0^1 |1-2t|^{p(1/r+\sigma+1-2(1-\alpha))} \ dt \right)^{\frac{1}{p}} \ , \tag{5.15}$$

and the integral in (5.15) is finite, provided $p\ (1/r + \sigma + 1 - 2(1-\alpha)) \geq p\ (1/r + 1 - 2(1-\alpha)) > -1$. The bound on $T_V\left\{\mathcal{G}\right\}[\varphi]$ now follows from the bounds on T_1 and T_2 by the triangle inequality. This completes the proof of Lemma 17. ■

5.2 Boundary terms

The following lemma gives bounds on $T_B\{\mathcal{G}\}$ which will be useful for the case $y > 1$.

Lemma 19 *Let $(p, q, r, \sigma) \in [1, \infty)^3 \times [0, \infty)$, such that $1 + 1/r = 1/p + 1/q$. Let $g \in L^p(\mathbb{R})$ and let $\mathcal{G} \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0, 0)\})$ with*

$$\int_1^\infty (1 + |y|^\sigma)^r \left(\int_{-\infty}^\infty |\partial_y \mathcal{G}(t, y)|^q \ dt \right)^{\frac{r}{q}} dy < \infty \ . \tag{5.16}$$

Then, $T_B\{\mathcal{G}\}[g] \in L^r_\sigma(\mathbb{R} \times (1, \infty))$, and we have the bound

$$|T_B\{\mathcal{G}\}[g]|_{r,\sigma} \leq C(p, q, r, \sigma, \mathcal{G}) \ |g|_p \ . \tag{5.17}$$

Proof. Since, for fixed $y > 0$, $T_B\{\mathcal{G}\}[g]$ is a convolution product with respect to the first variable, *i.e.*,

$$g *_x \partial_y \mathcal{G}(\cdot, y) \ ,$$

we can apply Young's inequality and the result follows. ■

In order to prove detailed bounds for $T_B\{\mathcal{G}\}$, we use the expressions (1.13) and (3.17) for ω_0 and T_B respectively. These expressions are based on the Fourier transform. We proceed in several steps. Our starting point is the following classical result in Fourier-multiplier theory [15, Theorem 3, p.96]:

Theorem 20 *Let $1 < p < \infty$ and let $\mu \in \mathcal{C}^1(\mathbb{R} \setminus \{0\})$ such that $|\mu|_\mathcal{M} < \infty$, where*

$$|\mu|_\mathcal{M} = \sup_{\xi \in \mathbb{R} \setminus \{0\}} \{|\mu(\xi)| + |\xi \mu'(\xi)|\} \ . \tag{5.18}$$

Then, the application $\mathcal{F}_\mu$, defined by

$$\mathcal{F}_\mu[f] = \mathcal{F}^{-1}[\mu \ \mathcal{F}[f] \ ] \ , \tag{5.19}$$

maps $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$. Moreover, there exists a constant $C(p) < \infty$, depending only on p, such that for all $f \in L^p(\mathbb{R})$,

$$|\mathcal{F}_\mu[f]|_{p,\mathbb{R}} \leq C(p) \ |\mu|_\mathcal{M} \ |f|_{p,\mathbb{R}} \ . \tag{5.20}$$

In view of Theorem 20, the expressions (1.13) and (3.17) for ω_0 and T_B can be considered multiplier transformations, but with a multiplier depending on two or three variables. For our purposes we therefore now generalize the definition (5.19) to the case of multipliers μ which depend on several variables. The Fourier transform is however always with respect to the first argument only.

The following lemma will allow us to analyze the function ω_0:

Lemma 21 *Let* $(p, r, s) \in (1, \infty) \times \mathbb{R}^2$, $\sigma > 1 - (1/p + s)$, $\mu \in \mathcal{C}^1((\mathbb{R} \setminus \{0\}) \times (0, \infty))$, *and define the norm* $|\cdot|_{\mathcal{M},r,s}$ *by*

$$|\mu|_{\mathcal{M},r,s} = \sup_{\eta \geq 0} \{s^r (1+\eta)^s |m(\cdot, \eta)|_{\mathcal{M}}\} \ . \tag{5.21}$$

Let $\mathcal{F}_\mu$ *be defined by*

$$\mathcal{F}_\mu[f] = \int_0^\infty \mathcal{F}^{-1}[\mu(\cdot, \eta)\ \mathcal{F}[f](\cdot, \eta)\ ]\ d\eta \ . \tag{5.22}$$

Then,

(*i*) *for all* $f \in \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2})$ *we have that* $\mathcal{F}_\mu[f] \in L^p(\mathbb{R})$, *and there exists a constant* $C(p, \sigma)$ *such that*

$$|\mathcal{F}_\mu[f]|_p \leq C(p, \sigma) |f|_{p,\sigma} \ , \tag{5.23}$$

provided

$$|\mu|_{\mathcal{M},0,s} < \infty \ . \tag{5.24}$$

(*ii*) *for all* $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+^2)$ *we have that* $\mathcal{F}_\mu[\varphi] \in L^p(\mathbb{R})$, *and there exists a constant* $C(p, \sigma)$ *such that*

$$|\mathcal{F}_\mu[\varphi]|_p \leq C(p, \sigma) \left(|\partial_y \varphi|_{p,\mathbb{R}\times(0,1)} + |\varphi|_{p,\sigma-1} \right) \ , \tag{5.25}$$

provided

$$|\mu|_{\mathcal{M},1,s} < \infty \ . \tag{5.26}$$

Proof. We first prove (*i*). Let $\alpha = s + \sigma$. Applying Hölder's inequality, we get from (5.22) that

$$|\mathcal{F}_\mu[f](x)|^p \leq \left(\int_0^\infty \frac{d\eta}{(1+\eta)^{\alpha q}} \right)^{\frac{p}{q}} \int_0^\infty \left| \mathcal{F}^{-1}[(1+\eta)^\alpha \mu(\cdot, \eta) \mathcal{F}[f](\cdot, \eta)](x) \right|^p \ d\eta \ , \tag{5.27}$$

with q the conjugate exponent of p, *i.e.*, $1/p + 1/q = 1$. The first integral on the right-hand side of (5.27) is finite because $\alpha > 1/q$. Therefore,

$$|\mathcal{F}_\mu[f]|_p^p \leq C \int_{\mathbb{R}_+^2} \left| \mathcal{F}^{-1}[(1+\eta)^\alpha \mu(\cdot, \eta) \mathcal{F}[f](\cdot, \eta)](x) \right|^p \ dx d\eta \ . \tag{5.28}$$

Using that $\alpha = s + \sigma$, we get from Theorem 20, that

$$|\mathcal{F}_\mu[f]|_p^p \leq C \int_0^\infty (1+\eta)^{sp} |\mu(\cdot, \eta)|_{\mathcal{M}}^p \ |(1+\eta)^\sigma f(\cdot, \eta)|_{p,\mathbb{R}}^p \ d\eta \ , \tag{5.29}$$

and, applying again Hölder's inequality we get that

$$|\mathcal{F}_\mu[f]|_p^p \leq C \ |\mu|_{\mathcal{M},0,s} \ |f|_{p,\sigma}^p \ , \tag{5.30}$$

as required. Wen now prove (*ii*). Applying the same technique as in the proof of (*i*) we get that

$$|\mathcal{F}_\mu[\varphi]|_p^p \leq C \ |\mu|_{\mathcal{M},1,s}^p \int_{\mathbb{R}_+^2} \left(\frac{(1+\eta)^\sigma |\varphi(x, \eta)|}{\eta} \right)^p dx d\eta \ , \tag{5.31}$$

and (*ii*) now follows using that, by a straightforward generalization of Hardy's inequality,

$$\int_{\mathbb{R}_+^2} \left(\frac{(1+\eta)^\sigma}{\eta} |\varphi(x, y)| \right)^p dx dy \leq C \ \left(|\partial_y \varphi|_{p,\mathbb{R}\times(0,1)}^p + |\varphi|_{p,\sigma-1}^p \right) \ , \tag{5.32}$$

This completes the proof of Lemma 21. ■

The following lemma will allow us to analyze the function ω_B:

Lemma 22 *Let* $(p,r,s) \in (1,\infty) \times \mathbb{R}^2$, $\sigma > 1 - (1/p + s)$, $\mu \in \mathcal{C}^1((\mathbb{R} \setminus \{0\}) \times (0,\infty)^2)$, *and define the norm* $|\cdot|_{\mathcal{M},r,s,p}$ *by*

$$|\mu|_{\mathcal{M},r,s,p} = \int_0^1 [|\mu(\cdot,\cdot,y)|_{\mathcal{M},r,s}]^p \ dy \ . \tag{5.33}$$

Let $\mathcal{F}_\mu$ *be defined by*

$$\mathcal{F}_\mu[f] = \int_0^\infty \mathcal{F}^{-1}[\mu(\cdot,\eta,\cdot) \ \mathcal{F}[f](\cdot,\eta) \ ] \ d\eta \ . \tag{5.34}$$

Then,

(*i*) *for all* $f \in \mathcal{C}_0^\infty(\overline{\mathbb{R}^2_+})$ *we have that* $\mathcal{F}_\mu[f] \in L^p(\mathbb{R} \times (0,1))$, *and there exists a constant* $C(p,\sigma)$ *such that*

$$|\mathcal{F}_\mu[f]|_{p,\mathbb{R}\times(0,1)} \leq C(p,\sigma)|f|_{p,\sigma} \ , \tag{5.35}$$

provided

$$|\mu|_{\mathcal{M},0,s,p} < \infty \ . \tag{5.36}$$

(*ii*) *for all* $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2_+)$ *we have that* $\mathcal{F}_\mu[\varphi] \in L^p(\mathbb{R} \times (0,1))$, *and there exists a constant* $C(p,\sigma)$ *such that*

$$|\mathcal{F}_\mu[\varphi]|_{p,\mathbb{R}\times(0,1)} \leq C(p,\sigma) \left(|\partial_y \varphi|_{p,\mathbb{R}\times(0,1)} + |\varphi|_{p,\sigma-1}\right) \ , \tag{5.37}$$

provided

$$|\mu|_{\mathcal{M},1,s,p} < \infty \ . \tag{5.38}$$

Proof. Proceeding exactly as in the proof of Lemma 21 we obtain for (*i*),

$$|\mathcal{F}_\mu[f]|^p_{p,\mathbb{R}\times(0,1)} \leq C \int_0^1 \int_0^\infty (1+\eta)^{sp} |\mu(\cdot,\eta,y)|^p_{\mathcal{M}} |(1+\eta)^\sigma f(\cdot,\eta)|^p_{p,\mathbb{R}} \ d\eta dy \ , \tag{5.39}$$

and equation (5.36) follows using Hölder's inequality. To prove (*ii*) we combine Hölder's inequality with the inequality (5.32). ■

6 Proof of Theorem 12

In the following two Lemmas we present our results for $\omega_V = T_V\{\mathcal{K}\}[\varphi]$ and $\omega_B = T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\varphi]]$. From these lemmas the Theorem 12 then follows using the triangle inequality.

Lemma 23 *Let* f *be a smooth function with compact support and let* $\varphi = \mathbf{a} \cdot \nabla f$, *with* $\mathbf{a} \in \mathbb{R}^2$. *Then,*

(*i*) $\omega_V \in L^p(\mathbb{R} \times (0,1))$, *for all* $p \in (3/2,\infty)$, *and for all* $q < 3p/(3+p)$,

$$|\omega_V|_{p,\mathbb{R}\times(0,1)} \leq C(\mathbf{a},p,q) \ (|\varphi|_p + |f|_q) \ . \tag{6.1}$$

(*ii*) $\nabla\omega_V \in L^p(\mathbb{R} \times (0,1))$, *for all* $p \in (1,\infty)$, *and for all* $q < p$,

$$|\nabla\omega_V|_{p,\mathbb{R}\times(0,1)} \leq C(\mathbf{a},p,q) \ (|f|_q + |\varphi|_p) \ . \tag{6.2}$$

In principle we can establish, instead of (6.1), (6.2), bounds that only involve the function φ, instead of φ and f. The price to be paid is that the constants in the bounds then also depend on the support of φ. Since the ultimate goal of the work that we start here is to use the present results for a proof of the existence of solutions to the Navier-Stokes equations, we have chosen to systematically establish bounds that are independent of the support of φ, in order to be able to generalize to functions φ of non compact support in a straightforward way.

Lemma 24 *Let* φ *be a smooth function with compact support. Then,* (*i*) $\omega_B \in L^p(\mathbb{R} \times (0,1))$, *for all* $p \in (1,\infty)$, *and for arbitrary* $\sigma > 2 - 1/p$,

$$|\omega_B|_{p,\mathbb{R}\times(0,1)} \leq C(p,\sigma)\ |\varphi|_{p,\sigma}\ . \tag{6.3}$$

(*ii*) $\nabla\omega_B \in L^p(\mathbb{R} \times (0,1))$, *for all* $p \in (1,\infty)$, *and for arbitrary* $\sigma_x > 1 - 1/p$ *and* $\sigma_y > 2 - 1/p$, *and*

$$|\partial_x \omega_B|_{p,\mathbb{R}\times(0,1)} \leq C(p,\sigma_x) \left[|\varphi|_{p,\sigma_x} + |\partial_y \varphi|_{p,\mathbb{R}\times(0,1)}\right]\ , \tag{6.4}$$

$$|\partial_y \omega_B|_{p,\mathbb{R}\times(0,1)} \leq C(p,\sigma_y) \left[|\varphi|_{p,\sigma_y} + |\partial_y \varphi|_{p,\mathbb{R}\times(0,1)}\right]\ . \tag{6.5}$$

In the remainder of this section we give a proof of these two lemmas.

6.1 Proof of Lemma 23.

Let $r = \sqrt{x^2+y^2}$, and let χ be a cut-off function in $\mathcal{C}^\infty(\mathbb{R}^2)$, *i.e.*, $\chi(x,y) = 1$ for $r \leq 1/2$, and $\chi(x,y) = 0$ for $r > 1$, say. We then set,

$$\omega_V = T_V\{\mathcal{K}\}[\varphi] = \omega_V^{\mathrm{c}} + \omega_V^{\mathrm{nc}}, \tag{6.6}$$

with $\omega_V^{\mathrm{c}} = T_V\{\chi\mathcal{K}\}[\varphi]$ and with $\omega_V^{\mathrm{nc}} = T_V\{(1-\chi)\mathcal{K}\}[\varphi]$. From Proposition 3 we have that

$$\chi\mathcal{K} \in L^1(\mathbb{R}^2)\ , \quad \text{and} \quad \nabla(\chi\mathcal{K}) \in L^1(\mathbb{R}^2)\ . \tag{6.7}$$

Therefore, we get from Lemma 16 with $q = 1$, that

$$\omega_V^{\mathrm{c}} \in L^p(\mathbb{R}^2_+)\ , \quad \text{and} \quad \nabla\omega_V^{\mathrm{c}} = T_V\{\nabla(\chi\mathcal{K})\}[\varphi] \in L^p(\mathbb{R}^2_+)\ , \tag{6.8}$$

and that

$$|\omega_V^{\mathrm{c}}|_p \leq C\ |\varphi|_p\ , \quad \text{and} \quad |\nabla\omega_V^{\mathrm{c}}|_p \leq C\ |\varphi|_p\ . \tag{6.9}$$

For the second term in (6.6) we use that $\varphi = \mathbf{a}\cdot\nabla f$, and obtain that

$$\omega_V^{\mathrm{nc}}(x,y) = \int_{\mathbb{R}^2_+} \mathbf{a}\cdot\nabla_0 \left[(1-\chi)\mathcal{K}(x-x_0,y-y_0) - (1-\chi)\mathcal{K}(x-x_0,y+y_0)\right]\ f(x_0,y_0)\ dx_0 dy_0\ , \tag{6.10}$$

where $\nabla_0 = (\partial_{x_0}, \partial_{y_0})$. We get that

$$T_V\{(1-\chi)\mathcal{K}\}[\varphi] = T_V\{a_1\partial_x(1-\chi)\mathcal{K}\}[f] + T_{V,+}\{a_2\partial_y(1-\chi)\mathcal{K}\}[f]\ , \tag{6.11}$$

where $\mathbf{a} = (a_1, a_2)$ and $T_{V,+}$ is defined as T_V, but with a plus sign instead of a minus sign between the two terms. Indeed, the integration by parts introduces a change of sign for the y_0-derivative. We therefore do not get any cancellation at large values of y, and it is therefore sufficient to estimate the two terms separately in a straightforward way. Applying Proposition 41, we get that, for all $\eta > 0$,

$$\nabla\left[(1-\chi)\mathcal{K}\right] \in L^{3/2+\eta}(\mathbb{R}^2)\ , \quad \text{and} \quad \nabla^2[(1-\chi)\mathcal{K}] \in L^{1+\eta}(\mathbb{R}^2)\ . \tag{6.12}$$

Proceeding as in the proof of Lemma 16 we therefore get that $\omega_V^{\mathrm{nc}} \in L^p(\mathbb{R}^2_+)$ for all $p > \frac{3}{2}$ and $q < 3p/(3+p)$, and that

$$|\omega_V^{\mathrm{nc}}|_p \leq C(p,q)|f|_q\ . \tag{6.13}$$

Similarly, we find that $\nabla\omega_V^{\mathrm{nc}} = T_V\{\nabla((1-\chi)\mathcal{K})\}[\varphi] \in L^p(\mathbb{R}^2_+)$, for all $p > 1$ and $q < p$, and that

$$|\nabla\omega_V^{\mathrm{nc}}|_p \leq C(p,q)|f|_p\ . \tag{6.14}$$

Combining (6.8,6.9) and (6.13,6.14), we obtain Lemma 23. This completes the proof of Lemma 23.

6.2 Proof of Lemma 24.

In order to study the behavior of ω_B in the strip $0 \leq y \leq 1$ we use the representation (3.22) for ω_B, which allows us to apply Lemma 22 in order to bound ω_B and $\nabla\omega_B$. We treat the two cases separately.

Bounds on ω_B. For all $(k,\eta) \in \mathbb{R} \times (0,\infty)$ we have that

$$|m_0(k,\eta)| \leq \eta e^{-|k|\eta} \ , \tag{6.15}$$

and therefore we find that m_B, as defined by (3.21), satisfies for all $(k,\eta,y) \in \mathbb{R} \times (0,\infty) \times (0,1)$,

$$|m_B(k,\eta,y)| \leq C\eta e^{-|k|(\eta+y)} \ . \tag{6.16}$$

Now, let $\dot{\kappa} = d\kappa/dk = (k - i/2)/\kappa$. Then,

$$\begin{aligned} \partial_k m_B(k,\eta,y) &= -y\dot{\kappa}e^{-\kappa y} m_0(k,\eta) \\ &\quad + e^{-\kappa y}\left[-\frac{\dot{\kappa} - \operatorname{sign}(k)}{\kappa - |k|} m_0(k,\eta) + \eta \frac{\dot{\kappa}e^{-\kappa\eta} - \operatorname{sign}(k)e^{-|k|\eta}}{\kappa - |k|}\right] \ . \end{aligned} \tag{6.17}$$

Using Lemma 42 we get that there is a constant C,

$$|\partial_k m_B(k,\eta,y)| \leq C \ \left[\eta y \left(1 + \frac{1}{\sqrt{|k|}}\right) + \frac{\eta}{\min(1,\sqrt{|k|})}\left(1 + \frac{1}{\sqrt{|k|}}\right)\right] e^{-|k|(\eta+y)} \ . \tag{6.18}$$

For a constant $C < \infty$. We therefore find that for all $(k,\eta,y) \in \mathbb{R} \times (0,\infty) \times (0,1)$,

$$|k\partial_k m_B(k,\eta,y)| \leq C\eta(|k|+1)e^{-|k|\eta} \ , \tag{6.19}$$

and that for all $(\eta,y) \in (0,\infty) \times (0,1)$,

$$|m_B(\cdot,\eta,y)|_{\mathcal{M}} \leq C \ (\eta+1) \ . \tag{6.20}$$

This shows that the function m_B is a multiplier in the sense of Lemma 22 with $s = -1$, $r = 0$, and arbitrary $p \in (1,\infty)$. Thus, applying (i) of Lemma 22, we obtain that $\omega_B \in L^p(\mathbb{R} \times (0,1))$ for arbitrary $p \in (1,\infty)$, and that, for all $\sigma > 2 - 1/p$,

$$|\omega_B|_{p,\mathbb{R}\times(0,1)} \leq |\varphi|_{p,\sigma} \ . \tag{6.21}$$

Bounds on $\nabla\omega_B$. Like ω_B, the functions $\partial_y\omega_B$, and $\partial_x\omega_B$ are associated with multipliers m_B^y and m_B^x respectively, which are defined by,

$$m_B^y(k,\eta,y) = -\kappa m_B(k,\eta,y) \ , \tag{6.22}$$

$$m_B^x(k,\eta,y) = -ikm_B(k,\eta,y) \ . \tag{6.23}$$

For m_B^x we apply, for fixed $(\eta,y) \in (0,\infty) \times (0,1)$, the equation (6.16), and obtain that

$$|m_B^x(k,\eta,y)| \leq C|k|\eta e^{-|k|\eta} \ . \tag{6.24}$$

We can therefore use a scaling argument to show that there exists a constant $C < \infty$ such that $|m_B^x(k,\eta,y)| \leq C$ for all $(k,\eta) \in \mathbb{R} \times (0,\infty)$. For $\partial_k m_B^x$ we get that for all $(k,\eta,y) \in \mathbb{R} \times (0,\infty) \times (0,1)$,

$$|k\partial_k m_B^x(k,\eta,y)| \leq |km_B(k,\eta,y)| + |k^2\partial_k m_B(k,\eta,y)| \ , \tag{6.25}$$

and we can apply (6.16) and (6.18) and we get that for all $(k,\eta,y) \in \mathbb{R} \times (0,\infty) \times (0,1)$,

$$|k\partial_k m_B^x(k,\eta,y)| \leq C|k|\eta(1+|k|)e^{-|k|(\eta+y)} \ . \tag{6.26}$$

Using again a scaling argument we find that for all $(\eta,y) \in (0,\infty) \times (0,1)$,

$$\sup_{k\in\mathbb{R}} |k\partial_k m_B^x(k,\eta,y)| \leq \frac{\eta}{\eta+y}\left(1 + \frac{1}{\eta+y}\right) \ , \tag{6.27}$$

and we therefore get that for all $(\eta,y) \in (0,\infty) \times (0,1)$,

$$|m_B^x(\cdot,\eta,y)|_{\mathcal{M}} \leq C \ \frac{1+\eta}{\eta} \ . \tag{6.28}$$

This shows that the function m_B^x satisfies the assumptions of Lemma 22 with $s = -1$, $r = 1$ and arbitrary $p \in (1, \infty)$. Thus, applying (ii) of Lemma 22, we get that $\partial_x \omega_B \in L^p(\mathbb{R} \times (0,1))$ for arbitrary $p > 1$, and for all $\sigma > 1 - 1/p$,

$$|\partial_x \omega_B|_{p,r,\mathbb{R}\times(0,1)} \leq C \left(|\varphi|_{p,\sigma} + |\partial_y \varphi|_{p,\mathbb{R}\times(0,1)} \right) . \tag{6.29}$$

For m_B^y we apply, for fixed $(\eta, y) \in (0,\infty) \times (0,1)$, (6.16) and (B.1), and obtain that for fixed $(\eta, y) \in (0,\infty) \times (0,1)$, and arbitrary $k \in \mathbb{R}$,

$$|m_B^y(k, \eta, y)| \leq C \, \eta \sqrt{|k|}(1 + \sqrt{|k|}) e^{-|k|\eta} . \tag{6.30}$$

As before, a scaling argument now implies that for all $(\eta, y) \in (0,\infty) \times (0,1)$,

$$\sup_{k \in \mathbb{R}} |m_B^y(k, \eta, y)| \leq (1 + \sqrt{\eta}) . \tag{6.31}$$

Therefore, by definition of κ and $\dot{\kappa}$, see (B.1) and (B.2), respectively, we get that for all $(k, \eta, y) \in \mathbb{R} \times (0,\infty) \times (0,1)$,

$$|\partial_k m_B^y(k, \eta, y)| \leq C \left(\sqrt{|k|}(1 + \sqrt{|k|}) |\partial_k m_B(k, \eta, y)| + \left(1 + \frac{1}{\sqrt{|k|}} \right) |m_B(k, \eta, y)| \right) . \tag{6.32}$$

Using (6.16) and (6.18), we therefore get that

$$|k \partial_k m_B^y(k, \eta, y)| \leq C \left(\eta \sqrt{|k|}(1 + \sqrt{|k|})(1 + |k|) \right) e^{-|k|(\eta + y)} , \tag{6.33}$$

and using again a scaling argument we get that for all $(\eta, y) \in (0,\infty) \times (0,1)$,

$$\sup_{k \in \mathbb{R}} |k \partial_k m_B^y(k, \eta, y)| \leq C \, \frac{(1+\eta)^{\frac{3}{2}}}{\sqrt{\eta}} . \tag{6.34}$$

Therefore, we finally get that for all $(\eta, y) \in (0,\infty) \times (0,1)$,

$$|m_B^y(\cdot, \eta, y)|_{\mathcal{M}} \leq C \, \frac{(1+\eta)^{\frac{3}{2}}}{\sqrt{\eta}} , \tag{6.35}$$

and m_B^y therefore satisfies the assumptions of Lemma 22 with $s = -2$, $r = 1$ and all $p > 1$. Thus, applying (ii) of this very lemma $\partial_y \omega_B \in L^p(\mathbb{R} \times (0,1))$ for arbitrary $p > 1$, and for all $\sigma > 2 - 1/p$, we have that

$$|\partial_y \omega_B|_{p,r,\mathbb{R}\times(0,1)} \leq C \left(|\varphi|_{p,\sigma} + |\partial_y \varphi|_{p,\mathbb{R}\times(0,1)} \right) . \tag{6.36}$$

This concludes the proof of Lemma 24.

7 Proof of Theorem 13

In this section we discuss the case of $\varphi = \partial_x f$. We again set

$$\omega_V = T_V\{\mathcal{K}\}[\varphi] = \omega_V^{\mathrm{c}} + \omega_V^{\mathrm{nc}} , \tag{7.1}$$

with

$$\omega_V^{\mathrm{c}} = T_V\{\chi \mathcal{K}\}[\varphi] , \tag{7.2}$$

$$\omega_V^{\mathrm{nc}} = T_V\{(1 - \chi)\mathcal{K}\}[\varphi] , \tag{7.3}$$

so that

$$\omega = \omega_V^{\mathrm{c}} + \omega_V^{\mathrm{nc}} + \omega_B . \tag{7.4}$$

We now proceed as in the previous section and estimate all the components in the decomposition (7.4) independently. Theorem 13 then follows by the triangle inequality.

Lemma 25 *Let f be a smooth function with compact support, and let $(p, \sigma) \in [1, \infty) \times [0, \infty)$. Then,* (*i*) $\omega_V^{\mathrm{c}} \in L^p_\sigma(\mathbb{R}^2_{++})$, *and there exists a constant* $C(\sigma)$ *such that*

$$|\omega_V^{\mathrm{c}}|_{p,\sigma,\mathbb{R}^2_{++}} \le C(\sigma)\,|\varphi|_{p,\sigma} \ . \tag{7.5}$$

(*ii*) $\nabla\omega_V^{\mathrm{c}} \in L^p_\sigma(\mathbb{R}^2_{++})$, *and there exists a constant* $C(\sigma)$ *such that*

$$|\nabla\omega_V^{\mathrm{c}}|_{p,\sigma,\mathbb{R}^2_{++}} \le C(\sigma)\,|\varphi|_{p,\sigma} \ . \tag{7.6}$$

Lemma 26 *Let* f *be a smooth function with compact support and let* $\varphi = \partial_x f$. *Let* $(m_x, m_y) \in \mathbb{N}^2$ *and* $(p,\sigma) \in [1,\infty)\times[0,\infty)$, *and let* $m_{xy} = 2m_x + m_y + 4$. *Then,*

(*i*) $\partial_x^{m_x}\partial_y^{m_y}\omega_V^{\mathrm{nc}} \in L^p_\sigma(\mathbb{R}^2_{++})$, *provided* $\sigma < m_{xy}$ *and* $p > 3/(m_{xy}-\sigma)$.

(*ii*) *for all* $1 \le q < q^\sigma_{m_x,m_y}$, *with*

$$q^\sigma_{m_x,m_y} = \left(\max\left\{1 + \frac{1}{p} - \frac{m_{xy}-\sigma}{3}, 0\right\}\right)^{-1} \in [1,\infty] \ , \tag{7.7}$$

there exists a constant $C(p,q,\sigma) < \infty$ *such that*

$$\left|\partial_x^{m_x}\partial_y^{m_y}\omega_V^{\mathrm{nc}}\right|_{p,\sigma,\mathbb{R}^2_{++}} \le C(p,q,\sigma)\,|f|_{q,1+\sigma} \ . \tag{7.8}$$

Lemma 27 *Let* f *be a smooth function with compact support and let* $\varphi = \partial_x f$. *Let* $(m_x, m_y) \in \mathbb{N}^2$ *and* $(p,\sigma) \in [1,\infty)\times[0,\infty)$, *and let* $m_{xy} = 2m_x + m_y + 4$. *Then,*

(*i*) $\partial_x^{m_x}\partial_y^{m_y}\omega_B \in L^p_\sigma(\mathbb{R}^2_{++})$, *provided* $\sigma < m_{xy}$ *and* $p > 3/(m_{xy}-\sigma)$.

(*ii*) *for all* $1 \le q < \widetilde{q}^\sigma_{m_x,m_y}$, *with*

$$\widetilde{q}^\sigma_{m_x,m_y} = \left(1 - \min\left\{1, \frac{m_{xy}-\sigma-3/p}{2}\right\}\right)^{-1} \in [1,\infty] \ , \tag{7.9}$$

there exists a constant $C(p,q,\sigma) < \infty$ *such that*

$$\left|\partial_x^{m_x}\partial_y^{m_y}\omega_B\right|_{p,\sigma,\mathbb{R}^2_{++}} \le C(p,q,\sigma)\,|f|_{q,2-1/q} \ . \tag{7.10}$$

The remainder of this section is devoted to the proof of these lemmas.

7.1 Proof of Lemma 25

Again we write ω_V^{c} as a convolution, *i.e.*, $\omega_V^{\mathrm{c}} = (\chi\mathcal{K}) * \widetilde{\varphi}$, with $\widetilde{\varphi}(x,y) = \varphi(x,y)$ for $x \in \mathbb{R}$, and $y \ge 0$ and $\widetilde{\varphi}(x,y) = -\varphi(x,-y)$ for $x \in \mathbb{R}$, and $y < 0$. Therefore, there exists a constant $C(\sigma) < \infty$ such that

$$|(1+y)^\sigma\omega_V^{\mathrm{c}}| \le C(\sigma)\,\left(|(1+y)^\sigma\chi\mathcal{K}| * |\widetilde{\varphi}| + |\chi\mathcal{K}| * |(1+y)^\sigma\widetilde{\varphi}|\right) \ . \tag{7.11}$$

Using Proposition 3, we see that $\chi\mathcal{K} \in L^1_\sigma(\mathbb{R}^2)$ for all $\sigma \in [0,\infty)$, and, using Young's inequality for convolution, we get that

$$\omega_V^{\mathrm{c}} \in L^p_\sigma(\mathbb{R}^2_+) \ , \quad \text{and that} \quad |\omega_V^{\mathrm{c}}|_{p,\sigma} \le C(\sigma)\,|\varphi|_{p,\sigma} \ . \tag{7.12}$$

Similarly, using again Proposition 3, we get that $\nabla(\chi\mathcal{K}) \in L^1_\sigma(\mathbb{R}^2)$ for all $\sigma \in [0,\infty)$, and therefore we get, using again Young's inequality, that

$$\nabla\omega_V^{\mathrm{c}} \in L^p_\sigma(\mathbb{R}^2_+) \ , \quad \text{and that} \quad |\nabla\omega_V^{\mathrm{c}}|_{p,\sigma} \le C(\sigma)\,|\varphi|_{p,\sigma} \ . \tag{7.13}$$

7.2 Proof of Lemma 26

Using that $\varphi = \partial_x f$, we find that $\omega_V^{\rm nc} = T_V\{\partial_x[(1-\chi)\mathcal{K}]\}[f]$, and therefore we have for arbitrary integers $(m_x, m_y) \in \mathbb{N}^2$, that

$$\partial_x^{m_x}\partial_y^{m_y}\omega_V^{\rm nc} = T_V\{\partial_x^{m_x+1}\partial_y^{m_y}[(1-\chi)\mathcal{K}]\}[f] \ . \tag{7.14}$$

Now, given $p > 3/(m_{xy} - \sigma)$, and $q^{\sigma}_{m_x,m_y}$ as defined in (7.7), and given q satisfying $1 \leq q < q^{\sigma}_{m_x,m_y}$, we define r by the equation $1/r = 1/p + 1 - 1/q$. Since

$$r \cdot (2(m_x + 1) + (m_y + 1) - \sigma) > 3 \ , \tag{7.15}$$

and since $1 - \chi$ truncates away from zero, Proposition 41 implies that

$$\partial_x^{m_x+1}\partial_y^{m_y+1}[(1-\chi)\mathcal{K}] \in L^r_\sigma(\mathbb{R}^2) \ . \tag{7.16}$$

We can now apply Lemma 17 with p, q, r as defined above and with $\mathcal{G} = \partial_x^{m_x+1}\partial_y^{m_y}[(1-\chi)\mathcal{K}]$, and we get that

$$\partial_x^{m_x}\partial_y^{m_y}\omega_V^{\rm nc} \in L^p_\sigma(\mathbb{R}^2_+) \ , \text{ with } \left|\partial_x^{m_x}\partial_y^{m_y}\omega_V^{\rm nc}\right|_{p,\sigma} \leq C \ \ |f|_{q,\sigma+1} \ . \tag{7.17}$$

7.3 Proof of Lemma 27

Since $\varphi = \partial_x f$, we have the following representation for ω_0 :

Lemma 28 *There exists a function $\Omega_0 \in \mathcal{C}^\infty(\mathbb{R})$, such that:*

(i) for all $1 < p < \infty$, there exists a constant $C(p) < \infty$ depending only on p such that

$$|\Omega_0[f]|_p \leq C(p) \ \ |f|_{p,2-1/p} \ . \tag{7.18}$$

(ii) $\omega_0 = \partial_x \Omega_0$.

Proof. Since $\varphi = \partial_x f$, we have $\mathcal{F}[\varphi] = -ik\mathcal{F}[f]$. Therefore. we have for all $k \in \mathbb{R}$, that

$$\mathcal{F}[\omega_0](k) = -ik\mathcal{F}[\Omega_0](k) \ , \tag{7.19}$$

where

$$\Omega_0 = \int_0^\infty \mathcal{F}^{-1}\left[m_0(\cdot,\eta) \ \mathcal{F}f(\cdot,\eta)\right] \ d\eta \ , \tag{7.20}$$

with m_0 as defined in (1.12). Proceeding exactly as in the proof of Lemma 4, we obtain that Ω_0 is smooth with all derivatives tending to zero at infinity, and (7.19) implies that $\omega_0 = \partial_x\Omega_0$. The function Ω_0 satisfies the assumptions of Lemma 21 with multiplier m_0. Namely, using Proposition 43 for all fixed $\eta \in (0,\infty)$, we get that $|m_0(k,\eta)| \leq \eta e^{-|k|\eta}$, for all $k \in \mathbb{R}$. Therefore, we find that $\sup_k |m_0(k,\eta)| \leq \eta$ for $\eta > 0$, and we conclude that, for $k \neq 0$,

$$\partial_k m_0(k,\eta) = \eta \ \frac{\dot{\kappa}e^{-\kappa\eta} - \mathrm{sign}(k)e^{-|k|\eta}}{\kappa - |k|} - \frac{\dot{\kappa} - \mathrm{sign}(k)}{\kappa - |k|} m_0(k,\eta) \ . \tag{7.21}$$

Therefore we get, using Lemma 42, that

$$|k\partial_k m_0(k,\eta)| \leq C \ \eta \, (1 + |k|) \, e^{-|k|\eta} \ , \tag{7.22}$$

from which we get using a scaling argument that

$$\sup_{k\in\mathbb{R}\setminus\{0\}} |k\partial_k m_0(k,\eta)| \leq C \ \ (1+\eta) \ . \tag{7.23}$$

From (7.23) the bound (7.18) on Ω_0 now follows using Theorem 21 with $s = -1$ and $r = 0$. ■

Using Lemma 28 we can now prove Lemma 27. First, we note that for arbitrary integers $(m_x, m_y) \in \mathbb{N}^2$,

$$\omega_B = T_B\left\{\partial_x^{m_x+1}\partial_y^{m_y}\mathcal{K}\right\}[\Omega_0] \ . \tag{7.24}$$

This shows that ω_B satisfies the assumptions of Lemma 19. Namely, let $\sigma < m_{xy}$, $p > 3/(m_{xy} - \sigma)$, and q such that $1 \leq q < \widetilde{q}^{\sigma}_{m_x,m_y}$, and define r by the equation $1/r = 1/p + 1 - 1/q$. By definition of $\widetilde{q}^{\sigma}_{m_x,m_y}$, we have for all $q < \widetilde{q}^{\sigma}_{m_x,m_y}$, that

$$1 - \frac{1}{q} < \frac{1}{2}\left(m_{xy} - \sigma - \frac{3}{p}\right) \ , \tag{7.25}$$

since the definition of p implies that $m_{xy} - \sigma - 3/p > 0$. Combining the above inequalities, we get that

$$m_{xy} - \sigma > 2\left(\frac{1}{p} + 1 - \frac{1}{q}\right) + \frac{1}{p} \ , \tag{7.26}$$

so that finally

$$p\ \left(\sigma - \left[2(m_x + 1) + (m_y + 1) + 4 - \frac{2}{r}\right]\right) < -1 \ . \tag{7.27}$$

Hence, Proposition 40 implies that

$$\int_1^{\infty} (1 + y^{\sigma})^p \left[\int_{-\infty}^{\infty} |\partial_x^{m_x}\partial_y^{m_y}\mathcal{K}(x,y)|^r \ dx\right]^{\frac{p}{r}} \ dy < \infty \ . \tag{7.28}$$

We can therefore apply Lemma 19 with p, q, r as defined above and with $\mathcal{G} = \partial_x^{m_x+1}\partial_y^{m_y}\left[(1-\chi)\mathcal{K}\right]$, and we get that

$$\partial_x^{m_x}\partial_y^{m_y}\omega_B \in L^p_{\sigma}(\mathbb{R}^2_{++}) \ , \text{ and that } \left|\partial_x^{m_x}\partial_y^{m_y}\omega_B\right|_{p,\sigma,\mathbb{R}^2_{++}} \leq C\left|\Omega_0\right|_q \ . \tag{7.29}$$

Since by Lemma 28 $\left|\Omega_0\right|_q \leq C\ \ |f|_{q,2-1/q}$ we get the stated result.

8 Proof of Theorem 14

In this section we treat the case of $\varphi = \partial_y f$. We again set

$$\omega_V = T_V\{\mathcal{K}\}[\varphi] = \omega_V^{\mathrm{c}} + \omega_V^{\mathrm{nc}} \ , \tag{8.1}$$

with ω_V^{c} and ω_V^{nc} as defined in (7.2), (7.3). The first term on the right-hand side of (8.1) has compact support and its properties as a function in $L^p_{\sigma}(\mathbb{R}^2_{++})$ have already been studied in Lemma 25. It is therefore sufficient to discuss the second term. Using that $\varphi = \partial_y f$, integration by parts leads to

$$\omega_V^{\mathrm{nc}}(x,y) = \int_{\mathbb{R}^2_+} \left(\partial_y[(1-\chi)\mathcal{K}](x - x_0, y - y_0) + \partial_y[(1-\chi)\mathcal{K}](x - x_0, y + y_0)\right)\ f(x_0,y_0)\ dx_0dy_0 \ . \tag{8.2}$$

When comparing with the case of $\varphi = \partial_x f$ (see (7.14)) we see that the two terms below the integral in (8.2) are added and not subtracted and we therefore loose one power of decay in y at large values of y, when compared to the case of $\varphi = \partial_x f$. Moreover, since $\partial_y\mathcal{K}$ already decays one power less fast in y than $\partial_x\mathcal{K}$ (see Proposition 41), we expect that ω_V^{nc} decays two powers of y less fast than for the case of $\varphi = \partial_x f$, *i.e.*, only like $1/y^2$ instead of $1/y^4$. This is in contradiction with our expectations which were motivated in Section 3.2, and which stipulated a decay like $1/y^3$. The reason for this apparent contradiction is that there is a compensation between ω_B and ω_V^{nc}, *i.e.*, both terms decay like $1/y^2$, and we therefore have to use a different splitting if we want to prove a decay like $1/y^3$.

8.1 New splitting

Since the function f is of compact support, we can define a function F by

$$F(x,y) = \int_y^{\infty} f(x,z)\ dz \ , \tag{8.3}$$

and in the remainder of this section we reserve the symbol F for this function. By definition $f = -\partial_y F$, F is of compact support in $\overline{\mathbb{R}^2_+}$, and, in particular, we have that $F \in L^p_{\sigma}(\mathbb{R}^2_+)$. Moreover, by Lemma 8, we have that for all $(p,\sigma) \in [1,\infty) \times [0,\infty)$, and for all $\sigma' > \sigma + 1$,

$$|F|_{p,\sigma} \leq C(p,\sigma,\sigma')|f|_{p,\sigma'} \ . \tag{8.4}$$

We now use that $f = -\partial_y F$ in order to integrate once more by parts in (8.2). Namely, using in addition that by (1.14) and (8.3),

$$F(\cdot,0) = \int_0^\infty f(\cdot,z)\ dz = \mathcal{F}_1[f]\ , \tag{8.5}$$

with 1 standing for the function identically equal to one. We get that

$$\omega_V^{\mathrm{nc}} = -\widetilde{\omega}_V^{\mathrm{nc}} + \omega_B^{\mathrm{nc}}\ , \tag{8.6}$$

where

$$\widetilde{\omega}_V^{\mathrm{nc}} = T_V\{\partial_{yy}[(1-\chi)\mathcal{K}]\}[F]\ , \tag{8.7}$$

$$\omega_B^{\mathrm{nc}} = T_B\{(1-\chi)\mathcal{K}\}[\mathcal{F}_1[f]]\ . \tag{8.8}$$

The new volume term $\widetilde{\omega}_V^{\mathrm{nc}}$ is now again an expression in terms of T_V for which we have a compensation at large values of y. Moreover, the corresponding Green's function $\partial_y^2\mathcal{K}$ behaves at large values of y like the Green's function $\partial_x\mathcal{K}$. Therefore, as we prove below, the new volume term behaves at large values of y like $1/y^4$. The term ω_B^{nc} is the above mentioned term which behaves for large y like $1/y^2$. We now isolate the compensating term in ω_B. Since $\chi(x,y) = 0$, for all $y > 1$ and $x \in \mathbb{R}$, we have that, for $(x,y) \in \mathbb{R}^2_{++}$,

$$T_B\{(1-\chi)\mathcal{K}\}[F(\cdot,0)] = T_B\{\mathcal{K}\}[F(\cdot,0)]\ . \tag{8.9}$$

Also, since $\varphi = \partial_y f$ we have by (3.6) that $\omega_B = T_B\{\mathcal{K}\}[\mathcal{F}_{m_0}[\partial_y f]]$. Integration by parts gives that

$$\mathcal{F}_{m_0}[\partial_y f] = -\mathcal{F}_{\partial_\eta m_0}[f]\ . \tag{8.10}$$

Therefore we have, for $(x,y) \in \mathbb{R}^2_{++}$, that $\omega_B = -\omega_B^{\mathrm{nc}} + \widetilde{\omega}_B$, where $\widetilde{\omega}_B = T_B\{\mathcal{K}\}[\mathcal{F}_{\widetilde{m}_0}[f]]$, with

$$\widetilde{m}_0(k,\eta) = 1 - \partial_\eta m_0(k,\eta)\ . \tag{8.11}$$

In order to analyze the behavior of the new boundary term $\widetilde{\omega}_B$ we split it into two parts, an explicit part $\widetilde{\omega}_B^{\mathrm{expl}}$ which shows the expected behavior like $1/y^3$ and a remainder $\widetilde{\omega}_B^{\mathrm{rest}}$ which decays again like $1/y^4$. Explicitly, we set $\widetilde{\omega}_B = \widetilde{\omega}_B^{\mathrm{expl}} + \widetilde{\omega}_B^{\mathrm{rest}}$, with $\widetilde{\omega}_B^{\mathrm{expl}} = T_B\{\mathcal{K}\}[\mathcal{F}_{\widetilde{m}_0^{\mathrm{expl}}}[f]]$ and $\widetilde{\omega}_B^{\mathrm{rest}} = T_B\{\mathcal{K}\}[\mathcal{F}_{\widetilde{m}_0^{\mathrm{rest}}}[f]]$, where

$$\widetilde{m}_0^{\mathrm{expl}}(k,\eta) = \kappa\eta e^{-\kappa\eta}\ , \tag{8.12}$$

$$\widetilde{m}_0^{\mathrm{rest}}(k,\eta) = \widetilde{m}_0(k,\eta) - \widetilde{m}_0^{\mathrm{expl}}(k,\eta)\ . \tag{8.13}$$

8.2 Formulation of main lemmas

With the above notation, we have that, in $\mathbb{R}^2_{++}$,

$$\omega = \omega_V^{\mathrm{c}} - \widetilde{\omega}_V^{\mathrm{nc}} + \widetilde{\omega}_B^{\mathrm{expl}} + \widetilde{\omega}_B^{\mathrm{rest}}\ .$$

We now prove bounds for each of these terms separately. The required result then follows using the triangle inequality. The term ω_V^{c} has already been estimated in the previous sections, see Lemma 25. To bound $\widetilde{\omega}_V^{\mathrm{nc}}$, we use that the bounds in Proposition 41 on $\partial_y^2\mathcal{K}$ are the same as the bounds on $\partial_x\mathcal{K}$. Therefore, the proof in Lemma 26 can be repeated, and using Lemma 8 to bound F we get:

Lemma 29 *Let f be a smooth function of compact support and let $\varphi = \partial_y f$. Let $(m_x, m_y) \in \mathbb{N}^2$, $m_{xy} = 2m_x + m_y + 4$, and let $(p,\sigma) \in [1,\infty) \times [0,\infty)$. Then,*

- *(i) $\partial_x^{m_x}\partial_y^{m_y}\widetilde{\omega}_V^{\mathrm{nc}} \in L^p_\sigma(\mathbb{R}^2_{++})$, provided $\sigma < m_{xy}$ and $p > 3/(m_{xy} - \sigma)$.*
- *(ii) for all q, $1 \le q < q^\sigma_{m_x,m_y}$ and $q^\sigma_{m_x,m_y}$ as in (7.7), and all $\sigma' > 2 + \sigma$, there exists a constant $C(p,q,\sigma,\sigma') < \infty$, such that*

$$|\widetilde{\omega}_V^{\mathrm{nc}}|_{p,\sigma,\mathbb{R}^2_{++}} \le C(p,q,\sigma,\sigma')\,|f|_{q,\sigma'}\ . \tag{8.14}$$

The boundary terms $\widetilde{\omega}_V^{\mathrm{nc}}$ can be dealt with as in the preceding section. We get:

Lemma 30 *Let f be a smooth function of compact support and let $\varphi = \partial_y f$. Let $(m_x, m_y) \in \mathbb{N}^2$, $m_{xy} = 2m_x + m_y + 4$, and let $(p,\sigma) \in [1,\infty) \times [0,\infty)$. Then,*

(*i*) $\partial_x^{m_x} \partial_y^{m_y} \widetilde{\omega}_B^{\mathrm{rest}} \in L^p_\sigma(\mathbb{R}^2_{++})$, *provided $\sigma < m_{xy}$ and $p > 3/(m_{xy} - \sigma)$.*

(*ii*) *for all q, $1 \le q < \widetilde{q}^\sigma_{m_x,m_y}$, with $\widetilde{q}^\sigma_{m_x,m_y}$ as in (7.9), there exists a constant $C(p,q,\sigma) < \infty$, such that*

$$|\widetilde{\omega}_B^{\mathrm{rest}}|_{p,\sigma,\mathbb{R}^2_{++}} \le C(p,q,\sigma)\, |f|_{q,2-1/q} \ . \tag{8.15}$$

The term $\widetilde{\omega}_B^{\mathrm{expl}}$, finally, is the term decaying only like $1/y^3$ at infinity:

Lemma 31 *Let f be a smooth function of compact support and let $\varphi = \partial_y f$. Let $(m_x, m_y) \in \mathbb{N}^2$, $m_{xy} = 2m_x + m_y + 4$, and let $(p,\sigma) \in [1,\infty) \times [0,\infty)$. Then,*

(*i*) $\partial_x^{m_x} \partial_y^{m_y} \widetilde{\omega}_B^{\mathrm{expl}} \in L^p_\sigma(\mathbb{R}^2_{++})$, *provided $\sigma < m_{xy} - 1$ and $p > 3/(m_{xy} - 1 - \sigma)$.*

(*ii*) *for all r, $1 \le r < 2\widetilde{q}^\sigma_{m_x,m_y}/(\widetilde{q}^\sigma_{m_x,m_y}+2)$, with $\widetilde{q}^\sigma_{m_x,m_y}$ as in (7.9), there exists a constant $C(p,q,\sigma) < \infty$, such that*

$$|\widetilde{\omega}_B^{\mathrm{expl}}|_{p,\sigma,\mathbb{R}^2_{++}} \le C(p,r,\sigma) \left(|f|_{r,1} + |\varphi|_r \right) \ . \tag{8.16}$$

The remainder of this section is devoted to the proof of Lemmas 30 and 31.

8.3 Proof of Lemma 30

We prove that $\widetilde{\omega}_B^{\mathrm{rest}}$ satisfies a lemma equivalent to 28. Lemma 30 then follows using the same arguments as in the proof for Lemma 27. With this idea in mind, we define for $(k,\eta) \in \mathbb{R} \times (0,\infty)$, the multiplier $\widetilde{M}_0^{\mathrm{rest}}$ by

$$\widetilde{M}_0^{\mathrm{rest}}(k,\eta) = -\frac{\widetilde{m}_0^{\mathrm{rest}}(k,\eta)}{ik} \ . \tag{8.17}$$

We have the following technical result:

Proposition 32 *The multiplier $\widetilde{M}_0^{\mathrm{rest}}$ satisfies:*

(*i*) $\widetilde{M}_0^{\mathrm{rest}} \in \mathcal{C}^1(\mathbb{R} \setminus \{0\} \times (0,\infty))$.

(*ii*) *for all $\eta > 0$*

$$\sup_{k \ne 0} |\widetilde{M}_0^{\mathrm{rest}}(k,\eta)| + \sup_{k \ne 0} |k \partial_k \widetilde{M}_0^{\mathrm{rest}}(k,\eta)| \le C \frac{(1+\eta^3)}{\eta} \ . \tag{8.18}$$

Proof. We first prove (*i*). By definition, we have that

$$\widetilde{m}_0^{\mathrm{rest}}(k,\eta) = -\frac{\kappa(e^{-\kappa\eta} - 1) - |k|(e^{-|k|\eta} - 1)}{\kappa - |k|} - \kappa e^{-\kappa\eta} \ .$$

Consequently

$$\widetilde{M}_0^{\mathrm{rest}}(k,\eta) = \frac{\kappa(e^{-\kappa\eta} - 1 + \eta(\kappa - |k|)e^{-\kappa\eta}) - |k|(e^{-|k|\eta} - 1)}{ik(\kappa - |k|)} \ .$$

Since κ is smooth away from $k = 0$, we find that $\widetilde{M}_0^{\mathrm{rest}} \in \mathcal{C}^1(\mathbb{R} \setminus \{0\} \times (0,\infty))$. To prove (*ii*) we set $\widetilde{M}_0^{\mathrm{rest}} = \widetilde{M}_{0,1}^{\mathrm{rest}} - \widetilde{M}_{0,2}^{\mathrm{rest}}$, where

$$\widetilde{M}_{0,1}^{\mathrm{rest}}(k,\eta) = \frac{\kappa(e^{-\kappa\eta} - 1 + \eta\kappa e^{-\kappa\eta})}{ik(\kappa - |k|)} \ ,$$

$$\widetilde{M}_{0,2}^{\mathrm{rest}}(k,\eta) = \frac{\mathrm{sign}(k)(\kappa\eta e^{-\kappa\eta} + (e^{-|k|\eta} - 1))}{i(\kappa - |k|)} \ .$$

and prove (ii) for $\widetilde{M}^{\rm rest}_{0,1}$ and $\widetilde{M}^{\rm rest}_{0,2}$ separately. For $\widetilde{M}^{\rm rest}_{0,1}$ we obtain by Lemma 42, that for all $(k,\eta)\in\mathbb{R}^2_+$,

$$
\begin{aligned}
|\widetilde{M}^{\rm rest}_{0,1}(k,\eta)| &\le C\left(1+\frac{1}{\sqrt{|k|}}\right)\frac{1}{|\kappa-|k||}\left[e^{-\kappa\eta}-1+\eta\kappa e^{-\kappa\eta}\right],\\
&\le C\left(1+\frac{1}{\sqrt{|k|}}\right)\frac{1}{|\kappa-|k||}\Bigg\{\left|e^{-|k|\eta}-1+\eta|k|e^{-|k|\eta}\right|\\
&\qquad\qquad+\left|e^{-\kappa\eta}-e^{-|k|\eta}+\eta\kappa e^{-\kappa\eta}-\eta|k|e^{-|k|\eta}\right|\Bigg\}.
\end{aligned}
$$

Since $1/|\kappa-|k||\le C\,(1+1/\sqrt{|k|})$ we have that

$$\left(1+\frac{1}{\sqrt{|k|}}\right)\frac{1}{|\kappa-|k||}\le C\,\left(1+\frac{1}{|k|}\right).$$

Since furthermore $\left|e^{-|k|\eta}-1+\eta|k|e^{-|k|\eta}\right|$ vanishes like $|k|^2$ at $k=0$ and is bounded at infinity, we get that, for all $\eta>0$,

$$\sup_{k\neq 0}\left(1+\frac{1}{\sqrt{|k|}}\right)\frac{1}{|\kappa-|k||}\left|e^{-|k|\eta}-1+\eta|k|e^{-|k|\eta}\right|<\infty .$$

Using a scaling argument, we therefore find that there exists a constant C independent of η for which for all $\eta>0$,

$$\sup_{k\neq 0}\left(1+\frac{1}{\sqrt{|k|}}\right)\frac{1}{|\kappa-|k||}\left|e^{-|k|\eta}-1+\eta|k|e^{-|k|\eta}\right|<C(1+\eta) .$$

We also have that

$$\left|e^{-\kappa\eta}-e^{-|k|\eta}+\eta\kappa e^{-\kappa\eta}-\eta|k|e^{-|k|\eta}\right|\le \eta^2|\kappa-|k||\;|\kappa|\;e^{-|k|\eta} .$$

Therefore we get, using the bound in (B.1), that

$$\left(1+\frac{1}{\sqrt{|k|}}\right)\frac{1}{|\kappa-|k||}\left|e^{-\kappa\eta}-e^{-|k|\eta}+\eta\kappa e^{-\kappa\eta}-\eta|k|e^{-|k|\eta}\right|\le C\eta^2\,(1+|k|)^2\,e^{-|k|\eta} .$$

Using again a scaling argument we find that for all $\eta>0$,

$$\sup_{k\neq 0}\left(1+\frac{1}{\sqrt{|k|}}\right)\frac{1}{|\kappa-|k||}\left|e^{-\kappa\eta}-e^{-|k|\eta}+\eta\kappa e^{-\kappa\eta}-\eta|k|e^{-|k|\eta}\right|\le C(1+\eta)^2 .$$

As a consequence,

$$\left|\partial_k\widetilde{M}^{\rm rest}_{0,1}(k,\eta)\right|\le\left|\frac{|\dot\kappa-\mathrm{sign}(k)|}{|\kappa-|k||}+\frac{1}{|k|}\right|\left|\widetilde{M}^{\rm rest}_{0,1}(k,\eta)\right|+\left|\frac{\dot\kappa(e^{-\kappa\eta}-1+\eta\kappa e^{-\kappa\eta})}{k(\kappa-|k|)}\right|+\left|\frac{\eta^2\dot\kappa\kappa^2e^{-\kappa\eta}}{k(\kappa-|k|)}\right| .$$

By Lemma 42 we have that

$$
\begin{aligned}
|k|\left|\frac{|\dot\kappa-\mathrm{sign}(k)|}{|\kappa-|k||}+\frac{1}{|k|}\right|&\le C\,\left(1+\min\left\{|k|,\frac{1}{|k|}\right\}\right),\\
|k|\left|\frac{\eta^2\dot\kappa\kappa^2e^{-\kappa\eta}}{k(\kappa-|k|)}\right|&\le C\eta^2|k|(1+|k|)^2e^{-|k|\eta} .
\end{aligned}
$$

Therefore, we find that, for all $\eta>0$,

$$\sup_{k\neq 0}|k|\left|\frac{|\dot\kappa-\mathrm{sign}(k)|}{|\kappa-|k||}+\frac{1}{|k|}\right|\left|\widetilde{M}^{\rm rest}_{0,1}(k,\eta)\right|\le\sup_{k\neq 0}|\widetilde{M}^{\rm rest}_{0,1}(k,\eta)| ,$$

and that

$$\sup_{k\neq 0}|k|\left|\frac{\eta^2\dot{\kappa}\kappa^2 e^{-\kappa\eta}}{k(\kappa-|k|)}\right| \leq C\frac{(1+\eta)^2}{\eta} \ .$$

Finally, we use that

$$|k|\left|\frac{\dot{\kappa}(e^{-\kappa\eta}-1+\eta\kappa e^{-\kappa\eta})}{k(\kappa-|k|)}\right| \leq C\ \left(1+\frac{1}{|k|}\right)|e^{-\kappa\eta}-1+\eta\kappa e^{-\kappa\eta}| \ ,$$

and get by a scaling argument that

$$\sup_{k\neq 0}|k|\left|\frac{\dot{\kappa}(e^{-\kappa\eta}-1+\eta\kappa e^{-\kappa\eta})}{k(\kappa-|k|)}\right| \leq C\ (1+\eta) \ .$$

Collecting all the bounds we find that we have proved that

$$\sup_{k\neq 0}|k\partial_k\widetilde{M}_{0,1}^{\mathrm{rest}}(k,\eta)| \leq C\frac{(1+\eta)^3}{\eta} \ .$$

For $\widetilde{M}_{0,2}^{\mathrm{rest}}$, we obtain by Lemma 42, that for all $(k,\eta)\in\mathbb{R}_+^2$,

$$|\widetilde{M}_{0,2}^{\mathrm{rest}}(k,\eta)| \leq C\ \left[(1+|k|)e^{-|k|\eta}+\left(1+\frac{1}{\sqrt{|k|}}\right)|e^{-|k|\eta}-1|\right] \ . \tag{8.19}$$

The last term in (8.19) is bounded, since $e^{-|k|\eta}-1$ vanishes like $|k|$ at $k=0$. Therefore, we find by a scaling argument, that for all $\eta>0$,

$$\sup_{k\neq 0}|\widetilde{M}_{0,2}^{\mathrm{rest}}(k,\eta)| \leq C\ \left[\frac{1}{\eta}+\sqrt{\eta}\right] \ .$$

Next,

$$|\partial_k\widetilde{M}_{0,2}^{\mathrm{rest}}(k,\eta)| \leq \left|\frac{\dot{\kappa}-\mathrm{sign}(k)}{\kappa-|k|}\right||\widetilde{M}_{0,2}^{\mathrm{rest}}(k,\eta)|+\left|\frac{\dot{\kappa}\eta e^{-\kappa\eta}-\dot{\kappa}\kappa\eta^2e^{-\kappa\eta}-\mathrm{sign}(k)\eta e^{-|k|\eta}}{(\kappa-|k|)}\right| \ .$$

As before we therefore find that

$$\sup_{k\neq 0}|k|\left|\frac{\dot{\kappa}-\mathrm{sign}(k)}{\kappa-|k|}\right||\widetilde{M}_{0,2}^{\mathrm{rest}}(k,\eta)| \leq C\sup_{k\neq 0}|\widetilde{M}_{0,2}^{\mathrm{rest}}(k,\eta)| \ ,$$

and that

$$|k|\left|\frac{\dot{\kappa}\eta e^{-\kappa\eta}-\dot{\kappa}\kappa\eta^2e^{-\kappa\eta}-\mathrm{sign}(k)\eta e^{-|k|\eta}}{(\kappa-|k|)}\right| \leq C\left[\eta\,(1+|k|)+\eta^2(1+|k|)^2\right]e^{-|k|\eta} \ .$$

Therefore, using again a scaling argument, we finally get that for all $\eta>0$,

$$\sup_{k\neq 0}|k\partial_k\widetilde{M}_{0,2}^{\mathrm{rest}}(k,\eta)| \leq C\,(1+\eta)^2 \ .$$

■

With these technical results at hand we can now prove the following lemma.

Lemma 33 *Let f be a smooth function with compact support and let*

$$\Omega_0^{\mathrm{rest}} = \int_0^\infty \mathcal{F}^{-1}\left[\widetilde{M}_0^{\mathrm{rest}}(\cdot,\eta)\mathcal{F}f(\cdot,\eta)\right]\ d\eta \ . \tag{8.20}$$

Then,

(*i*) $\Omega_0^{\mathrm{rest}}\in\mathcal{C}^\infty(\mathbb{R})$, (*ii*) *for all* $p > 1$ *there exists a constant* $C(p) < \infty$ *such that* $|\Omega_0^{\text{rest}}|_p \leq C(p)\ |f|_{p,2-1/p}$.

(*iii*) $\mathcal{F}_{\widetilde{m}_0^{\text{rest}}}[\varphi] = \partial_x \Omega_0^{\text{rest}}$.

Proof. (*i*) follows because f is a smooth function with compact support, and since by definition $\mathcal{F}[\Omega_0^{\text{rest}}](k) = -ik\mathcal{F}_{\widetilde{m}_0^{rest}}[\varphi]$, for all $k \in \mathbb{R}$, we get (*iii*). To prove (*ii*) we apply Lemma 32. The multiplier $\widetilde{M}_0^{\text{rest}}$ satisfies the assumptions of Theorem 21 with $s = 3$, $r = 1$ and any p. Therefore, $\Omega_0^{\text{rest}} \in L^p(\mathbb{R})$, for all $p \in (1, \infty)$, and

$$\left|\Omega_0^{\text{rest}}\right|_p \leq C \left(|\varphi|_p + |f|_{p,2-1/p} \right) \ . \tag{8.21}$$

This completes the proof of Lemma 33. ∎

The remainder of the proof of Lemma 30 is identical to the proof in Section 7.3.

8.4 Proof of Lemma 31

For $\widetilde{\omega}_B^{\text{expl}}$ the multiplier technique does not give sufficient information. We therefore first find a new representation for $\widetilde{\omega}_B^{\text{expl}}$.

Lemma 34 *Let* $\widetilde{\omega}_B^{\text{expl}}$ *be as above. Then,*

(*i*) $\widetilde{\omega}_B^{\text{expl}} = \partial_x \Omega_0^{\text{expl}}$, *where for* $x \in \mathbb{R}$,

$$\Omega_0^{\text{expl}}(x) = \int_{\mathbb{R}_+^2} (\partial_x \mathcal{K}(x - x_0, \eta) - \mathcal{K}(x - x_0, \eta)) f(x_0, \eta)\ d\eta \ . \tag{8.22}$$

(*ii*) $\Omega_0^{\text{expl}} \in L^p(\mathbb{R})$ *for all* $p > 2$.

(*iii*) *for all* $p > 2$ *there exists* $C(p) < \infty$ *such that for* $r < 2p/(2 + p)$

$$|\Omega_0^{\text{expl}}|_p \leq C(q) \left(|f|_{r,1} + |\varphi|_r \right) \ . \tag{8.23}$$

Proof. To prove (*i*) we first note that

$$\mathcal{F}[\mathcal{K}](k, y) = -\frac{e^{-\kappa y}}{\kappa} \ . \tag{8.24}$$

Therefore, we have for all $x \in \mathbb{R}$,

$$\Omega_0^{\text{expl}}(x) = \int_{\mathbb{R}_+^2} \partial_{yy} \mathcal{K}(x - x_0, y) f(x_0, y)\ dx_0 dy \ . \tag{8.25}$$

But, by definition, we have that away from $x = y = 0$, $\partial_{yy}\mathcal{K} = \partial_x \mathcal{K} - \partial_{xx}\mathcal{K}$. Substituting this identity into the integral (8.25) and using standard properties of convolution and parameter integrals we get the result. To prove (*ii*) we set $\Omega_0^{\text{expl}} = \Omega_{0,1}^{\text{expl}} + \Omega_{0,2}^{\text{expl}}$, where

$$\Omega_{0,1}^{\text{expl}}(x) = \int_0^1 \int_{-\infty}^{\infty} (\partial_x \mathcal{K}(x - x_0, \eta) - \mathcal{K}(x - x_0, \eta)) f(x_0, \eta)\ dx_0 d\eta \ , \tag{8.26}$$

$$\Omega_{0,2}^{\text{expl}}(x) = \int_1^{\infty} \int_{-\infty}^{\infty} (\partial_x \mathcal{K}(x - x_0, \eta) - \mathcal{K}(x - x_0, \eta)) f(x_0, \eta)\ dx_0 d\eta \ . \tag{8.27}$$

We have that

$$|\Omega_{0,1}^{\text{expl}}|_p^p \leq \int_{-\infty}^{\infty} \left| \int_0^1 \int_{-\infty}^{\infty} |\partial_x \mathcal{K}(x - x_0, \eta) - \mathcal{K}(x - x_0, \eta)|\ |f(x_0, \eta)|\ dx_0 d\eta \right|^p\ dx \ . \tag{8.28}$$

Since, for all $(x,\eta) \in \mathbb{R} \times (0,\infty)$,

$$|\partial_x \mathcal{K}(x,\eta) - \mathcal{K}(x,\eta)| \leq \frac{C}{\sqrt{x^2+\eta^2}} \ , \tag{8.29}$$

we find that for all $0 < \eta < 1$ and $x \in \mathbb{R}$

$$\eta|\partial_x \mathcal{K}(x,\eta) - \mathcal{K}(x,\eta)| \leq \frac{C}{\sqrt{x^2+1}} \ . \tag{8.30}$$

Therefore,

$$|\Omega_{0,1}^{\mathrm{expl}}|_p^p \leq \int_{-\infty}^{\infty} \left| \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+1}} \left| \int_0^1 \frac{|f(x_0,\eta)|}{\eta} \ d\eta \right| \ dx_0 \right) \right|^p dx \ . \tag{8.31}$$

Applying Young's inequality, with (p,q,r) satisfying $p > 2$, $r < 2p/(2+p)$ and $1/q + 1/r = 1 + 1/p$, we get, since $x \mapsto 1/\sqrt{x^2+1} \in L^q(\mathbb{R})$, that

$$|\Omega_{0,1}^{\mathrm{expl}}|_p^p \leq C \left[\int_{-\infty}^{\infty} \left(\int_0^1 \frac{|f(x,\eta)|}{\eta} d\eta \right)^r dx \right]^{\frac{p}{r}} \ , \tag{8.32}$$

where $r > 1$. Therefore, Jensen's inequality implies that

$$|\Omega_{0,1}^{\mathrm{expl}}|_p^p \leq C \left[\int_{-\infty}^{\infty} \int_0^1 \left| \frac{f(x,\eta)}{\eta} \right|^r d\eta \ dx \right]^{\frac{p}{r}} \ . \tag{8.33}$$

Applying Poincaré's inequality, and using that f vanishes a $y = 0$ and that $\partial_y f = \varphi$, we finally obtain that

$$|\Omega_{0,1}^{\mathrm{expl}}|_p \leq C \, |\varphi|_r \ .$$

For $\Omega_{0,2}^{\mathrm{expl}}$ we have, for $\eta > 1$, that $|\partial_x \mathcal{K}(x,\eta) - \mathcal{K}(x,\eta)| \leq \frac{C}{\sqrt{x^2+1}}$. Consequently,

$$|\Omega_{0,2}^{\mathrm{expl}}|_p^p \leq \int_{-\infty}^{\infty} \left| \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+1}} \left| \int_1^{\infty} |f(x_0,\eta)| \ d\eta \right| \ dx_0 \right) \right|^p dx \ . \tag{8.34}$$

Therefore, we find as above that

$$|\Omega_{0,2}^{\mathrm{expl}}|_p^p \leq C \left[\int_{-\infty}^{\infty} \left(\int_1^{\infty} |f(x,\eta)| d\eta \right)^r dx \right]^{\frac{p}{r}} \ . \tag{8.35}$$

By Hölder's inequality we have

$$\int_1^{\infty} |f(t,y)| \ dy \leq \left(\int_1^{\infty} \frac{1}{(1+\eta)^{r'}} d\eta \right)^{1/r'} \left(\int_1^{\infty} (1+\eta)^r |f(t,\eta)|^r \ dy \ dt \right)^{1/r} \ , \tag{8.36}$$

with $r' > 1$ such that $1/r' + 1/r = 1$. Therefore, we finally get that $\left|\Omega_{0,2}^{\mathrm{expl}}\right|_p \leq C \, |f|_{r,1}$. This completes the proof. ■

Using Lemma 34, the proof of Lemma 27 can now be repeated to prove Lemma 31, provided we choose q such that $2 < q \leq \widetilde{q}_{m_x,m_y}^{\sigma}$. This is possible, provided $\widetilde{q}_{m_x,m_y}^{\sigma} > 2$, which is equivalent to requiring that its conjugate exponent is smaller than two, *i.e.*, that

$$\frac{2}{m_{xy} - \sigma - 3/p} < 2 \ . \tag{8.37}$$

The condition (8.37) is satisfied in particular if $\sigma < m_{xy} - 1$, and if $p < (m_{xy} - 1 - \sigma)/3$. Now, given $1 < r < 2\widetilde{q}_{m_x,m_y}^{\sigma}/(2 + \widetilde{q}_{m_x,m_y}^{\sigma})$ let q be defined by $1/q = 1/r - 1/2$, hence, $2 < q < \widetilde{q}_{m_x,m_y}^{\sigma}$. Therefore, applying the method of proof of Lemma 27 we get that

$$|\partial_x^{m_x} \partial_y^{m_y} T_B\{\mathcal{K}\}[\widetilde{\omega}_B^{\mathrm{expl}}]|_{p,\sigma} \leq C \ |\Omega_0^{\mathrm{expl}}|_q \ , \tag{8.38}$$

and, with the above lemma that

$$|\partial_x^{m_x} \partial_y^{m_y} T_B\{\mathcal{K}\}[\widetilde{\omega}_B^{\mathrm{expl}}]|_{p,\sigma} \leq C \ \left(|f|_{r,1} + |\varphi|_r\right) \ . \tag{8.39}$$

This completes the proof.

A The Green's function $\mathcal{K}$

We start by recalling some well known properties of the Bessel functions K_0 and $K_1 = -K_0'$ (the prime denotes the derivative). See for example [13], sections 5.6 to 5.11, for more details. We have:

(i) the function K_0 is infinitely differentiable on $(0, \infty)$,

(ii) for small values of $z > 0$ we have that $K_0(z) = -\log(z) + O(1)$, and $K_1(z) = 1/z + O(z)$,

(iii) for large values of $z > 0$ we have that

$$K_0(z) = \sqrt{\frac{\pi}{2z}} e^{-z}(1 + o(1)) \quad \text{and} \quad K_1(z) = \sqrt{\frac{\pi}{2z}} e^{-z}(1 + o(1)) \ . \tag{A.1}$$

Let now $\mathcal{K}$ be as defined in (3.1). We divide the analysis of $\mathcal{K}$ into an analysis of the behavior close to the origin and an analysis far away from the origin.

A.1 Behavior close to the origin

We first prove the Proposition 3:
Proof. Let $\mathcal{B}_R \subset \mathbb{R}^2$ be the ball of radius R centered at zero and let $r = \sqrt{x^2 + y^2}$. Using the properties of K_0 we have by definition that $\mathcal{K} \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0, 0)\})$ and that $\mathcal{K} \in L^1(\mathcal{B}_R)$ for all $R < \infty$. For $\partial_x \mathcal{K}$ we have

$$\partial_x \mathcal{K}(x, y) = -\frac{e^{\frac{x}{2}}}{4\pi} \left(K_0(\frac{r}{2}) - \frac{x}{r} K_1(\frac{r}{2}) \right) \ , \tag{A.2}$$

and an explicit expression for $\partial_y \mathcal{K}$ is given in (3.9). Using the properties of the functions K_0 and K_1, we find that for all $R < \infty$,

$$|\partial_x \mathcal{K}(x, y)| + |\partial_y \mathcal{K}(x, y)| \leq C \frac{y}{x^2 + y^2} \ , \tag{A.3}$$

and therefore $\partial_x \mathcal{K}$ and $\partial_y \mathcal{K}$ are in $L^1(\mathcal{B}_R)$. Finally, since, by definition of $\mathcal{K}$, $\Delta \mathcal{K}(x, y) = \partial_x \mathcal{K}(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, the bound on $\partial_x \mathcal{K}$ implies the bound on $\Delta \mathcal{K}$. ■

A.2 Asymptotic behavior

We analyze separately the pointwise behavior and the behavior in the mean.

A.2.1 Pointwise decay at infinity

In order to state our results we use the following conventions concerning asymptotic expansions:

Definition 35 *Given $r_0 > 0$ and $\varepsilon \in \mathcal{C}^\infty((r_0, \infty) \times (0, \pi))$ we say that*

(i) *ε admits an asymptotic expansion of order p, as $\rho \to \infty$, if there exist trigonometric polynomials $(\gamma_0, \ldots, \gamma_{p-1})$ and a bounded function $\eta \in \mathcal{C}^\infty((r_0, \infty) \times (0, \pi))$ such that*

$$\varepsilon(\rho, \theta) = \sum_{k=0}^{p-1} \frac{\gamma_k(\theta)}{\rho^k} + \frac{\eta(\rho, \theta)}{\rho^p} \ . \tag{A.4}$$

(ii) *ε admits an infinite asymptotic expansion, if it admits asymptotic expansions of arbitrary order $p \geq 1$.*

Note that we require in our definition polynomial coefficients γ_i. This is for technical convenience only. Some straightforward properties of functions admitting infinite asymptotic expansions are:

(a) the set of functions admitting infinite asymptotic expansions is stable under sum, multiplication and multiplication by trigonometric polynomials.

(b) if ε and $\partial_\rho \varepsilon$ admit infinite asymptotic expansions then, the first order term γ_0 in the asymptotic expansion of $\partial_\rho \varepsilon$ vanishes.

(c) if $\varepsilon, \partial_\rho \varepsilon, \partial_\theta \varepsilon$ admit infinite asymptotic expansions, and if there is a trigonometric polynomial γ_0 and a remainder η, such that for all $(\rho, \theta) \in (r_0, \infty) \times (0, \pi)$

$$\varepsilon(\rho, \theta) = \gamma_0(\theta) + \frac{\eta(\rho, \theta)}{\rho} \ , \tag{A.5}$$

then $(\eta, \partial_\rho \eta, \partial_\theta \eta)$ also admit infinite asymptotic expansions.

For $\partial_x^m \mathcal{K}$ we have:

Lemma 36 *Let $m \geq 0$. Then, there exists $r_m > 0$, a family $(\alpha_n^m)_{n<m} \in [\mathcal{C}^\infty(0, \pi)]^m$, and functions $\varepsilon_m \in \mathcal{C}^\infty((r_m, \infty) \times (0, \pi))$ such that:*

(i) α_n^m is a trigonometric polynomial for all $n \in \{0, \ldots, m-1\}$.

(ii) there exists $\theta_m > 0$ and $(c_n^m)_{n \leq m} \in (0, \infty)$ such that $|\alpha_n^m(\theta)| \leq c_n^m \theta^{2(m-n)}$ for all $\theta \in (0, \theta_m)$.

(iii) ε_m and all its derivatives admit infinite asymptotic expansions.

(iv)

$$\partial_x^m \mathcal{K}(x, y) = \frac{e^{\rho(\cos(\theta)-1)/2}}{\sqrt{\rho}} \left[\sum_{n=0}^{m-1} \frac{\alpha_n^m(\theta)}{\rho^n} + \frac{\varepsilon_m(\rho, \theta)}{\rho^m} \right] \ . \tag{A.6}$$

Proof. We prove the lemma by induction over m. First, we have that $K_0(\rho/2) = e^{-\rho/2}\varepsilon_0(\rho)/\sqrt{\rho}$, with ε_0 admitting an infinite asymptotic expansion, and $K_1(\rho/2) = e^{-\rho/2}\widetilde{\varepsilon}_0(\rho)/\sqrt{\rho}$, with $\widetilde{\varepsilon}_1$ admitting an infinite asymptotic expansion (see [13, (5.11.9) p.123]). Therefore, the derivative ε_0', which is given by

$$\varepsilon_0'(\rho) = 2\widetilde{\varepsilon}_0(\rho) + \frac{1}{2}\varepsilon_0(\rho) + \frac{\varepsilon_0(\rho)}{2\rho} \ ,$$

also admits an infinite asymptotic expansions. We have that $K_1'(z) = -(K_0(z) + K_1(z)/z)$, and therefore ε_0'' admit infinite asymptotic expansions. Iterating this process, we get that all derivatives of ε_0 admit infinite asymptotic expansion. Consequently.

$$\mathcal{K}(\rho, \theta) = \frac{e^{\rho\cos(\theta)}}{2\pi} K_0(\rho/2) = \frac{e^{\rho(\cos(\theta)-1)}}{2\pi\sqrt{\rho}} \varepsilon_0(\rho) \ ,$$

satisfies the statement for $m = 0$ with ε_0 independent of θ. Assuming now that the assertion is satisfied by $\partial_x^m \mathcal{K}$, we have that

$$\partial_x^m \mathcal{K}(x, y) = e^{\rho\cos(\theta)/2} F(\rho, \theta) \ .$$

with

$$F(\rho, \theta) = e^{-\rho/2} \left[\sum_{n=0}^{m-1} \frac{\alpha_n^m(\theta)}{\rho^{n+1/2}} + \frac{\varepsilon_m(\rho, \theta)}{\rho^{m+1/2}} \right] \ .$$

Using polar coordinates, we get that

$$\partial_x^{m+1} \mathcal{K}(x, y) = e^{\rho\cos(\theta)/2} \left(\frac{F(\rho, \theta)}{2} + \cos(\theta)\partial_\rho F(\rho, \theta) - \frac{\sin(\theta)}{\rho} \partial_\theta F(\rho, \theta) \right) \ .$$

Substituting F yields

$$\begin{aligned} \partial_x^{m+1} \mathcal{K}(x, y) = \frac{e^{\rho(\cos(\theta)-1)/2}}{\sqrt{\rho}} & \left\{ \sum_{n=0}^{m-1} \frac{\alpha_n^{m+1}}{\rho^{n+\frac{1}{2}}} \right. \\ + \frac{1}{\rho^{m+\frac{1}{2}}} & \left[\frac{(1-\cos(\theta))\varepsilon_m(\rho, \theta)}{2} - \cos(\theta)\left(m - \frac{1}{2}\right)\alpha_{m-1}^m(\theta) - \sin(\theta)\partial_\theta \alpha_{m-1}^m(\theta) + \cos(\theta)\partial_\rho \varepsilon_m(\rho, \theta) \right] \\ & \left. - \frac{1}{\rho^{m+\frac{3}{2}}} \left[\left(m + \frac{1}{2}\right)\cos(\theta)\varepsilon_m(\rho, \theta) + \sin(\theta)\partial_\theta \varepsilon_m(\rho, \theta) \right] \right\} \ , \end{aligned}$$

where

$$\begin{cases} \alpha_0^{m+1}(\theta) &=& \dfrac{(1-\cos(\theta))\alpha_0^m(\theta)}{2} \ , \\ \alpha_n^{m+1}(\theta) &=& \dfrac{(1-\cos(\theta))\alpha_n^m(\theta)}{2} - \left(n-\frac{1}{2}\right)\cos(\theta)\alpha_{n-1}^m(\theta) - \sin(\theta)\partial_\theta\alpha_{n-1}^m(\theta) \ , \quad \forall\, 0<n<m \end{cases}$$

Using now that ε_m and $\partial_\rho\varepsilon_m$ admit asymptotic expansions of order one, and applying the above property (b) to $\partial_\rho\varepsilon_m$, we find that there exists a trigonometric polynomial γ and $(\widetilde{\varepsilon}_m, \widetilde{\varepsilon}'_m) \in [\mathcal{C}^\infty((r_{m+1},\infty)\times(0,\pi))]^2$ (for some $r_{m+1}>0$) such that for all $r>r_{m+1}$,

$$\varepsilon_m(\rho,\theta) = \gamma(\theta) + \frac{\widetilde{\varepsilon}_m(\rho,\theta)}{\rho} \ , \quad \text{and} \quad \partial_\rho\varepsilon_m(\rho,\theta) = \frac{\widetilde{\varepsilon}'_m(\rho,\theta)}{\rho} \ .$$

We therefore get that $\partial_x^{m+1}\mathcal{K}$ has the form (A.6) with

$$\begin{cases} \alpha_0^{m+1}(\theta) &=& \dfrac{(1-\cos(\theta))\alpha_0^m(\theta)}{2} \ , \\ \alpha_n^{m+1}(\theta) &=& \dfrac{(1-\cos(\theta))\alpha_n^m(\theta)}{2} - \left(n-\frac{1}{2}\right)\cos(\theta)\alpha_{n-1}^m(\theta) - \sin(\theta)\partial_\theta\alpha_{n-1}^m(\theta) \ , \\ \alpha_m^{m+1}(\theta) &=& \dfrac{(1-\cos(\theta))\gamma(\theta)}{2} - \cos(\theta)\left(m-\frac{1}{2}\right)\alpha_{m-1}^m(\theta) - \sin(\theta)\partial_\theta\alpha_{m-1}^m(\theta) \ , \\ \varepsilon_{m+1}(\rho,\theta) &=& \dfrac{(1-\cos(\theta))\widetilde{\varepsilon}_m(\rho,\theta)}{2} + \cos(\theta)\left(\widetilde{\varepsilon}'_m(\rho) - \left(m+\frac{1}{2}\right)\varepsilon_m(\rho,\theta)\right) - \sin(\theta)\partial_\theta\varepsilon_m(\rho,\theta) \ . \end{cases}$$

From these expressions, and using that $(\alpha_n^m)_{0\le n\le m-1}$ and γ are trigonometric polynomials, we obtain that $(\alpha_n^{m+1})_{0\le n\le m}$ are also trigonometric polynomials. This proves (i). Next, we have, applying the induction hypothesis, that there exists $\theta_m>0$ for which for all $\theta\in(0,\theta_m)$,

$$|\alpha_0^{m+1}(\theta)| \le |1-\cos(\theta)|c_m^0\theta^{2m}/2 \le c_{m+1}^0\theta^{2(m+1)} \ .$$

Then, for $n\ge1$, we have that

$$|\alpha_n^{m+1}(\theta)| \le \left|\frac{(1-\cos(\theta))\alpha_n^m(\theta)}{2}\right| + \left|\left(n-\frac{1}{2}\right)\alpha_{n-1}^m(\theta)\right| + \left|\frac{\sin(\theta)\partial_\theta\alpha_{n-1}^m(\theta)}{2}\right| \ .$$

Using the induction hypothesis, we can find constants $\hat{c}_{m+1}^n$ for which for all $\theta\in(0,\theta_m)$

$$\left|\frac{(1-\cos(\theta))\alpha_n^m(\theta)}{2}\right| \le \hat{c}_{m+1}^n\theta^{2(m+1-n)} \ ,$$

and

$$\left|\left(n-\frac{1}{2}\right)\alpha_{n-1}^m(\theta)\right| \le \hat{c}_{m+1}^n\theta^{2(m-(n-1))} \le \hat{c}_{m+1}^n\theta^{2(m+1-n)} \ .$$

Finally, since α_{n-1}^m is polynomial in $\cos(\theta)$ and $\sin(\theta)$, it can be written as a a power series in θ. Using again the induction hypothesis we find that the first $2(m-(n-1))$ terms in this power series vanish. Consequently, we get, that there exists a family of coefficients γ_p for which

$$\partial_\theta\alpha_{n-1}^m(\theta) = \sum_{p\ge 2(m-(n-1))} p\gamma_p\theta^{p-1} \ ,$$

and therefore there exist c_{m+1}^n, for which, for θ sufficiently small,

$$|\sin(\theta)\partial_\theta\alpha_{n-1}^m(\theta)| \le \hat{c}_{m+1}^n\theta^{2(m-(n-1))} \ .$$

Combining these inequalities, there exists θ_{m+1} sufficiently small for which for all $\theta\in(0,\theta_{m+1})$

$$|\alpha_n^{m+1}(\theta)| \le c_n^{m+1}\theta^{2(m+1-n)} \ .$$

Similarly, we obtain that

$$|\alpha_m^{m+1}(\theta)| \leq c_n^{m+1}\theta^{2(m+1-n)}, \quad \forall\,\theta \in (0,\theta_{m+1}) \ .$$

This proves (ii). Finally, applying the above property (c) to ε_m, we get that $\widetilde{\varepsilon}_m, \widetilde{\varepsilon}'_m, \partial_\theta \varepsilon_m$ and their derivatives admit also infinite asymptotic expansion. Consequently, the above property (a) implies that ε_{m+1} and its derivatives admit infinite asymptotic expansion, and therefore we get (iii). Finally, (iv) follows by construction. The stated result now follows by induction. ■

Using that ε_m in (A.6) admits an asymptotic expansion of order 0, we get that it is bounded and we have:

Proposition 37 *For all $m \in \mathbb{N}$, there exists $r_m > 0$, such that:*

(i) there exists a constant $C > 0$, such that

$$|\partial_x^m \mathcal{K}(x,y)| \leq C e^{\rho(\cos(\theta)-1)/2} \ , \quad \forall\,(\rho,\theta) \in (r_m,\infty)\times(0,\pi) \ . \tag{A.7}$$

(ii) there exists $\theta_m > 0$ and $0 < c_m < C_m < \infty$, such that

$$|\partial_x^m \mathcal{K}(x,y)| \leq C_m e^{-c_m\rho\theta^2} \sum_{n=0}^{m} \frac{\theta^{2(m-n)}}{\rho^{n+\frac{1}{2}}} \ , \quad \forall\,(\rho,\theta) \in (r_m,\infty)\times(0,\theta_m) \ . \tag{A.8}$$

In Lemma 36 we have already shown that $\partial_x^m \mathcal{K}(x,y) = e^{\rho\cos(\theta)/2}F(\rho,\theta)$, with

$$F(\rho,\theta) = Ce^{-\rho/2} \sum_{n\geq 0} \frac{\alpha_n^m(\theta)}{\rho^{n+1/2}} \ . \tag{A.9}$$

Using again polar coordinates, we get that

$$\partial_y \partial_x^m \mathcal{K}(x,y) = Ce^{\rho\cos(\theta)/2}\left(\sin(\theta)\partial_\rho F(\rho,\theta) + \frac{\cos(\theta)}{\rho}\partial_\theta F(\rho,\theta)\right) \ , \tag{A.10}$$

and substituting F we get that

$$\partial_y \partial_x^m \mathcal{K}(x,y) = Ce^{\rho(\cos(\theta)-1)/2}\left[\sum_{n=0}^{m} \frac{\beta_n^m(\theta)}{\rho^{n+1/2}} + \frac{\eta_m(\rho,\theta)}{\rho^{m+\frac{3}{2}}}\right] \ . \tag{A.11}$$

It therefore follows from Lemma 36 that there exists $\theta_m > 0$ and $(c_n)_{n\leq m} \in (0,\infty)$ such that $|\beta_n^m(\theta)| \leq c_n\theta^{2(m-n)+1}$, for all $\theta \in (0,\theta_m)$,and furthermore that η_m is uniformly bounded. We have therefore proved the following proposition:

Proposition 38 *For all $m \in \mathbb{N}$, there exists $r_m > 0$, such that:*

(i) there exists a constant $C > 0$, such that

$$|\partial_y \partial_x^m \mathcal{K}(x,y)| \leq C e^{\rho(\cos(\theta)-1)/2} \ , \quad \forall\,(\rho,\theta) \in (r_m,\infty)\times(0,\pi) \ . \tag{A.12}$$

(ii) there exists $\theta_m > 0$ and $0 < c_m < C_m < \infty$, such that

$$|\partial_y \partial_x^m \mathcal{K}(x,y)| \leq C_m e^{-c_m\rho\theta^2} \sum_{n=0}^{m} \frac{\theta^{2(m-n)+1}}{\rho^{n+\frac{1}{2}}} \ , \quad \forall\,(\rho,\theta) \in (r_m,\infty)\times(0,\theta_m) \ . \tag{A.13}$$

A.2.2 Mean decay at infinity

With the above pointwise informations, we can now analyze the asymptotic decay for fixed y as x goes to infinity. We have:

Proposition 39 *Given integers $(m_x, m_y) \in \mathbb{N}^2$, and $0 < y_m < y_M < \infty$, there exists a dominating function F_{m_x,m_y} such that for all $x \in \mathbb{R}$ and all $y \in (y_m, y_M)$,*

(i) $\left|\partial_x^{m_x} \partial_y^{m_y} \mathcal{K}(x,y)\right| \leq F_{m_x,m_y}(x)$.

(ii) $F_{m_x,m_y} \in L^q(\mathbb{R})$ *for all q such that*

$$q > \frac{1}{m_x + m_y + 1/2} \ . \tag{A.14}$$

Note that it follows in particular that $F_{m_x,m_y} \in L^1(\mathbb{R})$ provided $m_x \geq 1$.

Proof. We begin with the case $m_y = 0$ and we set $m_x = m$. We parametrize lines $y =$ const. by $\theta \in (0, \pi)$, so that

$$\rho = \frac{y}{\sin(\theta)}, \quad x = y\frac{\cos(\theta)}{\sin(\theta)} \ . \tag{A.15}$$

Then, we have, for arbitrary $y_m < y < y_M$ that $y_m \sin(\theta) < \rho < y_M \sin(\theta)$. Replacing in (A.7,A.8), we get that, restricting the size of θ_m if needed, there exists $0 < k_m < K_m < \infty$ for which, for all $y \in (y_m, y_M)$, $|\partial_x^m \mathcal{K}(x,y)| \leq F_{m,0}(x)$, where, in polar coordinates,

$$F_{m,0}(x) = \begin{cases} K_m \theta^{m+1/2} e^{-k_m \theta} \ , & \forall\, \theta \in (0, \theta_m) \ , \\ K_m \ , & \forall\, \theta \in (\theta_m, \pi - \theta_m) \ , \\ K_m e^{-k_m/\sin(\pi-\theta)} \ , & \forall\, \theta \in (\pi - \theta_m, \pi) \ . \end{cases} \tag{A.16}$$

Consequently, we have in polar coordinates, that

$$|F_{m,0}|_q^q = K_m^q \left[\frac{\pi}{(\sin(\theta_m))^2} + \int_0^{\theta_m} \theta^{q(m+1/2)} e^{-qk_m\theta} \frac{d\theta}{\sin(\theta)^2} + \int_{\pi-\theta_m}^{\pi} e^{-qk_m/\sin(\pi-\theta)} \frac{d\theta}{\sin(\pi-\theta)^2}\right] \ . \tag{A.17}$$

Since the integrand is a bounded function, the last integral is finite. In the first integral we change variables and set $\theta = 1/t$. We get that

$$\int_0^{\theta_m} \theta^{q(m+1/2)} e^{-qc_m\theta} \frac{d\theta}{\theta^2} \leq \int_{1/\theta_m}^{\infty} \frac{dt}{t^{q(m+1/2)}} \ . \tag{A.18}$$

The last integral is finite provided $q > 1/(m + 1/2)$. This completes the proof for this case.

For $m_y \neq 0$, we consider two cases. First, if m_y is even, we have that $m_y = 2n_y$, and, using that $\partial_x^2 \mathcal{K} + \partial_y^2 \mathcal{K} - \partial_x \mathcal{K} = 0$ away from $\{(0,0)\}$ we find that there exists a family of coefficients α_p such that

$$\partial_y^{m_y} \partial_x^{m_x} \mathcal{K} = \sum_{p=0}^{n_y} \alpha_p \partial_x^{m_x+n_y+k} \mathcal{K} \ . \tag{A.19}$$

Therefore we find that for all $(x,y) \in \mathbb{R} \times (y_m, y_M)$,

$$|\partial_y^{m_y} \partial_x^{m_x} \mathcal{K}(x,y)| \leq \sum_{p=0}^{n_y} \alpha_p F_{m_x+n_y+k,0}(x) \ . \tag{A.20}$$

The condition (A.14) is "decreasing in m_x "the most restrictive term being $F_{m_x+n_y}$. Consequently, this term fixes the condition that q must satisfy in order to get that the dominating function satisfies (A.23). At the same time we get that the assertion for even m_y is true and it therefore suffices to check the assertion for $m_y = 1$. We proceed as in the first case. Substituting (A.15) into (A.12, A.13), yields, that there exists $0 < k_m < K_m < \infty$ for which $|\partial_x^m \partial_y \mathcal{K}(x,y)| \leq F_{m,1}(x)$ for all $y \in (y_m, y_M)$, where in polar coordinates,

$$F_{m,1}(x) \leq \begin{cases} K_m \theta^{m+3/2} e^{-k_m \theta} \ , & \forall\, \theta \in (0, \theta_m) \ , \\ K_m \ , & \forall\, \theta \in (\theta_m, \pi - \theta_m) \ , \\ K_m e^{-k_m / \sin(\pi-\theta)} \ , & \forall\, \theta \in (\pi - \theta_m, \pi) \ . \end{cases} \tag{A.21}$$

With the same computation as above, we find the condition $q > 1/(m+3/2)$, which is the stated result for $m_y = 1$. ■

Finally, concerning the asymptotic decay when y goes to infinity, we have:

Proposition 40 *Let $(m_x, m_y) \in \mathbf{N}^2$, and $(r, q, \sigma) \in [1,\infty)^2 \times [0,\infty)$. Then,*

$$\int_1^\infty (1+y)^{\sigma r} \left[\int_{-\infty}^{\infty} |\partial_x^{m_x} \partial_y^{m_y} \mathcal{K}(x,y)|^q \ dx \right]^{\frac{r}{q}} \ dy \ , \tag{A.22}$$

provided

$$r \left(\sigma - \left(2m_x + m_y + 1 - \frac{2}{q} \right) \right) < -1 \quad \text{and} \quad q \left(m_x + m_y + \frac{1}{2} \right) > 1 \ . \tag{A.23}$$

Proof. As above, we do only need to check the cases $m_y \in \{0,1\}$. We begin with the case $m_y = 0$ and we set $m_x = m$. In polar coordinates we have

$$\int_1^\infty \left[(1+y)^\sigma \int_{-\infty}^{\infty} |\partial_x^m \mathcal{K}(\rho,\theta)|^q \ dx \right]^{\frac{r}{q}} \ dy = \int_1^\infty \left[(1+y)^\sigma \int_0^\pi \left| \partial_x^m \mathcal{K}\left(\frac{y}{\sin(\theta)}, \theta \right) \right|^q \frac{y \ d\theta}{\sin(\theta)^2} \right]^{\frac{r}{q}} \ dy \ . \tag{A.24}$$

First, we compute the function I,

$$I(y) = \int_0^\pi \left| \partial_x^m \mathcal{K}\left(\frac{y}{\sin(\theta)}, \theta \right) \right|^q \frac{y \ d\theta}{\sin(\theta)^2} \ . \tag{A.25}$$

Using the previous lemma we see that we can assume without further restriction that r_m as given by Proposition 37 is less than one. The bounds (A.7), (A.8) are therefore valid for arbitrary $(x,y) \in \mathbb{R}^2_{++}$. We now set $I(y) = I_-(y) + I_+(y)$, where, with the same notation as in Proposition 37,

$$I_-(y) = \int_{\theta_m}^\pi \left| \partial_x^m \mathcal{K}\left(\frac{y}{\sin(\theta)}, \theta \right) \right|^q \frac{y \ d\theta}{\sin(\theta)^2} \ , \tag{A.26}$$

$$I_+(y) = \int_0^{\theta_m} \left| \partial_x^m \mathcal{K}\left(\frac{y}{\sin(\theta)}, \theta \right) \right|^q \frac{y \ d\theta}{\sin(\theta)^2} \ . \tag{A.27}$$

From (A.7), we get the bound

$$I_-(y) \leq C \int_{\theta_m}^\pi \frac{e^{qy(\cos(\theta)-1)/\sin(\theta)}}{\sin(\theta)^2} y \ d\theta \leq \frac{1}{(1-\cos(\theta_m))} \int_{\theta_m}^\pi \frac{y(1-\cos(\theta))}{\sin(\theta)^2} e^{qy(\cos(\theta)-1)/\sin(\theta)} \ d\theta \ ,$$

where we use that

$$\frac{d}{d\theta} \left[\frac{\cos(\theta)-1}{\sin(\theta)} \right] = \frac{\cos(\theta)-1}{\sin(\theta)^2} \ .$$

Therefore,

$$I_-(y) \leq \frac{e^{q(\cos(\theta_m)-1)/\sin(\theta_m)}}{q(1-\cos(\theta_m))} \ .$$

For I_+, the bound (A.8) implies that

$$I_+(y) \leq C y \int_0^{\theta_m} \sum_{n=0}^m \frac{\theta^{q(2m-n+\frac{1}{2})-2}}{y^{q(n+1/2)}} d\theta \ ,$$

and the last integral is finite provided $q(m+1/2) > 1$. We therefore get that

$$I_+(y) \le Cy \sum_{n=0}^{m} \frac{\theta_m^{q(2m-n+\frac{1}{2})-1}}{y^{q(n+1/2)}} \ .$$

By definition we have that

$$\int_1^\infty (1+y)^{\sigma r} |I(y)|^{\frac{r}{q}} \ dy \le \int_1^\infty (1+y)^{\sigma r} |I_+(y)|^{\frac{r}{q}} \ dy + \int_1^\infty (1+y)^{\sigma r} |I_-(y)|^{\frac{r}{q}} \ dy \ .$$

We have that

$$\int_1^\infty (1+y)^{\sigma r} |I_+(y)|^{\frac{r}{q}} \ dy \le C \left[\int_1^\infty y^{r(\sigma+1/q)} \sum_{n=0}^{m} \frac{\theta_m^{r(2m-n+1/2)-r/q}}{y^{r(n+1/2)}} \ dy \right] \ .$$

If we assume that

$$r\left(\sigma - \left(2m + \left[1 - \frac{2}{q}\right]\right)\right) < -1 \ ,$$

then there exists $\alpha < 1$ sufficiently large, for which for all $n < m$,

$$r\left(\sigma + \frac{1}{q} - \alpha\left[2m - n + \frac{1}{2} - \frac{1}{q}\right] - \left(n + \frac{1}{2}\right)\right) < -1 \ .$$

Note that, provided θ_m is smaller than the choice made in Proposition 37, we could even make θ_m depend on y in the above arguments without any changes. Consequently, if we set $\theta_m = y^{-\alpha}$ for y sufficiently large, we get that

$$\int_1^\infty (1+y)^{\sigma r} |I_+(y)|^{\frac{r}{q}} \ dy \le C \int_1^\infty y^{r(\sigma+1/q-\alpha[2m-n+1/2-1/q]-(n+1/2)} \ dy \ .$$

This integral is finite because of our choice for α. With the choice $\theta_m = y^{-\alpha}$ we also get that

$$I_-(y) \le C \frac{e^{-q\frac{y^{1-\alpha}}{2}}}{y^{2\alpha}} \ , \text{ as } y \to \infty \ ,$$

and since $\alpha < 1$, the y-integral with I_- is also finite. This concludes the case $m_y = 0$.

$\partial_y \partial_x^m \mathcal{K}$, we apply the same splitting of $I(y)$ as for $\partial_x^m \mathcal{K}$ and use the bound (A.12, A.13). Since the bound outside $[0, \theta_m]$ is the same in all cases, we also get that

$$\int_1^\infty (1 + |y|^\sigma)^r |I_-(y)|^{\frac{r}{q}} \ dy < \infty \ ,$$

for all q, $\sigma > 0$ and $\theta_m = y^{-\alpha}$ with $\alpha < 1$. Finally, we note that the bounds (A.13) imply that we gain one factor of θ when compared with $\partial_x^m \mathcal{K}$. Therefore, we have

$$I_+(y) \le Cy \sum_{n=0}^{m} \frac{\theta_m^{q(2m-n+\frac{3}{2})-1}}{y^{q(n+1/2)}} \ ,$$

provided $q(m+3/2) - 1 > 0$. Thus, the same analysis as for $\partial_x^m \mathcal{K}$ implies that the integral

$$\int_1^\infty (1 + y^\sigma)^r |I_+(y)|^{\frac{q}{r}} \ dy$$

is finite, provided

$$r\left(\sigma - \left(2m + 1 + \left[1 - \frac{2}{q}\right]\right)\right) < -1 \ ,$$

and if we chose $\theta_m = y^{-\alpha}$ with α sufficiently close to 1. This completes the proof. ■

Note that for the case $q = r$ we obtain the following result:

Proposition 41 *Given $(m_x, m_y) \in \mathbf{N}^2$, and $(q, \sigma) \in [1, \infty) \times [0, \infty)$, we have that $\partial_x^{m_x} \partial_y^{m_y} \mathcal{K} \in L^q_\sigma(\mathcal{B}^R)$, for all $R > 0$, provided $q((2m_x + m_y + 1) - \sigma) > 3$.*

B Fourier multipliers

In this section we prove some properties of $\kappa = \sqrt{k^2 - ik}$, where the square root is always to be taken with a non negative real part. We denote the differentiation with respect to k with a dot.

Lemma 42 *There exists a constant $C < \infty$ for which, for all $k \in \mathbb{R} \setminus \{0\}$,*

$$|\kappa| \leq C \left(\sqrt{|k|} + |k| \right) \ , \quad \mathrm{Re}(\kappa)| \geq C \max(|k|, \sqrt{|k|/2}) \ , \quad |\kappa - |k|| \geq C \min(\sqrt{|k|}, 1) \ , \tag{B.1}$$

$$|\dot{\kappa}| \leq C \left(1 + \frac{1}{\sqrt{k}} \right) \ , \quad \text{and} \quad |\dot{\kappa} - \mathrm{sign}(k)| \leq C \min \left(\frac{1}{|k|}, 1 \right) \left(1 + \frac{1}{\sqrt{k}} \right) \ . \tag{B.2}$$

Proof. We consider the case $k > 0$ only. The case with negative k follows by symmetry. Note that $|\kappa| = (k^4 + k^2)^{\frac{1}{4}}$. Thus, (B.1) follows from bounds on both terms using the triangle inequality. Similarly, we find that for all $k > 0$

$$|\mathrm{Re}(\kappa)| = \sqrt{\frac{\sqrt{k^4 + k^2} + k^2}{2}} \geq \begin{cases} \sqrt{(\sqrt{k^4} + k^2)/2} = |k|, \\ \sqrt{\sqrt{k^2}/2} = \sqrt{|k|/2} \ . \end{cases} \tag{B.3}$$

Next we note that

$$|\kappa - |k|| \geq |\,\mathrm{Im}(\kappa)| \ , \quad \text{where } |\,\mathrm{Im}(\kappa)| = \sqrt{\frac{\sqrt{k^4 + k^2} - k^2}{2}} \ , \tag{B.4}$$

and $|\,\mathrm{Im}(\kappa)| > 0$ for arbitrary $k > 0$. Therefore, there exists a constant $C(\varepsilon, M) > 0$ such that for all k with $|k| \in [\varepsilon, M]$,

$$|\kappa - |k|| > C(\varepsilon, M) \ .$$

Therefore we have for $|k| < 1/2$, that$\sqrt{k^4 + k^2} - k^2 \geq |k|/2$. Consequently, we get $|\kappa - |k|| > \sqrt{|k|}/2$. For large values of k, *i.e.*, $|k| > M$, say, we have that

$$\sqrt{k^4 + k^2} = k^2 \sqrt{1 + \frac{1}{k^2}} \geq k^2 \left(1 + \frac{1}{3k^2} \right) \geq k^2 + \frac{1}{3} \ .$$

Substituting this bound into $\mathrm{Im}(\kappa)$ we obtain that $|\kappa - |k|| \geq \sqrt{1/6}$. For $\dot{\kappa}$, we have that

$$\dot{\kappa} = \frac{2k + i}{2\sqrt{k^2 - ik}} \ , \quad \text{and that} \quad |\dot{\kappa}| = \frac{\sqrt{4k^2 + 1}}{2(k^4 + k^2)^{\frac{1}{4}}} \ .$$

Using that $\sqrt{4k^2 + 1} \leq 2|k| + 1$ and replacing in $\dot{\kappa}$, we obtain the stated result. Finally, straightforward computations, lead to

$$|\dot{\kappa} - |k|| \leq C(1 + \frac{1}{\sqrt{|k|}}) \ .$$

Finally, as for the case of $|\dot{\kappa}|$, we use that $\kappa^2 - |k|^2 = ik$. Consequently, we have for all $k > 0$, that

$$(\dot{\kappa} - \mathrm{sign}(k))(\kappa + |k|) + (\dot{\kappa} + \mathrm{sign}(k))(\kappa - |k|) = i/2 \ .$$

Thus,

$$|\dot{\kappa} - \mathrm{sign}(k)| \leq \frac{1}{|\kappa + |k||} \left(\frac{1}{2} + |\dot{\kappa} + \mathrm{sign}(k)| \ |\kappa - |k|| \right) \ .$$

With this bound we obtain that

$$|\dot{\kappa} - \mathrm{sign}(k)| \leq \frac{1}{|k|} \left(\frac{1}{2} + \left[1 + \frac{1}{\sqrt{|k|}} \right] \ |\kappa - |k| \ | \right) \ .$$

Moreover, there exists a constant $C < \infty$ such that $|\kappa - |k|| \leq C$ for all $k > 0$. This is obvious for $k \in [0, M]$, since the function $k \mapsto |\kappa - |k||$ is continuous, and for large values $|k|$ we can use the asymptotic expansion of $\mathrm{Re}(\kappa)$ and $\mathrm{Im}(\kappa)$ and find that

$$\mathrm{Re}(\kappa) - |k| = O(1/|k|) \ , \quad \text{and} \quad \mathrm{Im}(\kappa) = \frac{1}{4} + o(1) \ .$$

We therefore finally have that

$$|\dot{\kappa} - \mathrm{sign}(k)| \leq \frac{C}{|k|}\left[1 + \frac{1}{\sqrt{|k|}}\right] .$$

■

As an immediate corollary, we get:

Proposition 43 *Given $t > 0$, and $k \in \mathbb{R}$, we have that*

$$\left|\frac{e^{-\kappa t} - e^{-|k|t}}{\kappa - |k|}\right| \leq te^{-|k|t} . \tag{B.5}$$

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