Regularity criteria involving the ratio of pressure and velocity for the Navier-Stokes equations

Zhengguang Guo

Department of Mathematics

East China Normal University

200241 Shanghai, P. R. China

gzgmath@gmail.com

Peter Wittwer

Département de Physique Théorique

Université de Genève, Switzerland

peter.wittwer@unige.ch

Abstract

We established some regularity criteria for the Navier-Stokes equations in a bounded domain concerning the quotients of $p/(1+|u|^\delta)$ and $\nabla p/(1+|u|^\delta)$ for some suitable δ. The results extend and generalize some previous ones given by Y. Zhou in the paper [Math. Ann. 328 (2004) 173-192].

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1 Introduction

In this article, we consider the following initial boundary value problem for the incompressible Navier-Stokes equations in $\Omega \times (0,T)$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \Delta u, & \text{in } \Omega \times (0,T) \\ \operatorname{div} u = 0, & \text{in } \Omega \times (0,T) \\ u = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases} \tag{1}$$

where $u = u(x,t) \in \mathbb{R}^3$ is the velocity field, $p(x,t)$ is a scalar pressure, $u_0(x)$ with $\operatorname{div} u_0 = 0$ in the sense of distributions is the initial velocity field, and Ω is a bounded domain with smooth boundary $\partial\Omega$.

The study of the incompressible Navier-Stokes equations in three space dimensions has a long history. In the pioneering work [11] and [17], Leray and Hopf proved the existence of weak solutions $u(x,t) \in L^\infty(0,T\,;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ for given $u_0(x) \in L^2(\mathbb{R}^3)$. However, it is not known yet whether or not the solution develops singularities in finite time even if the initial datum is C^∞-smooth. Therefore the study of the regularity of solutions becomes interesting and attracts many researchers' interest. On one hand, in [20], Scheffer began to study the partial regularity theory of the Navier-Stokes equations. Additional results were obtained by Caffarelli, Kohn and Nirenberg in [4]. Further result can be found in [23] and references therein. On the other hand, the regularity of a given weak solution u can be shown under additional conditions. In 1962, Serrin [21] proved that if u is a Leray-Hopf weak solution belonging to $L^{\alpha,\gamma} \equiv L^\alpha(0,T;L^\gamma(\mathbb{R}^3))$ with $2/\alpha + 3/\gamma \le 1$, $2 < \alpha < \infty$, $3 < \gamma < \infty$, then the solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T])$. From then on, there are many results with additional criterion added on u, see for instance [7, 14, 15, 22, 24].

It is well-known that if (u,p) solves the Navier-Stokes equations, then so does (u_λ, p_λ) for all $\lambda > 0$, with

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)\ ,$$
$$p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)\ .$$

The class of Serrin's type is important form a view point of scaling invariance which implies that $||u_\lambda||_{L^{\alpha,\gamma}} = ||u||_{L^{\alpha,\gamma}}$ holds true for all $\lambda > 0$ if and only if $2/\alpha + 3/\gamma = 1$ and we say that the norm $||u||_{L^{\alpha,\gamma}}$ has the scaling dimension zero [4]. One can find that if $2/\alpha + 3/\gamma = 2$, both $||\nabla u||_{L^{\alpha,\gamma}}$ and $||p||_{L^{\alpha,\gamma}}$ have scaling dimension zero. Related to this point, there are lots of papers devoted to the study of the regularity problem, see for instance [1, 5, 26, 27] and [3, 25, 28]. In [2], the same regularity of weak solutions was shown under the condition $p/(1+|u|)$ for a bounded domain, the generalization of which can be found in [28]. For further criteria, we refer the reader to [6, 8, 9, 29, 30, 31] and the recent publication [19] where the ratio $p/(1+|u|+|\nabla u|^r)$ is considered, for some $r < 1$.

It is well-known that for the Navier-Stokes equations in $\mathbb{R}^3$, we have the following equality by applying the "div" operator to both sides of the first equation of (1),

$$-\Delta p = \sum_{i,j=1}^{3} \partial_i \partial_j (u_i u_j). \tag{2}$$

Therefore the Calderón-Zygmund theorem gives

$$||p||_{L^\gamma} \le C_1 ||u||^2_{L^{2\gamma}}, \quad 1 < \gamma < \infty. \tag{3}$$

We find (roughly speaking) that the pressure behaves as velocity squared, *i.e.*, $|p| \lesssim |u|^2$. Because of this fact, not too much attention has been paid to one kind of additional condition, the so called "alternative natural assumption", $p/(1+|u|)$ which however, may also guarantee the regularity. An exception is the paper [2], see also [19] for the case of $p/(1+|u|+|\nabla u|^r)$ for some $r<1$. Again, taking "$\nabla \operatorname{div}$" on both sides of the first equation in (1) for smooth (u,p), one can obtain

$$-\Delta(\nabla p)=\sum_{i,j=1}^{3}\partial_i\partial_j(\nabla(u_iu_j)), \tag{4}$$

and the following inequality

$$||\nabla p||_{L^q}\le C_2|||u||\nabla u|||_{L^q}, \quad 1<q<\infty, \tag{5}$$

where C_2 is a constant depending only on q. Roughly speaking, the pressure gradient behaves like the inner product of u and its gradient, or simply, $|\nabla p|\lesssim |u||\nabla u|$. It should be mentioned that, to our knowledge, (5) is first used by Zhou in [25] to prove the regularity of weak solutions in $\mathbb{R}^3$, see also [29] where (5) played a very important role in the proof. However, if we consider the Navier-Stokes equations in a generic domain $\Omega \subsetneq \mathbb{R}^3$, although (2) and (4) are true, (3) and (5) don't hold anymore. This is the main difficulty for a generic domain, but this doesn't mean we can't establish regularity criteria in a generic domain. As far as we know, very little attention has been paid to discuss the regularity condition based on $p/(1+|u|)$ or $\nabla p/(1+|u|)$. We now make the following simple assumption: If $|u|$ is the leading term in the denominator of $p/(1+|u|)$, i.e., if $|u|\gg 1$, then $p/(1+|u|)\lesssim |u|$, with the regularity condition on the velocity u supposed to be known. By analogy, for $\nabla p/(1+|u|)$, it is interesting to consider this type of regularity condition for the Navier-Stokes equation. Therefore the purpose of this paper is to investigate what regularity for (1) can be inferred assuming some conditions on this kind of quotient. The following are our main results.

Theorem 1.1 *Let* $u_0(x)\in L^q(\Omega)$, $q\ge 3$, *and* $\operatorname{div} u_0=0$ *in the sense of distribution. Suppose* $u(x,t)$ *is a Leray-Hopf weak solution of (1) in* $[0,T)$. *If one of the following* *conditions is satisfied*

(C1) *For* $0 \le \delta < 2/3$,

$$\frac{p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \ \textit{with} \ \frac{2}{\alpha}+\frac{3}{\gamma}=\frac{4-3\delta}{2}, \ \frac{18}{8-9\delta} \le \gamma \le \frac{6}{2-3\delta},$$

$$\textit{or} \ \frac{p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \ \textit{with} \ \alpha \ge 2 \ \textit{and} \ \gamma > \frac{6}{(2-3\delta)(1-\delta)}.$$

(C2) *For* $2/3 \le \delta \le 8/9$,

$$\frac{p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \ \textit{with} \ \frac{2}{\alpha}+\frac{3}{\gamma}=\frac{4-3\delta}{2}, \ \frac{18}{8-9\delta} \le \gamma \le \infty.$$

Then $u(x,t)$ *is a smooth solution in* $[0,T]$.

Theorem 1.2 *Let the same assumption as Theorem 1.1 be true. If one of the following conditions is satisfied*

(H1) *For* $0 \le \delta < 2/3$,

$$\frac{\nabla p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \ \textit{with} \ \frac{2}{\alpha}+\frac{3}{\gamma}=\frac{6-3\delta}{2}, \ \frac{18}{14-9\delta} \le \gamma \le \frac{6}{2-3\delta},$$

$$\textit{or} \ \frac{\nabla p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \ \textit{with} \ \alpha \ge 1 \ \textit{and} \ \gamma > \frac{6}{(2-3\delta)(1-\delta)}.$$

(H2) *For* $2/3 \le \delta \le 14/9$,

$$\frac{\nabla p}{1+|u|^\delta} \in L^{\alpha,\gamma}, \ \textit{with} \ \frac{2}{\alpha}+\frac{3}{\gamma}=\frac{6-3\delta}{2}, \ \frac{18}{14-9\delta} \le \gamma \le \infty.$$

Then $u(x,t)$ *is a smooth solution in* $[0,T]$.

Remark 1.3 *The special cases of* $\delta = 0$ *in Theorem 1.1 and Theorem 1.2 have already been discussed, see the reference [28] where the author treated the problem for a generic domain (the half space, a bounded domain with smooth boundary and an exterior domain) or see [3] for partial results. Also, in [28], the author considered the case of some* $\delta \ge 1$*, but there was no information on the case of* $\delta < 1$*. Therefore, the present results extend and generalize the previous ones. It should be mentioned here that the current method used to prove the regularity, which is different from the one used in [2], can also be applied to higher dimensions* N, $N > 3$, *the corresponding results still hold, provided that appropriate modifications are made in our proof, see the references [10] and [28] for details.*

Before proceeding to prove the above theorems, we recall for convenience of the reader, the definition of Leray-Hopf weak solutions and collect some preliminary results.

Definition 1.4 *A measurable vector u is called a Leray-Hopf weak solution of the Navier-Stokes equations (1), if u satisfies the following properties*

(i) *u is weakly continuous from $[0,T)$ to $L^2(\Omega)$.*

(ii) *u verifies (1) in the sense of distributions, i.e.,*

$$\int_0^T \int_\Omega \left(\frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi \right) u dx dt + \int_\Omega u_0 \phi(x,0) dx = \int_0^T \int_\Omega \nabla u : \nabla \phi dx dt$$

for all $\phi \in C_0^\infty(\Omega \times [0,T))$ with $\operatorname{div} \phi = 0$, and

$$\int_0^T \int_\Omega u \cdot \nabla \phi dx dt = 0$$

for every $\phi \in C_0^\infty(\Omega \times [0,T))$.

(iii) *The energy inequality is satisfied, i.e.,*

$$||u(.,t)||_{L^2}^2 + 2\int_0^t ||\nabla u(.,\tau)||_{L^2}^2 d\tau \leq ||u_0||_{L^2}^2 \ , \quad 0 \leq t \leq T \ .$$

Lemma 1.5 *(1) Assume that Ω is a bounded domain with smooth boundary, $u_0 \in L^s(\Omega)$, $s \geq 3$. Then there exists T_0 and a unique classical solution $u \in BC([0,T_0); L^s(\Omega))$. Moreover, let $(0,T_*)$ be the maximal interval such that u solves (1) in $C((0,T_*); L^s(\Omega))$, $s > 3$. Then*

$$||u(.,\tau)||_{L^s} \geq \frac{C}{(T_* - \tau)^{\frac{s-3}{2s}}} \ , \tag{6}$$

with a constant C independent of T_* and s.

(2) Let u be a weak solution satisfying $u \in L^{r,s}$ with $2/r + 3/s \leq 1$, for $s > 3$. Then u belongs to $C^\infty((0,T] \times \Omega)$.

The proof is due to Giga [7] (See also [13]), see Kozono [16] for the case of a half space and Iwashita [12] for an exterior domain.

By a strong solution we mean a weak solution u such that

$$u \in L^\infty(0,T; H^1) \cap L^2(0,T; H^2) \ .$$

It is well-known that strong solutions are regular (say, classical) and unique in the class of weak solutions. In the following proofs we will adopt the convention that constants are different from section to section.

2 Proof of theorem 1.1

This section is devoted to the proof of Theorem 1.1, which is divided into three steps. First, we introduce an interpolation inequality which plays a very important role in the proof. Then we establish an a priori estimate for strong solutions. Finally, by a standard continuation argument we get the smoothness of solutions.

Lemma 2.1 *[28] Let $f \in L^{\infty,s} \cap L^{s,3s}$ be a measurable function on $\Omega \times [0,T)$, with $s > 1$. Then $f \in L^{p,q}$ with $s \le p$, $s \le q \le 3s$ and $\frac{s}{p} + \frac{3s}{2q} \ge \frac{3}{2}$*

$$\|f\|_{L^{p,q}} \le C(s,p,q,T)||f||_{L^{\infty,s}}^{\frac{3s-q}{2q}}||f||_{L^{s,3s}}^{\frac{3q-3s}{2q}},$$

where $C(s,p,q,T)$ depends on s, p, q, T, and $C(s,p,q,T) = 1$ if $\frac{s}{p} + \frac{3s}{2q} = \frac{3}{2}$.

This lemma can be proved by an application of an interpolation theorem and Hölder's inequality. For details we refer to [28]. A direct application of Lemma 2.1 is that if u is a Leray-Hopf weak solution of (1), then

$$u \in L^{p,q}\ ,\ \text{with}\ \frac{2}{p} + \frac{3}{q} \ge \frac{3}{2},\ p \ge 2,\ \text{and}\ 2 \le q \le 6\ .$$

Next we establish an a priori estimate.

Lemma 2.2 *Assume $u_0 \in L^q(\Omega)$, $q \ge 3$ with $\operatorname{div} u_0 = 0$ in the sense of distributions. Suppose u is a strong solution of (1) in $\Omega \times (0,T)$. If $p/(1+|u|^\delta) \in L^{\alpha,\gamma}$, for $0 \le \delta < 2/3$, with $2/\alpha + 3/\gamma = (4-3\delta)/2$, $18/(8-9\delta) \le \gamma \le 6/(2-3\delta)$ or if (C2) is satisfied, then*

$$\sup_{0 \le t < T} \|u(.,t)\|_{L^3}^3 + \|u\|_{L^{3,9}}^3 \le C\ , \tag{7}$$

where $C = C(T, \Omega, \|u_0\|_{L^3})$.

Remark 2.3 *Instead of (7) we would have preferred to get an estimate of the type*

$$\sup_{0 \le t \le T} ||u(.,t)||_{L^s} \le C\ ,$$

for some $s \ge 3$, but we could only show (7) (for $s = 3$) which is sufficient to prove Theorem 1.1.

Proof. We follow the idea proposed in [28] in order to get a bound for our solution. We multiply both sides of the first equation in (1) by $su|u|^{s-2}$, for $s \geq 3$, and integrate on $\Omega \times (0,t)$, for $0 < t \leq T$. After a suitable integration by parts we obtain

$$\begin{aligned} &||u(.,t)||_{L^s}^s + s\int_0^t \int_\Omega |u|^{s-2} |\nabla u|^2 \, dx d\tau + \frac{4(s-2)}{s} \int_0^t \int_\Omega \left|\nabla |u|^{\frac{s}{2}}\right|^2 dx d\tau \\ &\leq s \int_0^t \int_\Omega |\nabla p| \, |u|^{s-1} dx d\tau + ||u_0||_{L^s}^s \\ &\leq 2(s-2) \int_0^t \int_\Omega |p| \, |u|^{\frac{s}{2}-1} \left|\nabla \, |u|^{\frac{s}{2}}\right| dx d\tau + ||u_0||_{L^s}^s, \end{aligned} \tag{8}$$

where we have used that

$$\begin{aligned} s\int_\Omega u|u|^{s-2} \Delta u dx &= -s \int_\Omega \nabla \left(u|u|^{s-2}\right) \cdot \nabla u dx \\ &= -s \int_\Omega |u|^{s-2} |\nabla u|^2 \, dx - s(s-2) \int_\Omega |u|^{s-2} \left|\nabla |u|\right|^2 dx \\ &= -s \int_\Omega |u|^{s-2} |\nabla u|^2 \, dx - \frac{4(s-2)}{s} \int_\Omega \left|\nabla |u|^{\frac{s}{2}}\right|^2 dx \ , \end{aligned}$$

and that

$$\begin{aligned} &- s \int_0^t \int_\Omega \nabla p \cdot u|u|^{s-2} dx d\tau = s \int_0^t \int_\Omega p \cdot \operatorname{div}\left(u|u|^{s-2}\right) \\ &= s \int_0^t \int_\Omega p \cdot \left(u \cdot \nabla |u|^{s-2}\right) dx d\tau \leq 2(s-2) \int_0^t \int_\Omega |p| \, |u|^{\frac{s}{2}-1} \left|\nabla \, |u|^{\frac{s}{2}}\right| dx d\tau \ . \end{aligned}$$

If we use the fact that

$$|\nabla |u|^{\frac{s}{2}}| \leq \frac{s}{2} |u|^{\frac{s}{2}-1} |\nabla u| \ ,$$

then (8) becomes

$$||u(.,t)||_{L^s}^s + 2 \left\|\nabla |u|^{\frac{s}{2}}\right\|_{L^{2,2}}^2 \leq 2(s-2) \int_0^t \int_\Omega |p| \, |u|^{\frac{s}{2}-1} \left|\nabla \, |u|^{\frac{s}{2}}\right| dx d\tau + ||u_0||_{L^s}^s \ . \tag{9}$$

The goal is now to estimate the quantity $A := 2(s-2) \int_0^t \int_\Omega |p| \, |u|^{\frac{s}{2}-1} \left|\nabla \, |u|^{\frac{s}{2}}\right| dx d\tau.$

Case 1) : $0 \leq \delta < 2/3$.

If $p/(1+|u|^\delta) \in L^{\alpha,\gamma}$, with $2/\alpha+3/\gamma=(4-3\delta)/2$, $18/(8-9\delta) \le \gamma \le 6/(2-3\delta)$, we can estimate A as follows:

$$
\begin{aligned}
&2(s-2)\int_0^t\int_\Omega |p|\,|u|^{\frac{s}{2}-1}\left|\nabla |u|^{\frac{s}{2}}\right| dxd\tau \\
&\le C_1\int_0^t\int_\Omega |p|^2\,|u|^{s-2}dxd\tau+\frac{1}{2}\left\|\nabla |u|^{\frac{s}{2}}\right\|^2_{L^{2,2}} \\
&\le C_2\int_0^t\int_\Omega \left|\frac{p}{1+|u|^\delta}\right|^2(1+|u|)^{2\delta+s-2}\,dxd\tau+\frac{1}{2}\left\|\nabla |u|^{\frac{s}{2}}\right\|^2_{L^{2,2}} \\
&\le C_2\left\|\frac{p}{1+|u|^\delta}\right\|^2_{L^{\alpha,\gamma}}\|1+|u|\|^{2\delta}_{L^{\infty,2}}\|1+u\|^{s-2}_{L^{b,a}}+\frac{1}{2}\left\|\nabla |u|^{\frac{s}{2}}\right\|^2_{L^{2,2}} \\
&\le C_3\left\|\frac{p}{1+|u|^\delta}\right\|^2_{L^{\alpha,\gamma}}\|u\|^{s-2}_{L^{b,a}}+C_3\left\|\frac{p}{1+|u|^\delta}\right\|^2_{L^{\alpha,\gamma}}T^{\frac{s-2}{b}}|\Omega|^{\frac{s-2}{a}}+\frac{1}{2}\left\|\nabla |u|^{\frac{s}{2}}\right\|^2_{L^{2,2}}, \qquad (10)
\end{aligned}
$$

where C_3 is a constant depending on $\|u_0\|_{L^2}$ and on $|\Omega|$, the Lebesgue measure of the domain Ω and then Young's inequality and the Hölder's inequality are used. In the above inequalities, the constants α, γ, a and b which are to be determined later satisfy

$$\frac{2}{\alpha}+\frac{s-2}{b}=1\ , \qquad (11)$$

$$\frac{2}{\gamma}+\frac{s-2}{a}+\delta=1\ . \qquad (12)$$

Equation (11) and (12) can be solved easily and we obtain

$$\frac{1}{a}=\frac{1}{s-2}\left(1-\delta-\frac{2}{\gamma}\right),\quad \frac{1}{b}=\frac{1}{s-2}\left(1-\frac{2}{\alpha}\right). \qquad (13)$$

It is obvious that a and b given by (13) satisfy

$$
\begin{aligned}
\frac{s}{b}+\frac{3s}{2a} &= \frac{s}{s-2}\left[\frac{5-3\delta}{2}-\left(\frac{2}{\alpha}+\frac{3}{\gamma}\right)\right] \\
&= \frac{s}{2(s-2)}, \qquad (14)
\end{aligned}
$$

where we have used the condition on α and γ, $2/\alpha+3/\gamma=(4-3\delta)/2$. In order to apply Lemma 2.1, we use the additional conditions on a and b

$$\frac{s}{b}+\frac{3s}{2a}\ge\frac{3}{2}\ ,$$

$$s\le a\le 3s,\quad b\ge s.$$

Therefore

$$\frac{s}{2(s-2)}\ge\frac{3}{2}.$$

It is easy to see that $s=3$ is the only possible choice for s. Substituting $s=3$ into (10) we get

$$\begin{aligned}&||u(.,t)||_{L^3}^3+\frac{3}{2}\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2\\&\le C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2\|u\|_{L^{b,a}}+C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2T^{\frac{1}{b}}\left|\Omega\right|^{\frac{1}{a}}+||u_0||_{L^3}^3\ .\end{aligned}\tag{15}$$

It thus follows by Young's inequality and Lemma 2.1, that

$$\begin{aligned}&||u(.,t)||_{L^3}^3+\frac{3}{2}\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2\\&\le C_4\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2\|u\|_{L^{\infty,3}}^{\frac{6}{\alpha}-2}\|u\|_{L^{3,9}}^{3-\frac{6}{\alpha}}+C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2T^{\frac{1}{b}}\left|\Omega\right|^{\frac{1}{a}}+||u_0||_{L^3}^3\\&\le C_4\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^\alpha\|u\|_{L^{\infty,3}}^{3-\alpha}+C_5\|u\|_{L^{3,9}}^3+C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2T^{\frac{1}{b}}\left|\Omega\right|^{\frac{1}{a}}+||u_0||_{L^3}^3.\end{aligned}\tag{16}$$

Choosing a suitable constant C_5 such that

$$C_5\|u\|_{L^9}^3=C_5\left\||u|^{\frac{3}{2}}\right\|_{L^6}^2\le\frac{1}{2}\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^2}^2,$$

we note that for $\alpha\le 3$ we get from (16) that

$$\begin{aligned}||u(.,t)||_{L^3}^3\le{}&C_6\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^\alpha\|u\|_{L^{\infty,3}}^3\\&+C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^2T^{1-\frac{2}{\alpha}}\left|\Omega\right|^{1-\delta-\frac{2}{\gamma}}+||u_0||_{L^3}^3.\end{aligned}\tag{17}$$

By the integrability of $\left\|p/(1+|u|^\delta)\right\|_{L^\gamma}$ with respect to time t, we can always choose t_0, $0<t_0\le T$, such that

$$\int_0^{t_0}\left\|\frac{p}{1+|u|^\delta}\right\|_{L^\gamma}^\alpha d\tau\le\frac{1}{2C_6},$$

where C_6 depends on γ, δ, $\|u_0\|_{L^2}$ and $|\Omega|$. Therefore, it follows that

$$\sup_{0\le t\le t_0}||u(.,t)||_{L^3}^3\le\frac{2C_3}{(2C_6)^{2/\alpha}}T^{1-2/\alpha}\left|\Omega\right|^{1-\delta-2/\gamma}+2||u_0||_{L^3}^3.\tag{18}$$

Since u is a smooth solution, we can repeat the above process starting from t_0, and obtain a similar estimate as (17) for t satisfying $t_0\le t\le T$. By the integrability of $\left\|p/(1+|u|^\delta)\right\|_{L^\gamma}$, there exists a t_1, $t_0<t_1<T$, such that

$$\int_{t_0}^{t_1}\left\|\frac{p}{1+|u|^\delta}\right\|_{L^\gamma}^\alpha d\tau\le\frac{1}{2C_6}.$$

The sup norm for $||u(.,t)||_{L^3}$ can now be estimated as following

$$\sup_{t_0\le t\le t_1} ||u(.,t)||_{L^3}^3 \le \frac{2C_3}{(2C_6)^{2/\alpha}} T^{1-\frac{2}{\alpha}} |\Omega|^{1-\delta-\frac{2}{\gamma}} + \frac{4C_3}{(2C_6)^{2/\alpha}} T^{1-\frac{2}{\alpha}} |\Omega|^{1-\delta-\frac{2}{\gamma}} + 4||u_0||_{L^3}^3.$$

Since $p/(1+|u|^\delta) \in L^{\alpha,\gamma}$, the constant C_6 doesn't depend on t, and the above process can be repeated a finite number of times. Therefore, we have the estimate

$$\sup_{0\le t<T} ||u(.,t)||_{L^3}^3 \le C_7. \tag{19}$$

If we go back to (16), we find that thanks to (19) the following estimate holds true.

$$\int_0^T \left\| \nabla |u|^{\frac{3}{2}} \right\|_{L^2}^2 d\tau = M_1 < \infty.$$

Therefore

$$u \in L^3(0,T;L^9(\Omega)). \tag{20}$$

The combination of (19) and (20) gives the bound (7). ■

For $\gamma > 6/(2-3\delta)$, from (14), we can find that if $2/\alpha + 3/\gamma = \beta < (4-3\delta)/2$, then we can choose an s which is bigger than 3. More precisely, we obtain

$$\frac{s}{b} + \frac{3s}{2a} = \frac{s}{s-2}\left[\frac{5-3\delta}{2} - (\frac{2}{\alpha}+\frac{3}{\gamma})\right] = \frac{s\,(5-3\delta-2\beta)}{2\,(s-2)} > \frac{3}{2},$$

and if $\beta < (2-3\delta)/2$, then the above inequality holds for all $s \ge 3$. Otherwise, we have

$$s \le \frac{6}{3\delta - 2 + 2\beta}. \tag{21}$$

The inequality satisfied by a, $s \le a \le 3s$, implies that $(s-2)/3s \le (s-2)/a \le (s-2)/s$, and therefore

$$\frac{s-2}{3s} \le 1 - \delta - \frac{2}{\gamma} \le \frac{s-2}{s}. \tag{22}$$

Since $\gamma > 6/(2-3\delta)$, it is easy to check that the inequality on the left-hand side of (22) is always satisfied, from the inequality on the right-hand side it follows

$$\delta + \frac{2}{\gamma} \ge \frac{2}{s}, \tag{23}$$

which means that

$$\gamma \le \frac{2s}{2-s\delta}. \tag{24}$$

Note that if $2/s \le \delta < 2/3$, then (23) is obvious. Otherwise, we have (24). The combination of (21), (24) and the relation $2/\alpha + 3/\gamma = \beta$ therefore gives

$$\frac{3\alpha}{\alpha\beta - 2} = \gamma \le \frac{2s}{2 - s\delta} \le \frac{12}{2\left(3\delta - 2 + 2\beta\right) - 6\delta} = \frac{3}{\beta - 1}.$$

Finally, we get

$$\alpha \ge 2 \quad \text{and} \quad \beta \le 1 + \frac{3}{\gamma}.$$

The maximum of β is achieved when $\alpha = 2$ and $\gamma > 6/(2-3\delta)$. Now let in (9) $s = \gamma > 6/(2-3\delta)$. We find

$$\begin{aligned}
&||u(.,t)||_{L^\gamma}^\gamma + 2\left\|\nabla |u|^{\frac{\gamma}{2}}\right\|_{L^{2,2}}^2 \\
&\le 2(\gamma-2)\int_0^t\int_\Omega \left|\frac{p}{1+|u|^\delta}\right| (1+|u|)^{\delta+\frac{\gamma}{2}-1}\left|\nabla |u|^{\frac{\gamma}{2}}\right| dx d\tau + ||u_0||_{L^\gamma}^\gamma \\
&\le C_8\int_0^t \left|\frac{p}{1+|u|^\delta}\right|_{L^{\frac{\gamma}{1-\delta}}}^2 ||1+|u|||_{L^\gamma}^{2\delta+\gamma-2} d\tau + \left\|\nabla |u|^{\frac{\gamma}{2}}\right\|_{L^{2,2}}^2 + ||u_0||_{L^\gamma}^\gamma \\
&\le C_8\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{2,\gamma/(1-\delta)}}^2 \|1+|u|\|_{L^{\infty,\gamma}}^{2\delta+\gamma-2} + \left\|\nabla |u|^{\frac{\gamma}{2}}\right\|_{L^{2,2}}^2 + ||u_0||_{L^\gamma}^\gamma \\
&\le C_9\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{2,\gamma/(1-\delta)}}^2 \|u\|_{L^{\infty,\gamma}}^{2\delta+\gamma-2} + \left\|\nabla |u|^{\frac{\gamma}{2}}\right\|_{L^{2,2}}^2 \\
&+ C_9\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{2,\gamma/(1-\delta)}}^2 |\Omega|^{\frac{2\delta-2+\gamma}{\gamma}} + ||u_0||_{L^\gamma}^\gamma.
\end{aligned}$$

Consequently,

$$\sup_{0\le t<T} ||u||_{L^{\gamma'(1-\delta)}} \le C_{10}, \tag{25}$$

provided that $p/\left(1+|u|^\delta\right) \in L^{\alpha,\gamma'}$ with $\alpha \ge 2$ and $\gamma' = \gamma/(1-\delta) > 6/(2-3\delta)(1-\delta)$, where C_{10} is a constant depending on γ, $|\Omega|$, $||u_0||_{L^\gamma}$, $\left\|p/\left(1+|u|^\delta\right)\right\|_{L^{\alpha,\gamma'}}$.

Case 2) : $2/3 \le \delta \le 8/9$.

In this case, if (C2) is satisfied, for $\delta \ne 8/9$ and $18/(8-9\delta) \le \gamma < \infty$, then argument (17) still holds. Thus we can also obtain the bound (7).

For $(\alpha,\gamma) = (4/(4-3\delta), \infty)$, we note that a and b can be chosen such that

$$a = \frac{1}{1-\delta} \text{ and } b = \frac{2}{3\delta-2},\ 2/3 < \delta < 8/9; \quad b = \infty,\ \delta = 2/3.$$

We then have the following estimate

$$\begin{aligned}
||u(.,t)||_{L^3}^3 &\le C_6\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\frac{4}{4-3\delta},\infty}}^{\frac{4}{4-3\delta}} \|u\|_{L^{\infty,3}}^3 \\
&+ C_3\left\|\frac{p}{1+|u|^\delta}\right\|_{L^{\frac{4}{4-3\delta},\infty}}^2 T^{\frac{3\delta-2}{2}} |\Omega|^{1-\delta} + ||u_0||_{L^3}^3.
\end{aligned}$$

With a similar argument as for $\gamma < \infty$, we again get (7).

If $\delta = 8/9$, then $(\alpha, \gamma) = (3, \infty)$, and therefore $a = 9$, $b = 3$, and from (15), we have that

$$\begin{aligned} &||u(.,t)||_{L^3}^3 + \frac{3}{2} \left\| \nabla |u|^{\frac{3}{2}} \right\|_{L^{2,2}}^2 \\ &\leq C_3 \left\| \frac{p}{1+|u|^\delta} \right\|_{L^{3,\infty}}^2 \|u\|_{L^{3,9}} + C_3 \left\| \frac{p}{1+|u|^\delta} \right\|_{L^{3,\infty}}^2 T^{\frac{1}{3}} |\Omega|^{\frac{1}{9}} + ||u_0||_{L^3}^3, \end{aligned}$$

so (7) also follows immediately.

In order to complete the proof of our theorem, we apply Lemma 1.5 to our estimates. Since $u_0 \in L^q(\Omega)$ for some $q > 3$, and $u_0 \in L^s(\Omega)$ for all $s \in (3, q)$. There exists due to Lemma 1.5, a maximal interval $[0, T_*)$ such that there is a unique solution $\tilde{u}(x,t) \in BC([0,T_*); L^s(\Omega))$. Since u is a Leray-Hopf weak solution which satisfies the energy inequality, we have by the uniqueness criterion of Serrin-Masuda [21], [18] that

$$u(x,t) = \tilde{u}(x,t), \quad \text{on } [0, T_*).$$

Considering the first case of (C1) and (C2), we suppose that this interval is maximal and $T_* < T$. Then u is actually a strong solution on $[0, T_*)$, and we have the estimate (7). Combining (7) and (6), we obtain the following contradiction

$$\infty = \int_0^{T_*} \frac{C^3}{T_* - \tau} d\tau \leq \int_0^{T_*} ||u(.,\tau)||_{L^9}^3 d\tau < \infty \ .$$

Therefore $u(x,t)$ is a strong solution in $[0,T]$. If we are in the second case of (C1), then by the a priori estimate (25) and a standard continuation argument, we can continue our local solution corresponding to u_0 to obtain $u(x,t) \in BC([0,T]; L^s(\Omega)) \cap C^\infty(\Omega \times [0,T])$ for $s > 3$. The proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

The proof of Theorem 1.2 is very similar to the one of Theorem 1.1. We therefore present the main estimates. From (8), we have for $s \geq 3$,

$$\begin{aligned}
&\|u(.,t)\|_{L^s}^s + 2\left\|\nabla |u|^{\frac{s}{2}}\right\|_{L^{2,2}}^2 \\
&\leq s\int_0^t\int_\Omega |\nabla p|\,|u|^{s-1}dxd\tau + \|u_0\|_{L^s}^s \\
&\leq s\int_0^t\int_\Omega \left|\frac{\nabla p}{1+|u|^\delta}\right|(1+|u|)^{s+\delta-1}\,dxd\tau + \|u_0\|_{L^s}^s \\
&\leq s\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}\|1+|u|\|_{L^{a,b}}^{s-1}\,\|1+|u|\|_{L^{\infty,2}}^{\delta} + \|u_0\|_{L^s}^s \\
&\leq C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}\|u\|_{L^{a,b}}^{s-1} + C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}} T^{\frac{s-1}{a}}\,|\Omega|^{\frac{s-1}{b}} + \|u_0\|_{L^s}^s \ ,
\end{aligned}$$

where we have used Hölder's inequality and

$$\frac{1}{\alpha}+\frac{s-1}{a}=1\ ,$$
$$\frac{1}{\gamma}+\frac{s-1}{b}+\frac{\delta}{2}=1\ .$$

It follows that

$$\frac{s}{a}=\frac{s}{s-1}\left(1-\frac{1}{\alpha}\right)\ , \tag{26}$$
$$\frac{s}{b}=\frac{s}{s-1}\left(\frac{2-\delta}{2}-\frac{1}{\gamma}\right)\ . \tag{27}$$

Similarly, since a and b also have to satisfy the following additional conditions

$$\frac{s}{a}+\frac{3s}{2b}\geq\frac{3}{2}\ , \tag{28}$$

with

$$s\leq b\leq 3s\ . \tag{29}$$

If condition (H1) is satisfied for $0 \leq \delta < 2/3$, one obtains from (26), (27) and the condition $2/\alpha + 3/\gamma = (6-3\delta)/2$,

$$\begin{aligned}
\frac{s}{a}+\frac{3s}{2b} &= \frac{s}{s-1}\left(\frac{10-3\delta}{4}-\left(\frac{1}{\alpha}+\frac{3}{2\gamma}\right)\right) \\
&\geq \frac{s}{s-1}\ .
\end{aligned}$$

In order to apply Lemma 2.1, we have to take $s = 3$, and we get, using in addition a Sobolev inequality

$$\begin{aligned}
&\|u(.,t)\|_{L^3}^3 + 2\left\|\nabla |u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2 \\
&\leq C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}\|u\|_{L^{a,b}}^2 + C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}} T^{\frac{2}{a}}|\Omega|^{\frac{2}{b}} + \|u_0\|_{L^3}^3 \ , \\
&\leq C_{12}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}\|u\|_{L^{\infty,3}}^{\frac{3}{\alpha}-1}\|u\|_{L^{3,9}}^{3-\frac{3}{\alpha}} + C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}} T^{1-\frac{1}{\alpha}}|\Omega|^{\frac{2}{3\alpha}} + \|u_0\|_{L^3}^3 \ , \\
&\leq C_{13}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^{\alpha}\|u\|_{L^{\infty,3}}^{3-\alpha} + \left\|\nabla |u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2 + C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}} T^{1-\frac{1}{\alpha}}|\Omega|^{\frac{2}{3\alpha}} + \|u_0\|_{L^3}^3 \ .
\end{aligned}$$

Hence

$$\begin{aligned}
&\|u(.,t)\|_{L^3}^3 + \left\|\nabla |u|^{\frac{3}{2}}\right\|_{L^{2,2}}^2 \\
&\leq C_{13}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}}^{\alpha}\|u\|_{L^{\infty,3}}^{3-\alpha} + C_{11}\left\|\frac{\nabla p}{1+|u|^\delta}\right\|_{L^{\alpha,\gamma}} T^{1-\frac{1}{\alpha}}|\Omega|^{\frac{2}{3\alpha}} + \|u_0\|_{L^3}^3 \ . \qquad (30)
\end{aligned}$$

Then using again the argument as for (C1), by the integrability of $\nabla p/\left(1+|u|^\delta\right)$ with respect to the time variable, (7) is a easy consequence of (30).

If $\gamma > 6/(2-3\delta)$, we can find that if $2/\alpha + 3/\gamma = \beta_1 < (6-3\delta)/2$, then s can be chosen larger than 3. One obtains

$$\begin{aligned}
\frac{s}{a} + \frac{3s}{2b} &= \frac{s}{s-1}\left(\frac{10-3\delta}{4} - \frac{\beta_1}{2}\right) \\
&= \frac{s\,(10-3\delta-2\beta_1)}{4\,(s-1)} > \frac{3}{2}.
\end{aligned}$$

If $\beta_1 \leq (4-3\delta)/2$, the above inequality holds for all $s \geq 3$. Otherwise, we have

$$3 < s \leq \frac{6}{3\delta + 2\beta_1 - 4}. \qquad (31)$$

Note that the inequality (29) implies that

$$\frac{s-1}{3s} \leq \left(\frac{2-\delta}{2} - \frac{1}{\gamma}\right) \leq \frac{s-1}{s}, \qquad (32)$$

and

$$\frac{6}{2-3\delta} < \gamma \leq \frac{2s}{2-s\delta}. \qquad (33)$$

We notice that (32) is obvious for $2/s \leq \delta < 2/3$. Otherwise, we have (33). Combining (31) and (33), it follows that

$$\frac{3\alpha}{\alpha\beta_1 - 2} = \gamma \leq \frac{2s}{2-s\delta} \leq \frac{3}{\beta_1 - 2},$$

which gives

$$\alpha \geq 1 \quad \text{and} \quad \beta_1 \leq 2 + \frac{3}{\gamma}. \tag{34}$$

By (34) and arguments similar to the ones given in the second case of (C1), we have

$$\sup_{0 \leq t < T} ||u||_{L^{\gamma'(1-\delta)}} \leq C_{14}, \tag{35}$$

provided that $\nabla p / \left(1 + |u|^\delta\right) \in L^{\alpha,\gamma'}$ with $\alpha \geq 1$ and $\gamma' = \gamma / (1-\delta) > 6/(2-3\delta)(1-\delta)$, where C_{14} is a constant depending on γ, $|\Omega|$, $||u_0||_{L^\gamma}$, $\left\|\nabla p / \left(1 + |u|^\delta\right)\right\|_{L^{\alpha,\gamma'}}$.

If (H2) is satisfied for $2/3 \leq \delta \leq 14/9$, we obtain the estimate (7) by similar arguments in the case (C2) of Theorem 1.1.

The remaining part is very similar as the proof of Theorem 1.1, given the a priori estimates (7) and (35). This completes the proof.

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References

[1] H. Beirão da Veiga, A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$, Chinese Ann. Math. Ser. B. 16 (1995) 407-412.

[2] H. Beirão da Veiga, A sufficient condition on the pressure for the regularity of weak solutions to the Navier-Stokes equations, J. Math. Fluid Mech. 2 (2000) 99-106.

[3] L.C. Berselli, G.P. Galdi, Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations, Proc. Amer. Math. Soc. 130 (2002) 3585-3595.

[4] L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math. 35 (1982) 771-831.

[5] D. Chae, H-J. Choe, On the regularity criterion for the solutions of the Navier-Stokes equations, Electron. J. Differential Equations 1999 (1999) 1-7.

[6] C. Cao, E.S. Titi, Regularity criteria for the three dimensional Navier-Stokes equations, Indiana Univ. Math. J. 57 (2008) 2643-2661.

[7] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations 62 (1986) 186-212.

[8] Z. Guo and S. Gala, A regularity criterion for the Navier-Stokes equations in terms of one directional derivative of the velocity field. Submitted (2010).

[9] Z. Guo and S. Gala, Remarks on logarithmical regularity criteria for the Navier-Stokes equations, Submitted (2010).

[10] Z. Guo, Regularity criteria concerning the quotients of pressure, its gradient and gradient of velocity for the Navier-Stokes equations, Submitted (2010).

[11] E. Hopf, Über die Anfangwertaufgaben für die hydromischen Grundgleichungen, Math. Nachr. 4 (1951) 213-321.

[12] H. Iwashita, L^q-L^r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L^q spaces, Math. Ann. 285 (1989) 265-288.

[13] T. Kato, Strong L^p-solutions to the Navier-Stokes equations in $\mathbb{R}^m$, with applications to weak solutions, Math. Z. 187 (1984) 471-480.

[14] H. Kozono, H. Sohr, Regularity criterion on weak solutions to the Navier-Stokes equations, Adv. Differential. Equations 2 (1997) 535-554.

[15] H. Kozono, Y. Taniuchi, Bilinear estimates in BMO and the Navier-Stokes equations, Math. Z. 235 (2000) 173-194.

[16] H. Kozono, Global L^n-solution and its decay property for the Navier-Stokes equations in half space R^n_+, J. Differential Equations 79 (1989) 79-88.

[17] J. Leray, Étude de divers équations intégrales nonlinearies et de quelques problemes que posent lhydrodinamique, J. Math. Pures Appl. 12 (1931) 1-82.

[18] K. Masuda, Weak solutions of the Navier-Stokes equations, Tohoku Math. J. 36 (1984) 623-646.

[19] M. Núñez, Regularity criteria for the Navier-Stokes equations involving the ratio pressure-gradient of velocity, Math. Method. Appl. Sci. 33 (2010) 323-331.

[20] V. Scheffer, Partial regularity of solutions to the Navier-Stokes equations, Pacific J. Math. 66 (1976) 535-552.

[21] J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, Arch. Ration. Mech. Anal. 9 (1962) 187-195.

[22] M. Struwe, On partial regularity results for the Navier-Stokes equations, Comm. Pure Appl. Math. 41 (1988) 437-458.

[23] G. Tian, Z. Xin, Gradient estimation on Navier-Stokes equations, Comm. Anal. Geom. 7 (1999) 221-257.

[24] W. von Wahl, Regularity of weak solutions of the Navier-Stokes equations. Proceedings of the 1983 Summer Institute on Nonlinear Functional Analysis and Applications, Proc. Symposia in Pure Mathematics 45, Providence Rhode Island: Amer. Math. Soc. (1989) pp. 497-503.

[25] Y. Zhou, On regularity criteria in terms of pressure for the Navier-Stokes equations in $\mathbb{R}^3$, Proc. Amer. Math. Soc. 134 (2006) 149-156.

[26] Y. Zhou, A new regularity criterion for the Navier-Stokes equations in terms of the gradient of one velocity component, Methods Appl. Anal. 9 (2002) 563-578.

[27] Y. Zhou, A new regularity criterion for weak solutions to the Navier-Stokes equations, J. Math. Pures Appl. 84 (2005) 1496-1514.

[28] Y. Zhou, Regularity criteria in terms of pressure for the 3-D Navier-Stokes equations in a generic domain, Math. Ann. 328 (2004) 173-192.

[29] Y. Zhou, On a regularity criterion in terms of the gradient of pressure for the Navier-Stokes equations in $\mathbb{R}^N$, Z. Angew. Math. Phys. 57 (2006) 384-392.

[30] Y. Zhou, M. Pokorný, On a regularity criterion for the Navier-Stokes equations involving gradient of one velocity component, J. Math. Phys. 50 (2009) 123514.

[31] Y. Zhou, M. Pokorný, On the regularity of the solutions of the Navier-Stokes equations via one velocity component, Nonlinearity 23 (2010) 1097-1107.