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Regularity Issue of the Navier-Stokes Equations Involving the Combination of Pressure and Velocity Field

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Abstract We establish some regularity criteria for the incompressible Navier-Stokes equations in a bounded three-dimensional domain concerning the quotients of the pressure, the velocity field and the pressure gradient.

Keywords Navier-Stokes equations · Regularity criterion · A priori estimates

Mathematics Subject Classification 35B45 · 35B65 · 76D05

1 Introduction

In this article, we consider the following initial boundary value problem for the incompressible Navier-Stokes equations in $\Omega \times (0, T)$

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \Delta u, & \text{in } \Omega \times (0, T) \\ \operatorname{div} u = 0, & \text{in } \Omega \times (0, T) \\ u = 0, & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x), & \text{in } \Omega \end{cases} \tag{1}$$

where $u = u(x, t) \in \mathbb{R}^3$ is the velocity field, $p(x, t)$ is a scalar pressure, $u_0(x)$ with $\operatorname{div} u_0 = 0$ in the sense of distributions is the initial velocity field, and Ω is a bounded domain with smooth boundary $\partial\Omega$.

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The study of the incompressible Navier-Stokes equations in three space dimensions has a long history. In the pioneering work [12] and [20], Leray and Hopf proved the existence of weak solutions $u(x,t) \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3))$ for given $u_0(x) \in L^2(\mathbb{R}^3)$. However, it is not known yet whether or not the solution develops singularities in finite time even if the initial datum is C^∞-smooth. Therefore the study of the regularity of solutions becomes interesting and attracts many researchers' interest. On one hand, in [23], Scheffer began to study the partial regularity theory of the Navier-Stokes equations. Additional results were obtained by Caffarelli, Kohn and Nirenberg in [4]. Further result can be found in [26] and references therein. On the other hand, the regularity of a given weak solution u can be shown under additional conditions. In 1962, Serrin [24] proved that if u is a Leray-Hopf weak solution belonging to $L^{\alpha,\gamma} \equiv L^\alpha(0,T; L^\gamma(\mathbb{R}^3))$ with $2/\alpha + 3/\gamma \le 1$, $2 < \alpha < \infty$, $3 < \gamma < \infty$, then the solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T])$. From then on, there are many results with additional criterion added on u, see for instance [9, 17, 18, 25, 27].

It is well-known that if (u,p) solves the Navier-Stokes equations, then so does (u_λ, p_λ) for all $\lambda > 0$ in $\mathbb{R}^3$, with

$$u_\lambda(x,t) = \lambda u\big(\lambda x, \lambda^2 t\big),$$
$$p_\lambda(x,t) = \lambda^2 p\big(\lambda x, \lambda^2 t\big).$$

The class of Serrin's type is important form a view point of scaling invariance which implies that $\|u_\lambda\|_{L^{\alpha,\gamma}} = \|u\|_{L^{\alpha,\gamma}}$ holds true for all $\lambda > 0$ if and only if $2/\alpha + 3/\gamma = 1$ and we say that the norm $\|u\|_{L^{\alpha,\gamma}}$ has the scaling dimension zero [4]. One can find that if $2/\alpha + 3/\gamma = 2$, both $\|\nabla u\|_{L^{\alpha,\gamma}}$ and $\|p\|_{L^{\alpha,\gamma}}$ have scaling dimension zero. Related to this point, there are lots of papers devoted to the study of the regularity problem, see for instance [1, 5, 29, 30] and [3, 28, 31]. In [2], the same regularity of weak solutions was shown under the condition $p/(1+|u|)$ for a bounded domain, the generalization of which can be found in [2, 31]. For further criteria, we refer the reader to [6–8, 10, 11, 32–34] and the recent publication [21] where the ratio $p/(1+|u|+|\nabla u|^r)$ is considered, for some $r < 1$.

It is well-known that for the Navier-Stokes equations in $\mathbb{R}^3$, we have the following equality by applying the "div" operator to both sides of the first equation of (1),

$$-\Delta p = \sum_{i,j=1}^{3} \partial_i \partial_j (u_i u_j). \tag{2}$$

Therefore the Calderón-Zygmund theorem gives

$$\|p\|_{L^\gamma} \le C_1 \|u\|^2_{L^{2\gamma}}, \quad 1 < \gamma < \infty. \tag{3}$$

We find (roughly speaking) that the pressure behaves as velocity squared, i.e., $|p| \lesssim |u|^2$. Because of this fact, not too much attention has been paid to a kind of additional condition, the so called "alternative natural assumption", $p/(1+|u|)$ which however, may also guarantee the regularity. An exception is the paper [2], see also [21] for the case of $p/(1+|u|+|\nabla u|^r)$ for some $r < 1$. Again, taking "∇ div" on both sides of the first equation in (1) for smooth (u,p), one can obtain

$$-\Delta(\nabla p) = \sum_{i,j=1}^{3} \partial_i \partial_j \big(\nabla(u_i u_j)\big), \tag{4}$$

and the following inequality

$$\|\nabla p\|_{L^q} \le C_2 \big\| |u||\nabla u| \big\|_{L^q}, \quad 1 < q < \infty, \tag{5}$$

where C_2 is a constant depending only on q. Roughly speaking, the pressure gradient behaves like the inner product of u and its gradient, or simply, $|\nabla p| \lesssim |u||\nabla u|$. It should be mentioned that, to our knowledge, (5) was skillfully used in [22, 28, 32] to prove the regularity of weak solutions in $\mathbb{R}^3$, where (5) played a very important role in the proof. However, if we consider the Navier-Stokes equations in a generic domain $\Omega \subsetneq \mathbb{R}^3$, although (2) and (4) are true, (3) and (5) don't hold anymore. This is the main difficulty for a generic domain, but this doesn't mean we can't establish regularity criteria in a generic domain. As far as we know, Kang and Lee in [14] and its erratum [15] established regularity criteria for Navier-Stokes equations on pressure and its gradient by borrowing Stokes estimates in bounded domains. We notice that very little attention has been paid to discuss the regularity criterion based on $p/(1+|u|)$ or $\nabla p/(1+|u|)$ in a bounded domain with smooth boundaries except the ones [2, 31]. This consideration is reasonable. We make the following simple assumption: If $|u|$ is the leading term in the denominator of $p/(1+|u|)$, i.e., if $|u| \gg 1$, then $p/(1+|u|) \lesssim |u|$, with the regularity condition on the velocity u supposed to be known. By analogy, for $\nabla p/(1+|u|)$. Note that in this paper we will discuss the quotients $p/(1+|u|^\delta)$ and $\nabla p/(1+|u|^\delta)$ for certain δ, especially for small δ, since the limiting cases of $p/(1+|u|^\delta)$ and $\nabla p/(1+|u|^\delta)$ with δ goes to 0 are nothing else but the investigation in [14]. Therefore the purpose of this paper is to investigate what regularity for (1) can be inferred assuming some conditions on these kinds of quotients. The following are our main results.

Theorem 1.1 *Let $u_0(x) \in L^q(\Omega)$, $q \ge 3$, and $\operatorname{div} u_0 = 0$ in the sense of distribution. Suppose $u(x,t)$ is a Leray-Hopf weak solution of (1) in $[0,T)$. If one of the following conditions is satisfied*

(C1) *For $0 \le \delta < 2/3$,*

$$\frac{p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \quad \textit{with } \frac{2}{\alpha} + \frac{3}{\gamma} = \frac{4-3\delta}{2}, \ \frac{18}{8-9\delta} \le \gamma \le \frac{6}{2-3\delta},$$

$$\textit{or} \quad \frac{p}{1+|u|^\delta} \in L^{\alpha,\eta}, \quad \textit{with } \alpha \ge 2 \textit{ and } \eta > \frac{6}{(2-3\delta)(1-\delta)}.$$

(C2) *For $2/3 \le \delta \le 8/9$,*

$$\frac{p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \quad \textit{with } \frac{2}{\alpha} + \frac{3}{\gamma} = \frac{4-3\delta}{2}, \ \frac{18}{8-9\delta} \le \gamma \le \infty.$$

Then $u(x,t)$ is a smooth solution in $[0,T]$.

Theorem 1.2 *Let the same assumption as Theorem 1.1 be true. If one of the following conditions is satisfied*

(H1) *For $0 \le \delta < 2/3$,*

$$\frac{\nabla p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \quad \textit{with } \frac{2}{\alpha} + \frac{3}{\gamma} = \frac{6-3\delta}{2}, \ \frac{18}{14-9\delta} \le \gamma \le \frac{6}{2-3\delta},$$

$$or \quad \frac{\nabla p}{1+|u|^\delta} \in L^{\alpha,\gamma'}, \quad with\ \alpha \geq 1\ and\ \gamma' > \frac{6}{(2-3\delta)(1-\delta)}.$$

(H2) *For* $2/3 \leq \delta \leq 14/9$,

$$\frac{\nabla p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \quad with\ \frac{2}{\alpha}+\frac{3}{\gamma}=\frac{6-3\delta}{2}, \ \frac{18}{14-9\delta} \leq \gamma \leq \infty.$$

Then $u(x,t)$ *is a smooth solution in* $[0,T]$.

Remark 1.3 We remark here that the effect of pressure on the boundary can be roughly regarded as the linear part of δ. Our results complement the previous work. Less information about the cases of small δ is obtained in [2, 31]. We will apply different methods from [10, 21] to prove Theorem 1.1 and Theorem 1.2. Moreover, results in higher dimensions can be achieved by a suitable modification of the method in our proof.

We recall for convenience of the reader, the definition of Leray-Hopf weak solutions and collect some preliminary results.

Definition 1.4 A measurable vector u is called a Leray-Hopf weak solution of the Navier-Stokes equations (1), if u satisfies the following properties

(i) u is weakly continuous from $[0,T)$ to $L^2(\Omega)$.
(ii) u verifies (1) in the weak sense, i.e.,

$$\int_0^T\int_\Omega\left(\frac{\partial\phi}{\partial t}+(u\cdot\nabla)\phi\right)u\,dxdt+\int_\Omega u_0\phi(x,0)dx=\int_0^T\int_\Omega \nabla u:\nabla\phi\,dxdt$$

for all $\phi \in C_0^\infty(\Omega\times[0,T))$ with $\operatorname{div}\phi=0$, and

$$\int_0^T\int_\Omega u\cdot\nabla\phi\,dxdt=0$$

for every $\phi\in C_0^\infty(\Omega\times[0,T))$.
(iii) The energy inequality is satisfied, i.e.,

$$\|u(.,t)\|_{L^2}^2+2\int_0^t\|\nabla u(.,\tau)\|_{L^2}^2d\tau\leq\|u_0\|_{L^2}^2, \quad 0\leq t\leq T.$$

Lemma 1.5

(1) *Assume that* Ω *is a bounded domain with smooth boundary,* $u_0\in L^s(\Omega)$, $s\geq 3$. *Then there exists* T_0 *and a unique classical solution* $u\in BC([0,T_0);L^s(\Omega))$. *Moreover, let* $(0,T_*)$ *be the maximal interval such that* u *solves* (1) *in* $C((0,T_*);L^s(\Omega))$, $s>3$. *Then*

$$\|u(.,\tau)\|_{L^s}\geq\frac{C}{(T_*-\tau)^{\frac{s-3}{2s}}}, \tag{6}$$

with a constant C *independent of* T_* *and* s.
(2) *Let* u *be a weak solution satisfying* $u\in L^{r,s}$ *with* $2/r+3/s\leq 1$, *for* $s>3$. *Then* u *belongs to* $C^\infty((0,T]\times\Omega)$, *and* u *is even smooth up to the boundary if* Ω *is smooth.*

The proof is due to Giga [9] (see also [16]), see Kozono [19] for the case of a half space and Iwashita [13] for an exterior domain.

By a strong solution we mean a weak solution u such that

$$u \in L^{\infty}\big(0,T;H^{1}\big) \cap L^{2}\big(0,T;H^{2}\big).$$

It is well-known that strong solutions are regular (say, classical) and unique in the class of weak solutions. In the following proofs we will adopt the convention that constants are different from section to section.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, which is divided into three steps. First, we introduce an interpolation inequality which plays a very important role in the proof. Then we establish the a priori estimates for strong solutions. Finally, by a standard continuation argument we get the smoothness of solutions.

Lemma 2.1 *Let $f \in L^{\infty,s} \cap L^{s,3s}$ be a measurable function on $\Omega \times [0,T)$, with $s > 1$. Then $f \in L^{p,q}$ with $s \le p$, $s \le q \le 3s$ and $\frac{s}{p} + \frac{3s}{2q} \ge \frac{3}{2}$*

$$\|f\|_{L^{p,q}} \le C(s,p,q,T)\|f\|_{L^{\infty,s}}^{\frac{3s-q}{2q}} \|f\|_{L^{s,3s}}^{\frac{3q-3s}{2q}},$$

where $C(s,p,q,T)$ depends on s,p,q,T, and $C(s,p,q,T) = 1$ if $\frac{s}{p} + \frac{3s}{2q} = \frac{3}{2}$.

This lemma was proved by an application of an interpolation theorem and Hölder's inequality. For details we refer to [31]. A direct application of Lemma 2.1 is that if u is a Leray-Hopf weak solution of (1), then

$$u \in L^{p,q}, \quad \text{with } \frac{2}{p} + \frac{3}{q} \ge \frac{3}{2},\ p \ge 2,\ \text{and } 2 \le q \le 6.$$

Now we are in a position to prove Theorem 1.1.

Proof By the previous known results on weak solutions, the calculations we are going to do are completely justified and, in particular, all the boundary integrals arising in the integrations by parts vanish. We multiply both sides of the first equation in (1) by $su|u|^{s-2}$, for $s \ge 3$, and integrate on $\Omega \times (0,t)$, for $0 < t \le T$. After a suitable integration by parts we obtain

$$\begin{aligned}
&\|u(.,t)\|_{L^s}^s + s\int_0^t\int_\Omega |u|^{s-2}|\nabla u|^2 dx d\tau + \frac{4(s-2)}{s}\int_0^t\int_\Omega \big|\nabla|u|^{\frac{s}{2}}\big|^2 dx d\tau \\
&\le s\int_0^t\int_\Omega |\nabla p||u|^{s-1} dx d\tau + \|u_0\|_{L^s}^s \\
&\le 2(s-2)\int_0^t\int_\Omega |p||u|^{\frac{s}{2}-1}\big|\nabla|u|^{\frac{s}{2}}\big| dx d\tau + \|u_0\|_{L^s}^s,
\end{aligned} \tag{7}$$

where we have used that

$$
\begin{aligned}
s\int_{\Omega} u|u|^{s-2}\Delta u dx &= -s\int_{\Omega}\nabla\big(u|u|^{s-2}\big)\cdot\nabla u dx\\
&= -s\int_{\Omega}|u|^{s-2}|\nabla u|^2 dx - s(s-2)\int_{\Omega}|u|^{s-2}\big|\nabla|u|\big|^2 dx\\
&= -s\int_{\Omega}|u|^{s-2}|\nabla u|^2 dx - \frac{4(s-2)}{s}\int_{\Omega}\big|\nabla|u|^{\frac{s}{2}}\big|^2 dx,
\end{aligned}
$$

and that

$$
\begin{aligned}
&-s\int_0^t\int_{\Omega}\nabla p\cdot u|u|^{s-2}dxd\tau\\
&= s\int_0^t\int_{\Omega} p\cdot \operatorname{div}\big(u|u|^{s-2}\big)\\
&= s\int_0^t\int_{\Omega} p\cdot\big(u\cdot\nabla|u|^{s-2}\big)dxd\tau \le 2(s-2)\int_0^t\int_{\Omega}|p||u|^{\frac{s}{2}-1}\big|\nabla|u|^{\frac{s}{2}}\big|dxd\tau.
\end{aligned}
$$

If we use the fact that

$$
|\nabla|u|^{\frac{s}{2}}| \le \frac{s}{2}|u|^{\frac{s}{2}-1}|\nabla u|,
$$

then (7) becomes

$$
\big\|u(.,t)\big\|_{L^s}^s + 2\big\|\nabla|u|^{\frac{s}{2}}\big\|_{L^{2,2}}^2 \le 2(s-2)\int_0^t\int_{\Omega}|p||u|^{\frac{s}{2}-1}\big|\nabla|u|^{\frac{s}{2}}\big|dxd\tau + \|u_0\|_{L^s}^s. \tag{8}
$$

The goal is now to estimate the quantity

$$
A := 2(s-2)\int_0^t\int_{\Omega}|p||u|^{\frac{s}{2}-1}\big|\nabla|u|^{\frac{s}{2}}\big|dxd\tau.
$$

Case (1): $0 \le \delta < 2/3$.

If $p/(1+|u|^{\delta}) \in L^{\alpha,\gamma}$, with $2/\alpha + 3/\gamma = (4-3\delta)/2$, $18/(8-9\delta) \le \gamma \le 6/(2-3\delta)$, we can estimate A as follows

$$
\begin{aligned}
&2(s-2)\int_0^t\int_{\Omega}|p||u|^{\frac{s}{2}-1}\big|\nabla|u|^{\frac{s}{2}}\big|dxd\tau\\
&\le C_1\int_0^t\int_{\Omega}|p|^2|u|^{s-2}dxd\tau + \frac{1}{2}\big\|\nabla|u|^{\frac{s}{2}}\big\|_{L^{2,2}}^2\\
&\le C_2\int_0^t\int_{\Omega}\left|\frac{p}{1+|u|^{\delta}}\right|^2\big(1+|u|\big)^{2\delta+s-2}dxd\tau + \frac{1}{2}\big\|\nabla|u|^{\frac{s}{2}}\big\|_{L^{2,2}}^2\\
&\le C_2\left\|\frac{p}{1+|u|^{\delta}}\right\|_{L^{\alpha,\gamma}}^2\big\|1+|u|\big\|_{L^{\infty,2}}^{2\delta}\|1+u\|_{L^{b,a}}^{s-2} + \frac{1}{2}\big\|\nabla|u|^{\frac{s}{2}}\big\|_{L^{2,2}}^2\\
&\le C_3\left\|\frac{p}{1+|u|^{\delta}}\right\|_{L^{\alpha,\gamma}}^2\|u\|_{L^{b,a}}^{s-2} + C_3\left\|\frac{p}{1+|u|^{\delta}}\right\|_{L^{\alpha,\gamma}}^2 T^{\frac{s-2}{b}}|\Omega|^{\frac{s-2}{a}} + \frac{1}{2}\big\|\nabla|u|^{\frac{s}{2}}\big\|_{L^{2,2}}^2,
\end{aligned} \tag{9}
$$

where C_3 is a constant depending on $\|u_0\|_{L^2}$ and on $|\Omega|$, the Lebesgue measure of the domain Ω and then Young's inequality and the Hölder's inequality are used. In the above inequalities, the parameters α, γ, a and b which are to be determined later satisfy

$$\frac{2}{\alpha}+\frac{s-2}{b}=1, \tag{10}$$

$$\frac{2}{\gamma}+\frac{s-2}{a}+\delta=1. \tag{11}$$

Equation (10) and (11) can be easily solved and we obtain

$$\frac{1}{a}=\frac{1}{s-2}\left(1-\delta-\frac{2}{\gamma}\right), \qquad \frac{1}{b}=\frac{1}{s-2}\left(1-\frac{2}{\alpha}\right). \tag{12}$$

It is obvious that a and b given by (12) satisfy

$$\begin{aligned}\frac{s}{b}+\frac{3s}{2a}&=\frac{s}{s-2}\left[\frac{5-3\delta}{2}-\left(\frac{2}{\alpha}+\frac{3}{\gamma}\right)\right]\\&=\frac{s}{2(s-2)},\end{aligned} \tag{13}$$

where we have used the condition on α and γ, $2/\alpha+3/\gamma=(4-3\delta)/2$. In order to apply Lemma 2.1 with additional restrictions on a and b

$$\frac{s}{b}+\frac{3s}{2a}\geq\frac{3}{2},$$

$$s\leq a\leq 3s, \qquad b\geq s. \tag{14}$$

We find that

$$\frac{s}{2(s-2)}\geq\frac{3}{2}.$$

It is easy to see that $s=3$ is the only possible choice of s. Substituting $s=3$ into (9) we obtain

$$\begin{aligned}&\|u(.,t)\|_{L^3}^3+\frac{3}{2}\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2\\&\quad\leq C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2\|u\|_{L^{b,a}}+C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2 T^{\frac{1}{b}}|\Omega|^{\frac{1}{a}}+\|u_0\|_{L^3}^3.\end{aligned} \tag{15}$$

It thus follows by Young's inequality and Lemma 2.1, that

$$\begin{aligned}&\|u(.,t)\|_{L^3}^3+\frac{3}{2}\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2\\&\quad\leq C_4\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2\|u\|_{L^{\infty,3}}^{\frac{6}{\alpha}-2}\|u\|_{L^{3,9}}^{3-\frac{6}{\alpha}}+C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2 T^{\frac{1}{b}}|\Omega|^{\frac{1}{a}}+\|u_0\|_{L^3}^3\\&\quad\leq C_4\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^\alpha\|u\|_{L^{\infty,3}}^{3-\alpha}+C_5\|u\|_{L^{3,9}}^3+C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2 T^{\frac{1}{b}}|\Omega|^{\frac{1}{a}}+\|u_0\|_{L^3}^3.\end{aligned} \tag{16}$$

Choosing a suitable constant C_5 such that

$$C_5\|u\|^3_{L^9} = C_5\big\||u|^{\frac{3}{2}}\big\|^2_{L^6} \le \frac{1}{2}\big\|\nabla|u|^{\frac{3}{2}}\big\|^2_{L^2},$$

we note that for $\alpha \le 3$ we get from (16) that

$$\begin{aligned}\big\|u(.,t)\big\|^3_{L^3} \le C_6\left\|\frac{p}{1+|u|^\delta}\right\|^\alpha_{L^{\alpha,\gamma}}\|u\|^3_{L^{\infty,3}} \\ + C_3\left\|\frac{p}{1+|u|^\delta}\right\|^2_{L^{\alpha,\gamma}} T^{1-\frac{2}{\alpha}}|\Omega|^{1-\delta-\frac{2}{\gamma}} + \|u_0\|^3_{L^3}.\end{aligned} \tag{17}$$

By the integrability of $\|p/(1+|u|^\delta)\|_{L^\gamma}$ with respect to time t, we can always choose t_0, $0 < t_0 \le T$, such that

$$\int_0^{t_0}\left\|\frac{p}{1+|u|^\delta}\right\|^\alpha_{L^\gamma} d\tau \le \frac{1}{2C_6},$$

where C_6 depends on γ, δ, $\|u_0\|_{L^2}$ and $|\Omega|$. Therefore, it follows that

$$\sup_{0\le t\le t_0}\big\|u(.,t)\big\|^3_{L^3} \le \frac{2C_3}{(2C_6)^{2/\alpha}}T^{1-2/\alpha}|\Omega|^{1-\delta-2/\gamma} + 2\|u_0\|^3_{L^3}. \tag{18}$$

Since u is a smooth solution, we can repeat the above process starting from t_0, and obtain a similar estimate as (17) for t satisfying $t_0 \le t \le T$. By the integrability of $\|p/(1+|u|^\delta)\|_{L^\gamma}$, there exists a t_1, $t_0 < t_1 < T$, such that

$$\int_{t_0}^{t_1}\left\|\frac{p}{1+|u|^\delta}\right\|^\alpha_{L^\gamma} d\tau \le \frac{1}{2C_6}.$$

The sup norm for $\|u(.,t)\|_{L^3}$ can now be estimated as following

$$\begin{aligned}\sup_{t_0\le t\le t_1}\big\|u(.,t)\big\|^3_{L^3} \le \frac{2C_3}{(2C_6)^{2/\alpha}}T^{1-\frac{2}{\alpha}}|\Omega|^{1-\delta-\frac{2}{\gamma}} \\ + \frac{4C_3}{(2C_6)^{2/\alpha}}T^{1-\frac{2}{\alpha}}|\Omega|^{1-\delta-\frac{2}{\gamma}} + 4\|u_0\|^3_{L^3}.\end{aligned}$$

Since $p/(1+|u|^\delta) \in L^{\alpha,\gamma}$, the constant C_6 doesn't depend on t, and the above process can be repeated a finite number of times. Therefore, we have the estimate

$$\sup_{0\le t<T}\big\|u(.,t)\big\|^3_{L^3} \le C_7. \tag{19}$$

If we go back to (16), we find that thanks to (19) the following estimate holds true.

$$\int_0^T\big\|\nabla|u|^{\frac{3}{2}}\big\|^2_{L^2}d\tau = M_1 < \infty.$$

Therefore

$$u \in L^3\big(0,T;L^9(\Omega)\big). \tag{20}$$

Note that it is the combination of (11) and (14) that gives the bound of γ in our condition.

For $\gamma > 6/(2-3\delta)$, from (13), we can find that if $2/\alpha + 3/\gamma = \beta < (4-3\delta)/2$, then we can choose a suitable s which is bigger than 3. More precisely, we obtain

$$\begin{aligned}\frac{s}{b} + \frac{3s}{2a} &= \frac{s}{s-2}\left[\frac{5-3\delta}{2} - \left(\frac{2}{\alpha} + \frac{3}{\gamma}\right)\right]\\ &= \frac{s(5-3\delta-2\beta)}{2(s-2)} > \frac{3}{2},\end{aligned}$$

and if $\beta < (2-3\delta)/2$, then the above inequality holds for all $s \geq 3$. Otherwise, we have

$$s \leq \frac{6}{3\delta - 2 + 2\beta}. \tag{21}$$

The inequality satisfied by a, $s \leq a \leq 3s$, implies that $(s-2)/3s \leq (s-2)/a \leq (s-2)/s$, and therefore

$$\frac{s-2}{3s} \leq 1 - \delta - \frac{2}{\gamma} \leq \frac{s-2}{s}. \tag{22}$$

Since $\gamma > 6/(2-3\delta)$, it is easy to check that the inequality on the left-hand side of (22) is always satisfied, from the inequality on the right-hand side it follows

$$\delta + \frac{2}{\gamma} \geq \frac{2}{s}, \tag{23}$$

which means that

$$\gamma \leq \frac{2s}{2 - s\delta}. \tag{24}$$

Note that if $2/s \leq \delta < 2/3$, then (23) is obvious. Otherwise, we have (24). The combination of (21), (24) and the relation $2/\alpha + 3/\gamma = \beta$ therefore gives

$$\frac{3\alpha}{\alpha\beta - 2} = \gamma \leq \frac{2s}{2 - s\delta} \leq \frac{12}{2(3\delta - 2 + 2\beta) - 6\delta} = \frac{3}{\beta - 1}.$$

Finally, we get

$$\alpha \geq 2 \quad \text{and} \quad \beta \leq 1 + \frac{3}{\gamma}.$$

The maximum of β is achieved when $\alpha = 2$ and $\gamma > 6/(2-3\delta)$. Now let in (8) $s = \gamma > 6/(2-3\delta)$. We find

$$\begin{aligned}&\left\|u(.,t)\right\|_{L^\gamma}^{\gamma} + 2\left\|\nabla |u|^{\frac{\gamma}{2}}\right\|_{L^{2,2}}^2\\ &\quad \leq 2(\gamma-2)\int_0^t \int_\Omega \left|\frac{p}{1+|u|^\delta}\right| \left(1+|u|\right)^{\delta+\frac{\gamma}{2}-1} \left|\nabla |u|^{\frac{\gamma}{2}}\right| dx d\tau + \|u_0\|_{L^\gamma}^{\gamma}\\ &\quad \leq C_8 \int_0^t \left|\frac{p}{1+|u|^\delta}\right|_{L^{\frac{\gamma}{1-\delta}}}^2 \left\|1+|u|\right\|_{L^\gamma}^{2\delta+\gamma-2} d\tau + \left\|\nabla |u|^{\frac{\gamma}{2}}\right\|_{L^{2,2}}^2 + \|u_0\|_{L^\gamma}^{\gamma}\\ &\quad \leq C_8 \left\|\frac{p}{1+|u|^\delta}\right\|_{L^{2,\gamma/(1-\delta)}}^2 \left\|1+|u|\right\|_{L^{\infty,\gamma}}^{2\delta+\gamma-2} + \left\|\nabla |u|^{\frac{\gamma}{2}}\right\|_{L^{2,2}}^2 + \|u_0\|_{L^\gamma}^{\gamma}\end{aligned}$$

$$\leq C_9 \left\| \frac{p}{1+|u|^\delta} \right\|^2_{L^{2,\gamma/(1-\delta)}} \|u\|^{2\delta+\gamma-2}_{L^{\infty,\gamma}} + \left\| \nabla |u|^{\frac{\gamma}{2}} \right\|^2_{L^{2,2}}$$
$$+ C_9 \left\| \frac{p}{1+|u|^\delta} \right\|^2_{L^{2,\gamma/(1-\delta)}} |\Omega|^{\frac{2\delta-2+\gamma}{\gamma}} + \|u_0\|^\gamma_{L^\gamma}.$$

Consequently,

$$\sup_{0\leq t<T} \|u\|_{L^\gamma} \leq C_{10}, \tag{25}$$

provided that $p/(1+|u|^\delta) \in L^{\alpha,\gamma/(1-\delta)}$ with $\alpha \geq 2$ and $\gamma > 6/(2-3\delta)$, where C_{10} is a constant depending on γ, $|\Omega|$, $\|u_0\|_{L^\gamma}$, $\|p/(1+|u|^\delta)\|_{L^{\alpha,\gamma/(1-\delta)}}$. For simplicity, denoting $\gamma/(1-\delta)$ by η, we obtain the regularity if $p/(1+|u|^\delta) \in L^{\alpha,\eta}$ with $\alpha \geq 2$ and $\eta > 6/(2-3\delta)(1-\delta)$.

Case (2): $2/3 \leq \delta \leq 8/9$.

In this case, if (C2) is satisfied, for $\delta \neq 8/9$ and $18/(8-9\delta) \leq \gamma < \infty$, then argument (17) still holds. Thus we can also obtain the bound (20).

For $(\alpha,\gamma) = (4/(4-3\delta), \infty)$, we observe that a and b can be chosen such that

$$a = \frac{1}{1-\delta} \quad \text{and} \quad b = \frac{2}{3\delta-2}, \quad 2/3 < \delta < 8/9; \qquad b = \infty, \quad \delta = 2/3.$$

Then we have the following estimate

$$\left\|u(.,t)\right\|^3_{L^3} \leq C_6 \left\| \frac{p}{1+|u|^\delta} \right\|^{\frac{4}{4-3\delta}}_{L^{\frac{4}{4-3\delta},\infty}} \|u\|^3_{L^{\infty,3}}$$
$$+ C_3 \left\| \frac{p}{1+|u|^\delta} \right\|^2_{L^{\frac{4}{4-3\delta},\infty}} T^{\frac{3\delta-2}{2}} |\Omega|^{1-\delta} + \|u_0\|^3_{L^3}.$$

With a similar argument as for $\gamma < \infty$ in case 1, we again get (20).

If $\delta = 8/9$, then $(\alpha,\gamma) = (3,\infty)$, and therefore $a = 9$, $b = 3$, and from (15), we have that

$$\left\|u(.,t)\right\|^3_{L^3} + \frac{3}{2} \left\| \nabla |u|^{\frac{3}{2}} \right\|^2_{L^{2,2}}$$
$$\leq C_3 \left\| \frac{p}{1+|u|^\delta} \right\|^2_{L^{3,\infty}} \|u\|_{L^{3,9}} + C_3 \left\| \frac{p}{1+|u|^\delta} \right\|^2_{L^{3,\infty}} T^{\frac{1}{3}} |\Omega|^{\frac{1}{9}} + \|u_0\|^3_{L^3},$$

so it also follows immediately that

$$u \in L^\infty\big(0,T;L^3(\Omega)\big) \cap L^3\big(0,T;L^9(\Omega)\big).$$

In order to complete the proof of our theorem, we apply Lemma 1.5 to our estimates. Since (C1) and (C2) all give the bound of $u \in L^3(0,T;L^9(\Omega))$, then we find that u falls into the Serrin's regularity class satisfying Lemma 1.5. Therefore, u is a smooth solution in $[0,T]$. $\square$

3 Proof of Theorem 1.2

The proof of Theorem 1.2 is very similar to the one of Theorem 1.1. We therefore present the main estimates. From (7), we have for $s \geq 3$,

$$
\begin{aligned}
&\left\|u(.,t)\right\|_{L^s}^s + 2\left\|\nabla |u|^{\frac{s}{2}}\right\|_{L^{2,2}}^2 \\
&\leq s\int_0^t \int_\Omega |\nabla p||u|^{s-1} dx d\tau + \|u_0\|_{L^s}^s \\
&\leq s\int_0^t \int_\Omega \left|\frac{\nabla p}{1+|u|^\delta}\right| \left(1+|u|\right)^{s+\delta-1} dx d\tau + \|u_0\|_{L^s}^s \\
&\leq s\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}} \left\|1+|u|\right\|_{L^{a,b}}^{s-1} \left\|1+|u|\right\|_{L^{\infty,2}}^{\delta} + \|u_0\|_{L^s}^s \\
&\leq C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}} \|u\|_{L^{a,b}}^{s-1} + C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}} T^{\frac{s-1}{a}} |\Omega|^{\frac{s-1}{b}} + \|u_0\|_{L^s}^s,
\end{aligned}
$$

where we have used Hölder's inequality and

$$
\begin{aligned}
\frac{1}{\alpha} + \frac{s-1}{a} &= 1, \\
\frac{1}{\gamma} + \frac{s-1}{b} + \frac{\delta}{2} &= 1.
\end{aligned}
$$

It follows that

$$
\frac{s}{a} = \frac{s}{s-1}\left(1 - \frac{1}{\alpha}\right), \tag{26}
$$

$$
\frac{s}{b} = \frac{s}{s-1}\left(\frac{2-\delta}{2} - \frac{1}{\gamma}\right). \tag{27}
$$

Similarly, since a and b also have to satisfy the following additional conditions

$$
\frac{s}{a} + \frac{3s}{2b} \geq \frac{3}{2}, \tag{28}
$$

with

$$
s \leq b \leq 3s. \tag{29}
$$

If condition (H1) is satisfied for $0 \leq \delta < 2/3$, one obtains from (26), (27) and the condition $2/\alpha + 3/\gamma = (6-3\delta)/2$,

$$
\begin{aligned}
\frac{s}{a} + \frac{3s}{2b} &= \frac{s}{s-1}\left(\frac{10-3\delta}{4} - \left(\frac{1}{\alpha} + \frac{3}{2\gamma}\right)\right) \\
&\geq \frac{s}{s-1}.
\end{aligned}
$$

In order to apply Lemma 2.1, we have to take $s = 3$, and we get, using in addition a Sobolev inequality

$$
\begin{aligned}
&\left\|u(.,t)\right\|_{L^3}^3+2\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2\\
&\quad\le C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}\|u\|_{L^{a,b}}^2+C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}T^{\frac{2}{a}}|\Omega|^{\frac{2}{b}}+\|u_0\|_{L^3}^3,\\
&\quad\le C_{12}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}\|u\|_{L^{\infty,3}}^{\frac{3}{\alpha}-1}\|u\|_{L^{3,9}}^{3-\frac{3}{\alpha}}+C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}T^{1-\frac{1}{\alpha}}|\Omega|^{\frac{2}{3\alpha}}+\|u_0\|_{L^3}^3,\\
&\quad\le C_{13}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^{\alpha}\|u\|_{L^{\infty,3}}^{3-\alpha}+\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2+C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}T^{1-\frac{1}{\alpha}}|\Omega|^{\frac{2}{3\alpha}}+\|u_0\|_{L^3}^3.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\|u(.,t)\right\|_{L^3}^3+\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2\\
&\quad\le C_{13}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^{\alpha}\|u\|_{L^{\infty,3}}^{3-\alpha}+C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}T^{1-\frac{1}{\alpha}}|\Omega|^{\frac{2}{3\alpha}}+\|u_0\|_{L^3}^3.
\end{aligned}
\tag{30}
$$

Then using again the argument as for (C1), by the integrability of $\nabla p/(1+|u|^\delta)$ with respect to the time variable, (20) is a easy consequence of (30).

If $\gamma>6/(2-3\delta)$, we can find that if $2/\alpha+3/\gamma=\beta_1<(6-3\delta)/2$, then s can be chosen larger than 3. One obtains

$$
\begin{aligned}
\frac{s}{a}+\frac{3s}{2b}&=\frac{s}{s-1}\left(\frac{10-3\delta}{4}-\frac{\beta_1}{2}\right)\\
&=\frac{s(10-3\delta-2\beta_1)}{4(s-1)}>\frac{3}{2}.
\end{aligned}
$$

If $\beta_1\le(4-3\delta)/2$, the above inequality holds for all $s\ge3$. Otherwise, we have

$$
3<s\le\frac{6}{3\delta+2\beta_1-4}.
\tag{31}
$$

Note that the inequality (29) implies that

$$
\frac{s-1}{3s}\le\left(\frac{2-\delta}{2}-\frac{1}{\gamma}\right)\le\frac{s-1}{s},
\tag{32}
$$

and

$$
\frac{6}{2-3\delta}<\gamma\le\frac{2s}{2-s\delta}.
\tag{33}
$$

We notice that (32) is obvious for $2/s\le\delta<2/3$. Otherwise, we have (33). Combining (31) and (33), it follows that

$$
\frac{3\alpha}{\alpha\beta_1-2}=\gamma\le\frac{2s}{2-s\delta}\le\frac{3}{\beta_1-2},
$$

which gives

$$
\alpha\ge1\quad\text{and}\quad\beta_1\le2+\frac{3}{\gamma}.
\tag{34}
$$

By (34) and arguments similar to the ones given in the second case of (C1), we have

$$\sup_{0\le t<T} \|u\|_{L^\gamma} \le C_{14}, \tag{35}$$

provided that $\nabla p/(1+|u|^\delta) \in L^{\alpha,\gamma'}$ with $\alpha \ge 1$ and $\gamma' = \gamma/(1-\delta) > 6/(2-3\delta)(1-\delta)$, where C_{14} is a constant depending on γ, $|\Omega|$, $\|u_0\|_{L^\gamma}$, $\|\nabla p/(1+|u|^\delta)\|_{L^{\alpha,\gamma'}}$.

If (H2) is satisfied for $2/3 \le \delta \le 14/9$, we obtain the estimate (20) by similar arguments in the case (C2) of Theorem 1.1.

The remaining part is very similar as the proof of Theorem 1.1, given the a priori estimates (20) and (35). This completes the proof.

Remark 3.1 A regularity criterion added on the quantity $\nabla p/(1+|\nabla u|)$ can also be obtained similarly. This can be of certain interest due to the relation (5), but we skip it to prevent repetitiveness.

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References

1. Beirão da Veiga, H.: A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$. Chin. Ann. Math., Ser. B **16**, 407–412 (1995)
2. Beirão da Veiga, H.: A sufficient condition on the pressure for the regularity of weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech. **2**, 99–106 (2000)
3. Berselli, L.C., Galdi, G.P.: Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations. Proc. Am. Math. Soc. **130**, 3585–3595 (2002)
4. Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier-Stokes equations. Commun. Pure Appl. Math. **35**, 771–831 (1982)
5. Chae, D., Choe, H.-J.: On the regularity criterion for the solutions of the Navier-Stokes equations. Electron. J. Differ. Equ. **1999**, 1–7 (1999)
6. Cao, C., Titi, E.S.: Regularity criteria for the three dimensional Navier-Stokes equations. Indiana Univ. Math. J. **57**, 2643–2661 (2008)
7. Cao, C., Qin, J.L., Titi, E.S.: Regularity criterion for solutions of three-dimensional turbulent channel flows. Commun. Partial Differ. Equ. **33**, 419–428 (2008)
8. Fan, J., Jiang, S., Nakamura, G., Zhou, Y.: Logarithmically improved regularity criteria for the Navier-Stokes and MHD equations. J. Math. Fluid Mech. **13**, 557–571 (2011)
9. Giga, Y.: Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. J. Differ. Equ. **62**, 186–212 (1986)
10. Guo, Z., Gala, S.: A note on the regularity criteria for the Navier-Stokes equations. Appl. Math. Lett. **25**, 305–309 (2012)
11. Guo, Z., Gala, S.: Remarks on logarithmical regularity criteria for the Navier-Stokes equations. J. Math. Phys. **52**, 063503 (2011)
12. Hopf, E.: Über die Anfangwertaufgaben für die hydromischen Grundgleichungen. Math. Nachr. **4**, 213–321 (1951)
13. Iwashita, H.: L^q-L^r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L^q spaces. Math. Ann. **285**, 265–288 (1989)
14. Kyungkeun, K., Lee, J.: On regularity criteria in conjunction with the pressure of Navier-Stokes equations. Int. Math. Res. Not., 80762 (2006)
15. Kyungkeun, K., Lee, J.: Erratum: on regularity criteria in conjunction with the pressure of the Navier-Stokes equations. Int. Math. Res. Not. **9**, 1772–1774 (2010). doi:10.1093/imrn/rnq073

16. Kato, T.: Strong L^p-solutions to the Navier-Stokes equations in $\mathbb{R}^m$, with applications to weak solutions. Math. Z. **187**, 471–480 (1984)
17. Kozono, H., Sohr, H.: Regularity criterion on weak solutions to the Navier-Stokes equations. Adv. Differ. Equ. **2**, 535–554 (1997)
18. Kozono, H., Taniuchi, Y.: Bilinear estimates in BMO and the Navier-Stokes equations. Math. Z. **235**, 173–194 (2000)
19. Kozono, H.: Global L^n-solution and its decay property for the Navier-Stokes equations in half space R^n_+. J. Differ. Equ. **79**, 79–88 (1989)
20. Leray, J.: Étude de divers équations intégrales nonlinearies et de quelques problemes que posent lhydrodinamique. J. Math. Pures Appl. **12**, 1–82 (1931)
21. Núñez, M.: Regularity criteria for the Navier-Stokes equations involving the ratio pressure-gradient of velocity. Math. Methods Appl. Sci. **33**, 323–331 (2010)
22. Penel, P., Pokorný, M.: Some new regularity criteria for the Navier-Stokes equations containing gradient of velocity. Appl. Math. **5**, 483–493 (2004)
23. Scheffer, V.: Partial regularity of solutions to the Navier-Stokes equations. Pac. J. Math. **66**, 535–552 (1976)
24. Serrin, J.: On the interior regularity of weak solutions of the Navier-Stokes equations. Arch. Ration. Mech. Anal. **9**, 187–195 (1962)
25. Struwe, M.: On partial regularity results for the Navier-Stokes equations. Commun. Pure Appl. Math. **41**, 437–458 (1988)
26. Tian, G., Xin, Z.: Gradient estimation on Navier-Stokes equations. Commun. Anal. Geom. **7**, 221–257 (1999)
27. von Wahl, W.: Regularity of weak solutions of the Navier-Stokes equations. In: Proceedings of the 1983 Summer Institute on Nonlinear Functional Analysis and Applications. Proc. Symposia in Pure Mathematics, vol. 45, pp. 497–503. Am. Math. Soc., Providence (1989)
28. Zhou, Y.: On regularity criteria in terms of pressure for the Navier-Stokes equations in $\mathbb{R}^3$. Proc. Am. Math. Soc. **134**, 149–156 (2006)
29. Zhou, Y.: A new regularity criterion for the Navier-Stokes equations in terms of the gradient of one velocity component. Methods Appl. Anal. **9**, 563–578 (2002)
30. Zhou, Y.: A new regularity criterion for weak solutions to the Navier-Stokes equations. J. Math. Pures Appl. **84**, 1496–1514 (2005)
31. Zhou, Y.: Regularity criteria in terms of pressure for the 3-D Navier-Stokes equations in a generic domain. Math. Ann. **328**, 173–192 (2004)
32. Zhou, Y.: On a regularity criterion in terms of the gradient of pressure for the Navier-Stokes equations in $\mathbb{R}^N$. Z. Angew. Math. Phys. **57**, 384–392 (2006)
33. Zhou, Y., Pokorný, M.: On a regularity criterion for the Navier-Stokes equations involving gradient of one velocity component. J. Math. Phys. **50**, 123514 (2009)
34. Zhou, Y., Pokorný, M.: On the regularity of the solutions of the Navier-Stokes equations via one velocity component. Nonlinearity **23**, 1097–1107 (2010)