Time periodic solutions of the Navier-Stokes equations with nonzero constant boundary conditions at infinity

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Abstract

We construct solutions for the Navier-Stokes equations in three dimensions with a time periodic force which is of compact support in a frame that moves at constant speed. These solutions are related to solutions of the problem of a body which moves within an incompressible fluid at constant speed and rotates around an axis which is aligned with the motion. In contrast to other authors who analayze stationary solutions in a frame of reference attached to the body, the analysis for the present problem is done in a frame which is moving at constant speed but is non rotating. This avoids the unpleasant unbounded linear terms which are present in a description in a rotating frame.

Contents

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1 Introduction

The classic paper of H. F. Weinberger [9] concerning the steady fall of a body in a Navier-Stokes liquid starts with the definition: “We say a body undergoes a steady falling motion in an infinite viscous fluid if the motion of the fluid as seen by an observer attached to the body is independent of time.”

One of the interesting possible cases is a body that is falling steadily, and is rotating around an axis that is parallel to the direction in which the body is falling.

A first proof of the existence of such solutions for this case has been given only recently in the three papers by G. P. Galdi and Ana Silvestre [4, 3, 2]. Their method for solving the problem is to consider the equations, as proposed by Weinberger, in a frame attached to the body, where the flow is stationary. In this frame the Navier-Stokes equations have an additional linear term with unbounded coefficients, which is due to the transformation into the rotating frame. This complicates the problem considerably when compared to the situation without rotation.

It is important to note that even without the rotation the problem is difficult because of the slow decay of the vorticity in the downstream region. This leads to a very strong asymmetry in the behavior at infinity and the main difficulty is to encode this behavior when choosing function spaces.

Once the existence of a solution is established one is interested in giving detailed information concerning its behavior at infinity. Like in related problems this behavior is expected to be independent of the details of the body. It turns out to be possible to use this fact in order to simplify the analysis of the asymptotic behavior, by considering first the problem in the whole space, and to mimic the body by a smooth force of compact support (see the end of this section for details). The case with a body, i.e. , the case of an exterior domain, can then be treated in a second step, once the behavior at infinity is understood. For a related problem in 2D, this strategy has been implemented in [7].

The present paper contains the results for the first step of the strategy in [7]. Namely, we give a proof of the existence of a solution for the problem in the whole space, where the body is replaced by a force term of compact support.

Our strategy for constructing solutions is different from the one used in [4, 3, 2]. Instead of constructing stationary solutions in a frame which moves with constant speed and rotates, we choose a frame that is only moving but not rotating, i.e. , we consider the Navier-Stokes equations (the $\partial_x u$ term is due to the translational motion of the frame),

$$\partial_t \mathbf{u} = -\partial_x \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} + \Delta\mathbf{u} - \nabla p + \mathbf{f} \ , \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \ , \tag{2}$$

$$\lim_{x\to\infty} \mathbf{u}(x) = 0 \ , \tag{3}$$

in $\Omega = \mathbb{R}^3$, with $\mathbf{f}$ a given vector field, compactly supported and time periodic with some frequency λ.

This choice of coordinate system avoids the unpleasant unbounded linear terms mentioned before, but the solution is not time independent but time periodic in our coordinates and we expand it therefore into a Fourier series. The resulting equations are then treated with the methods that we have developed in [8, 5], which allow to treat the asymmetric behavior of the solutions in an optimal way. Namely, we choose a coordinate system where the body falls along the (negative) $x-$axis, and then consider the $x-$axis as a “time”-coordinate and look at the “time” evolution of the resulting system for positive times and negative times, choosing Sobolev type norms in the variables transverse to the “time” direction and weighted supremum norms in the “time”-coordinate. These ideas lead to a very natural functional setting and allow for a very detailed description of the behavior at positive and negative “times”. We show in particular that the vorticity decays exponentially fast in the negative x-direction and like $x^{-3/2}$ within the wake in the positive x-direction. The description of the velocity field is equally precise.

Instead of looking at (1)–(3) we rather look at the equations for the vorticity, i.e. , we solve the system

$$\partial_t \boldsymbol{\omega} + \partial_x \boldsymbol{\omega} - \Delta \boldsymbol{\omega} = \nabla \times [(\mathbf{u} \times \boldsymbol{\omega}) + \mathbf{f}] \ , \tag{4}$$

$$\nabla \cdot \mathbf{u} = 0 \ , \tag{5}$$

$$\nabla \times \mathbf{u} = \boldsymbol{\omega} \ . \tag{6}$$

In order to solve equations (4)–(6) we proceed as follows. We define the Biot-Savart map $\mathcal{K}$,

$$\mathcal{K}\boldsymbol{\omega} = \mathbf{u} \ , \tag{7}$$

with $\mathbf{u}$ defined as the solution of (5)–(6), and the map $\mathcal{C}$ (the nonlinearity),

$$\mathcal{C}(\mathbf{u}, \boldsymbol{\omega}) = \mathbf{u} \times \boldsymbol{\omega} \ . \tag{8}$$

We further define the map $\mathcal{S}$,

$$\mathcal{S}\mathbf{q} = \nabla \times (\partial_t + \partial_x - \Delta)^{-1} \mathbf{q} \ . \tag{9}$$

Then, we consider, for a given $\mathbf{f}$ that is $\frac{2\pi}{\lambda}$-periodic in time, the map $\mathcal{N}$ defined by

$$\mathcal{N}(\boldsymbol{\omega}) = \mathcal{S}\left(\mathcal{C}(\mathcal{K}\boldsymbol{\omega}, \boldsymbol{\omega}) + \mathbf{f}\right) \ . \tag{10}$$

Below we show that $\mathcal{N}$ defines a differentiable map on a certain Banach space, and contracts, for small $\mathbf{f}$, a neighborhood of zero into itself. By the contraction mapping principle there exists therefore a (locally unique) solution to the equation $\boldsymbol{\omega} = \mathcal{N}(\boldsymbol{\omega})$. The Banach space chosen below will give detailed decay rates for the vorticity at large spatial distances. In a future publication, we plan to show how this information can be used to give an asymptotic expansion of the vorticity and velocity fields in the downstream direction, see [10, 8] for similar results in the stationary case in 3D, and in the time-periodic case in 2D.

The paper is organized as follows. In Section 2, we give our functional setting and formulate our existence result. In Section 3, we study the Biot-Savart map $\mathcal{K}$ and in Section 4 the nonlinear map $\mathbf{u} \times \boldsymbol{\omega}$. The core of the paper is in Section 5, where we study the map $\mathcal{S} = \nabla \times (\partial_t + \partial_x - \triangle)^{-1}$.

Before concluding this section, we explain in more details how the model (1)–(3) relates to flows in exterior domains. To do so, let $\Omega(t) \subset \mathbb{R}^3$ be an exterior domain, i.e. $\Omega(t)^c = \mathbb{R}^3 \setminus \Omega(t)$ is compact (and smooth) for all times, and denote by $\delta(\Omega(t)^c)$ the radius of the 'obstacle' $\Omega(t)^c$. Using a time dependent Ω is necessary to describe, say, a rotating obstacle. For readability, this explicit dependence is omitted below. Consider then the following Navier-Stokes system in Ω

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \triangle \mathbf{u} + \nabla p = 0 \ , \tag{11}$$

$$\nabla \cdot \mathbf{u} = 0 \ , \tag{12}$$

$$\lim_{|\mathbf{x}| \to \infty} \mathbf{u}(\mathbf{x}, t) = \mathbf{u}_\infty \neq (0, 0, 0) \ , \tag{13}$$

$$\mathbf{u}|_{\partial\Omega} = 0 \ . \tag{14}$$

In Appendix C (see Proposition 21), we prove the existence of a so-called extension map $\mathrm{E}_{a,b}$ from Ω to $\mathbb{R}^3$. Namely we prove that if $\nabla \mathbf{u} \in \mathrm{L}^2(\Omega)$, there exists $\mathrm{E}_{a,b}[\mathbf{u}] : \mathbb{R}^3 \to \mathbb{R}^3$ which interpolates smoothly between $\mathrm{E}_{a,b}[\mathbf{u}](\mathbf{x}) = 0$ if $|\mathbf{x}| \leq a\delta(\Omega^c)$ and $\mathrm{E}_{a,b}[\mathbf{u}](\mathbf{x}) = \mathbf{u}(\mathbf{x})$ if $|\mathbf{x}| \geq b\delta(\Omega^c)$ for some $1 < a < b$, and satisfies $\nabla \cdot \mathrm{E}_{a,b}[\mathbf{u}](\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^3$.

Assume now that a solution of (11)–(14) exists, and set $\tilde{\mathbf{u}} = \mathrm{E}_{a,b}[\mathbf{u}]$ and $\tilde{\boldsymbol{\omega}} = \nabla \times \tilde{\mathbf{u}}$. Consider then

$$\mathbf{F}[\mathbf{u}] = \partial_t \tilde{\mathbf{u}} - \triangle \tilde{\mathbf{u}} - \tilde{\mathbf{u}} \times \tilde{\boldsymbol{\omega}} \ . \tag{15}$$

Clearly, $\mathbf{F}[\mathbf{u}] : \mathbb{R}^3 \to \mathbb{R}^3$ is supported in the annulus $a\delta(\Omega^c) \leq |\mathbf{x}| \leq b\delta(\Omega^c)$. Namely, if $|\mathbf{x}| \geq b\delta(\Omega^c)$, the r.h.s. of (15) is the l.h.s. of (11), and so $\mathbf{F}[\mathbf{u}](\mathbf{x}) = 0$ for $|\mathbf{x}| \geq b\delta(\Omega^c)$. Similarly, if $|\mathbf{x}| \leq a\delta(\Omega^c)$, $\tilde{\mathbf{u}}(\mathbf{x}) = \tilde{\boldsymbol{\omega}}(\mathbf{x}) = 0$, and so $\mathbf{F}[\mathbf{u}](\mathbf{x}) = 0$ for $|\mathbf{x}| \leq a\delta(\Omega^c)$.

The idea is now to consider $\mathbf{F}[\mathbf{u}]$ as given and $\tilde{\boldsymbol{\omega}}$ and $\tilde{u}$ as *unknowns* solving the following system holding in the whole space $\mathbb{R}^3$:

$$\partial_t \tilde{\boldsymbol{\omega}} = \triangle \tilde{\boldsymbol{\omega}} + \nabla \times (\tilde{\mathbf{u}} \times \tilde{\boldsymbol{\omega}} + \mathbf{F}) \,, \quad \nabla \cdot \tilde{\mathbf{u}} = 0 \quad \text{and} \quad \tilde{\boldsymbol{\omega}} = \nabla \times \tilde{\mathbf{u}} \tag{16}$$

where $\mathbf{F} = \mathbf{F}[\mathbf{u}]$ is compactly supported and the curl of a function. Up to translation of the velocities by $\mathbf{u}_\infty$ and choice of axes/units where $\mathbf{u}_\infty = (1,0,0)$, (16) is the same as (4)–(6). Since $\tilde{\mathbf{u}} = \mathbf{u}$ and $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}$ for $|\mathbf{x}| \geq b\delta(\Omega^c)$, studying large-distance behavior of solutions of (16) in the whole space $\mathbb{R}^3$ is a first step in understanding large-distance asymptotics of solutions of the Navier-Stokes equations in exterior domains. We now briefly sketch the strategy that we hope to implement in a future paper. The first step is is to start from a weak formulation of the problem with a body and to prove existence of a solution. We expect such solutions to be smooth, but, generally, to know little information about their behavior at infinity. The second step is then to truncate such a weak solution as described above. This provides a source term for the problem in the whole space which is then treated with the techniques in the present paper. The third step is to use the information obtained from the present paper to prove a weak-strong uniqueness result. This last result would then allow us to conclude that the original weak solution and the solution discussed here coincide outside the cut-off region. The above steps will be done in a later paper. This strategy has been implemented with success in a similar problem in [7].

2 Main result

In order to formulate precisely our main result, we first introduce some notation and function spaces. Let $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$, $r = \sqrt{x^2+y^2+z^2}$, and for $0 \leq \alpha \leq 1$,

$$W_\alpha(x,y,z) = \mathrm{e}^{\frac{|x|-x}{2}\alpha + \frac{r-x}{2}(1-\alpha)} \,.$$

Since $W_1(x,y,z) = \mathrm{e}^{\frac{|x|-x}{2}}$, we often simply write $W_1(x)$ instead of $W_1(x,y,z)$. It will be important later on that $W_\alpha \geq 1$ and $W_\alpha \geq W_1$ for all $\alpha \in [0,1]$. We will use the symbols $\triangle_{\mathrm{T}}$ and ∇_{T} to describe the *transverse* Laplacian, i.e. $\triangle_{\mathrm{T}} f(x,y,z) = (\partial_y^2 + \partial_z^2) f(x,y,z)$ and *transverse* gradient $\nabla_{\mathrm{T}} f(x,y,z) = (0, \partial_y f(x,y,z), \partial_z f(x,y,z))$.

Definition 1 *For fixed $0 \leq \alpha \leq 1$ and $\lambda > 0$, we define:*

1. $\mathcal{C}^\infty_{0,\mathrm{per}}(\mathbb{R}^4,\mathbb{R}^3)$ *to be the space of smooth (vector) functions that are compactly supported in their first three arguments and $\frac{2\pi}{\lambda}$-periodic in their last argument. For any $\mathbf{f} \in \mathcal{C}^\infty_{0,\mathrm{per}}(\mathbb{R}^4,\mathbb{R}^3)$, we define its Fourier coefficient $\mathbf{f}_n \in \mathcal{C}^\infty_0(\mathbb{R}^3,\mathbb{R}^3)$ for any $n \in \mathbb{Z}$ by*

$$\mathbf{f}_n(\mathbf{x}) = \frac{\lambda}{2\pi}\int_0^{\frac{2\pi}{\lambda}} \mathrm{d}t\, \mathrm{e}^{-in\lambda t}\, \mathbf{f}(\mathbf{x},t) \quad \Leftrightarrow \quad \mathbf{f}(\mathbf{x},t) = \sum_{n\in\mathbb{Z}} \mathbf{f}_n(\mathbf{x}) e^{i\lambda n t} \,.$$

2. $\mathcal{C}^\infty_{0,\mathrm{per,sol}}(\mathbb{R}^4,\mathbb{R}^3)$ *to be the subset of those $\mathbf{f} \in \mathcal{C}^\infty_{0,\mathrm{per}}(\mathbb{R}^4,\mathbb{R}^3)$ satisfying $\nabla \cdot \mathbf{f} = 0$.*

3. $\mathcal{W}_\alpha$ *to be the Banach space obtained by completing* $\mathcal{C}^\infty_{0,\mathrm{per,sol}}(\mathbb{R}^4,\mathbb{R}^3)$*, with respect to the norm*

$$\|\mathbf{f};\mathcal{W}_\alpha\| = \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^{\frac{1}{2}}}{(1+|x|-x)^{\frac{1}{2}}} \|W_\alpha(x,\cdot)\mathbf{f}_n(x,\cdot)\|_1 + \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^{\frac{3}{2}}}{(1+|x|-x)^{\frac{1}{2}}} \|W_\alpha(x,\cdot)\mathbf{f}_n(x,\cdot)\|_\infty + \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^2}{(1+|x|-x)^{\frac{1}{2}}} \|W_1(x)\triangle_{\mathrm{T}}\mathbf{f}_n(x,\cdot)\|_2 \, .$$

4. $\mathcal{U}$ *to be the Banach space obtained by completing* $\mathcal{C}^\infty_{0,\mathrm{per,sol}}(\mathbb{R}^4,\mathbb{R}^3)$*, with respect to the norm*

$$\|\mathbf{f};\mathcal{U}\| = \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^{\frac{1}{2}}}{\ln(2+|x|+x)} \|\mathbf{f}_n(x,\cdot)\|_2 + \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)(1+|x|-x)^{\frac{1}{2}} \Big((\,\|\mathbf{f}_n(x,\cdot)\|_\infty + \|\nabla_{\mathrm{T}}\mathbf{f}_n(x,\cdot)\|_2 \Big) + \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{3}{2}}(1+|x|-x)^{\frac{1}{2}} \|\triangle_{\mathrm{T}}\mathbf{f}_n(x,\cdot)\|_2 \, .$$

5. $\mathcal{Q}_\alpha$ *to be the Banach space obtained by completing* $\mathcal{C}^\infty_{0,\mathrm{per}}(\mathbb{R}^4,\mathbb{R}^3)$*, with respect to the norm*

$$\|\mathbf{f};\mathcal{Q}_\alpha\| = \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{5}{2}} \|W_\alpha(x,\cdot)\mathbf{f}_n(x,\cdot)\|_\infty + \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{3}{2}} \|W_\alpha(x,\cdot)\mathbf{f}_n(x,\cdot)\|_1 + \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^3 \|W_1(x)\triangle_{\mathrm{T}}\mathbf{f}_n(x,\cdot)\|_2 \, .$$

We then have:

Lemma 2 *For all* $0 \le \alpha \le 1$*, the (linear) Biot-Savart map* $\mathcal{K}$

$$\mathcal{K}\colon \mathcal{W}_\alpha \to \mathcal{U}, \quad \boldsymbol{\omega} \mapsto \mathbf{u}$$

such that $\mathcal{K}\boldsymbol{\omega} = u$ *is the (unique) solution of* $\nabla\cdot\mathbf{u} = 0$ *and* $\nabla\times\mathbf{u} = \boldsymbol{\omega}$ *is well defined and continuous.*

Lemma 3 *For all* $0 \le \alpha \le 1$*, the bilinear map*

$$\mathcal{C}\colon \mathcal{U}\times\mathcal{W}_\alpha \to \mathcal{Q}_\alpha, \quad (\mathbf{u},\boldsymbol{\omega}) \mapsto \mathbf{u}\times\boldsymbol{\omega}$$

is well defined and continuous.

Lemma 4 *Let* $0 < \alpha \le 1$*. Then, the linear map* $\mathcal{S}$

$$\mathcal{S}\colon \mathcal{Q}_\alpha \to \mathcal{W}_\alpha, \quad \mathbf{q} \mapsto \boldsymbol{\omega}$$

such that $\boldsymbol{\omega} = \mathcal{S}\mathbf{q}$ *solves* $(\partial_t + \partial_x + \triangle)\boldsymbol{\omega} = \nabla\times\mathbf{q}$*, is well defined and continuous.*

Lemma 2 is proved in Section 3, Lemma 3 in Section 4 and Lemma 4 in Section 5. It follows from the Lemmata 2, 3 and 4 that the quantities

$$C_1(\alpha) \equiv \|\mathcal{S}; \mathcal{L}(\mathcal{Q}_\alpha; \mathcal{W}_\alpha)\| \cdot \|\mathcal{C}; \mathcal{L}(\mathcal{U}, \mathcal{W}_\alpha; \mathcal{Q}_\alpha)\| \cdot \|\mathcal{K}; \mathcal{L}(\mathcal{W}_\alpha; \mathcal{U})\| ,$$
$$C_2(\alpha) \equiv \|\mathcal{S}; \mathcal{L}(\mathcal{Q}_\alpha; \mathcal{W}_\alpha)\|$$

are finite for all $0 < \alpha \leq 1$, where, as usual, $\|\mathcal{A}; \mathcal{L}(\mathcal{B}; \mathcal{D})\|$ denotes the operator norm of $\mathcal{A} \in \mathcal{L}(\mathcal{B}, \mathcal{D})$, the space of continuous linear maps from $\mathcal{B}$ to $\mathcal{D}$.

Definition 5 *Let $0 < \alpha \leq 1$ and $\mathbf{f} \in \mathcal{Q}_\alpha$ with $\|\mathbf{f}; \mathcal{Q}_\alpha\| < \infty$. Then $\boldsymbol{\omega}$ is called an α-solution of (4)–(6) if:*

(i) $\boldsymbol{\omega} \in \mathcal{W}_\alpha$,

(ii) $\mathcal{N}(\boldsymbol{\omega}) \equiv \mathcal{S}(\mathcal{C}(\mathcal{K}\boldsymbol{\omega}, \boldsymbol{\omega}) + \mathbf{f}) = \boldsymbol{\omega}$.

With this definition at hand we can now give a precise formulation of our main result:

Theorem 6 *(Existence) Let $0 < \alpha \leq 1$, $\mathbf{f} \in \mathcal{Q}_\alpha$ and assume that $\|\mathbf{f}; \mathcal{Q}_\alpha\| \leq (4C_1(\alpha)C_2(\alpha))^{-1}$. Then there exists an α-solution $\boldsymbol{\omega}$ in $\mathcal{W}_\alpha$. This solution is unique in the ball*

$$\mathcal{B}(\alpha, \|\mathbf{f}; \mathcal{Q}_\alpha\|) = \left\{ \boldsymbol{\omega} \in \mathcal{W}_\alpha \text{ s.t. } \|\boldsymbol{\omega}; \mathcal{W}_\alpha\| \leq \tfrac{1 - \sqrt{1 - 4C_1(\alpha)C_2(\alpha)\|\mathbf{f}; \mathcal{Q}_\alpha\|}}{2C_1(\alpha)} \right\} .$$

Proof. Lemmata 2, 3 and 4 imply that for all $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathrm{B}_\rho(\mathcal{W}_\alpha) = \{\boldsymbol{\omega} \in \mathcal{W}_\alpha \text{ s.t. } \|\boldsymbol{\omega}; \mathcal{W}_\alpha\| \leq \rho\}$,

$$\|\mathcal{N}(\boldsymbol{\omega}_i); \mathcal{W}_\alpha\| \leq C_1(\alpha)\rho^2 + C_2(\alpha)\|\mathbf{f}; \mathcal{Q}_\alpha\| ,$$
$$\|\mathcal{N}(\boldsymbol{\omega}_1) - \mathcal{N}(\boldsymbol{\omega}_2); \mathcal{W}_\alpha\| \leq 2C_1(\alpha)\rho\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2; \mathcal{W}_\alpha\| .$$

Since $\|\mathbf{f}; \mathcal{Q}_\alpha\| \leq (4C_1(\alpha)C_2(\alpha))^{-1}$, $\mathcal{N}$ is a contraction on the ball $\mathrm{B}_\rho(\mathcal{W}_\alpha)$ for

$$\rho = \frac{1 - \sqrt{1 - 4C_1(\alpha)C_2(\alpha)\|\mathbf{f}; \mathcal{Q}_\alpha\|}}{2C_1(\alpha)} < \frac{1}{2C_1(\alpha)} ,$$

which completes the proof. ■

3 The Biot-Savart map $\mathcal{K}$

Our purpose in this section is to derive estimates on the solution of

$$\nabla \times \mathbf{u} = \boldsymbol{\omega} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 , \tag{17}$$

where $\boldsymbol{\omega} \in \mathcal{W}_\alpha$ for $0 \leq \alpha \leq 1$. We will show the

Proposition 7 *For any $0 \leq \alpha \leq 1$, the solution map $\mathcal{K}$ of (17) is a continuous linear map from $\mathcal{W}_\alpha$ to $\mathcal{U}$.*

From now on, we will use the letter C to denote a numerical constant, whose value may change even within the same line, but is independent of the functions that are estimated.

Proof. Since $\nabla \cdot \boldsymbol{\omega} = 0$, taking the Fourier transform on $\mathbb{R}^3$, we find for the Fourier coefficients $\hat{\mathbf{u}}_n$ that

$$\hat{\mathbf{u}}_n(\mathbf{k}) = \frac{i\mathbf{k} \times \hat{\boldsymbol{\omega}}_n(\mathbf{k})}{|\mathbf{k}|^2} .$$

Performing the inverse Fourier transform with respect to k_1, we get the following pointwise estimate:

$$|\hat{\mathbf{u}}_n(x,k_2,k_3)| \leq \int_{-\infty}^{\infty} \mathrm{d}y \; \mathrm{e}^{-\sqrt{k_2^2+k_3^2}\;|x-y|} \; |\hat{\boldsymbol{\omega}}_n(y,k_2,k_3)| \; .$$

Let $\hat{K}(x,k_2,k_3) = \mathrm{e}^{-\sqrt{k_2^2+k_3^2}\;|x|}$. Since $\|\hat{K}(x,\cdot)\|_\infty = 1$ and $\|\hat{K}(x,\cdot)\|_2 \leq 2|x|^{-1}$, we find

$$\sum_{n\in\mathbb{Z}} \|\mathbf{u}_n(x,\cdot)\|_2 \leq \int_{|x-y|\geq 2} \frac{2C_1(\boldsymbol{\omega})\,\mathrm{d}y}{|x-y|\sqrt{1+|y|}} + \int_{|x-y|\leq 2} \frac{C_2(\boldsymbol{\omega})\,\mathrm{d}y}{(1+|y|)} \; , \tag{18}$$

$$\sum_{n\in\mathbb{Z}} \|\hat{\mathbf{u}}_n(x,\cdot)\|_1 + \|\nabla_{\mathrm{T}}\mathbf{u}_n(x,\cdot)\|_2 + \|\triangle_{\mathrm{T}}\mathbf{u}_n(x,\cdot)\|_2 \leq \int_{-\infty}^{\infty} \frac{C_3(\boldsymbol{\omega}) + C_4(\boldsymbol{\omega})\,\mathrm{d}y}{(1+|y|)^{\frac{3}{2}}} \; , \tag{19}$$

where

$$\begin{aligned}
C_1(\boldsymbol{\omega}) &\equiv \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{1}{2}}(1+|x|-x)\|\boldsymbol{\omega}_n(x,\cdot)\|_1 \; , \\
C_2(\boldsymbol{\omega}) &\equiv \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)(1+|x|-x)\|\boldsymbol{\omega}_n(x,\cdot)\|_2 \; , \\
C_3(\boldsymbol{\omega}) &\equiv \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{3}{2}}(1+|x|-x)\left(\|\hat{\boldsymbol{\omega}}_n(x,\cdot)\|_1 + \|\nabla_{\mathrm{T}}\boldsymbol{\omega}_n(x,\cdot)\|_2\right) \; , \\
C_4(\boldsymbol{\omega}) &\equiv \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^2(1+|x|-x)\|\triangle_{\mathrm{T}}\boldsymbol{\omega}_n(x,\cdot)\|_2 \; .
\end{aligned}$$

We show below that $\sum_{i=1}^4 C_i(\boldsymbol{\omega}) \leq C\|\boldsymbol{\omega};\mathcal{W}_\alpha\|$. For the moment, we note that

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \|\mathbf{u}_n(x,\cdot)\|_2 + \|\hat{\mathbf{u}}_n(x,\cdot)\|_1 + \|\nabla_{\mathrm{T}}\mathbf{u}_n(x,\cdot)\|_2 + \|\triangle_{\mathrm{T}}\mathbf{u}_n(x,\cdot)\|_2 \leq \sum_{i=1}^4 C_i(\boldsymbol{\omega}) \; ,$$

which does not yet give the decay as $|x| \to \infty$ needed for $\mathbf{u}$ to be in $\mathcal{U}$.

We first improve (18) and (19) for $x \geq 4$ by writing $|\hat{\mathbf{u}}_n(x,k_2,k_3)| \leq \hat{U}_n(x,k_2,k_3) + \hat{V}_n(x,k_2,k_3)$ where

$$\begin{aligned}
\hat{U}_n(x,k_2,k_3) &= \int_{x-\sqrt{|x|}}^{x+\sqrt{|x|}} \mathrm{d}y \; |\hat{\boldsymbol{\omega}}_n(y,k_2,k_3)| \; , \\
\hat{V}_n(x,k_2,k_3) &= \int_{|y-x|\geq\sqrt{|x|}} \mathrm{d}y \; \mathrm{e}^{-\sqrt{k_2^2+k_3^2}\;|x-y|} \; |\hat{\boldsymbol{\omega}}_n(y,k_2,k_3)| \; .
\end{aligned}$$

For convenience, we also define

$$\begin{aligned}
\mathrm{M}_1[\hat{F}](x) &= \|\hat{F}(x,\cdot)\|_2 \; , \\
\mathrm{M}_2[\hat{F}](x) &= \|\hat{F}(x,\cdot)\|_1 + \|k_2\hat{F}(x,\cdot)\|_2 + \|k_3\hat{F}(x,\cdot)\|_2 \; , \\
\mathrm{M}_3[\hat{F}](x) &= \|(k_2^2+k_3^2)\hat{F}(x,\cdot)\|_2 \; , \\
\mathrm{J}_p(x) &= |x|^{\frac{p}{2}} \begin{cases} \ln(|x|)^{-1} & \text{if} \quad p = 1 \\ 1 & \text{if} \quad p > 1 \end{cases} \; .
\end{aligned}$$

Since $\|\hat{K}(z,\cdot)\|_\infty = 1$, $\mathrm{M}_1[\hat{K}](z) \leq 2|z|^{-1}$, $\mathrm{M}_2[\hat{K}](z) \leq 10|z|^{-2}$, $\mathrm{M}_3[\hat{K}](z) \leq 4|z|^{-3}$ and

$$\sum_{n\in\mathbb{Z}} \sup_{z\in\mathbb{R}} (1+|z|)^{\frac{1+p}{2}} \mathrm{M}_p[\boldsymbol{\omega}_n](z) \leq C_{1+p}(\boldsymbol{\omega}) \quad \text{for} \quad p = 1,2,3 \; ,$$

we find that for all $p \in \{1,2,3\}$,

$$\sum_{n\in\mathbb{Z}} \sup_{x\geq 4} |x|^{\frac{p}{2}} \, \mathrm{M}_p[\hat{U}_n](x) \leq C_{1+p}(\boldsymbol{\omega}) \sup_{x\geq 4} |x|^{\frac{p}{2}} \int_{x-\sqrt{|x|}}^{x+\sqrt{|x|}} \frac{\mathrm{d}y}{(1+|y|)^{\frac{1+p}{2}}} \leq C\, C_{1+p}(\boldsymbol{\omega}) \;,$$
$$\sum_{n\in\mathbb{Z}} \sup_{x\geq 4} \mathrm{J}_p(x) \, \mathrm{M}_p[\hat{V}_n](x) \leq C_1(\boldsymbol{\omega}) \sup_{x\geq 4} \mathrm{J}_p(x)\mathrm{I}_p(x) \;,$$

where

$$\mathrm{I}_p(x) = \int_{-\infty}^{x-\sqrt{|x|}} \frac{\mathrm{d}y}{|x-y|^p\sqrt{1+|y|}} + \int_{x+\sqrt{|x|}}^{\infty} \frac{\mathrm{d}y}{|x-y|^p\sqrt{1+|y|}} \;. \tag{20}$$

We now claim that for any $p \geq 1$, we have $\sup_{x\geq 4} \mathrm{J}_p(x)\mathrm{I}_p(x) < \infty$. Changing variables in (20), we find

$$\mathrm{I}_p(x) = |x|^{\frac{1}{2}-p} \int_{1+\frac{1}{\sqrt{|x|}}}^{\infty} \frac{\mathrm{d}z}{|1-z|^p\sqrt{|z|}} + |x|^{\frac{1}{2}-p} \int_{-\infty}^{1-\frac{1}{\sqrt{|x|}}} \frac{\mathrm{d}z}{|1-z|^p\sqrt{|z|}} \;.$$

We consider the case $p=1$ first, finding

$$\begin{aligned}\mathrm{I}_1(x) &\leq |x|^{-\frac{1}{2}} \left(\int_{1+\frac{1}{\sqrt{|x|}}}^{2} \frac{\mathrm{d}z}{z-1} + \int_{2}^{\infty} \frac{\mathrm{d}z}{(z-1)\sqrt{z}} + \int_{\frac{1}{2}}^{1-\frac{1}{\sqrt{|x|}}} \frac{\mathrm{d}z}{1-z} + \int_{-\infty}^{\frac{1}{2}} \frac{\mathrm{d}z}{(1-z)\sqrt{|z|}} \right) \\ &\leq 6|x|^{-\frac{1}{2}} \ln(|x|) \;,\end{aligned}$$

while if $p > 1$,

$$\begin{aligned}\mathrm{I}_p(x) &\leq |x|^{\frac{1}{2}-p} \left(\int_{1+\frac{1}{\sqrt{|x|}}}^{\infty} \frac{\mathrm{d}z}{|1-z|^p} + \int_{\frac{1}{2}}^{1-\frac{1}{\sqrt{|x|}}} \frac{\mathrm{d}z}{|1-z|^p} + \int_{-\infty}^{\frac{1}{2}} \frac{\mathrm{d}z}{|1-z|^p\sqrt{|z|}} \right) \\ &\leq |x|^{-\frac{p}{2}} \left(5|x|^{\frac{1-p}{2}} + \frac{2}{p-1} \right) \;.\end{aligned}$$

Finally, we improve (18) and (19) for $x \leq -4$ by writing $|\hat{\mathbf{u}}_n(x,k_2,k_3)| \leq \hat{U}_n(x,k_2,k_3) + \hat{V}_n(x,k_2,k_3)$ where, this time, we set

$$\begin{aligned}\hat{U}_n(x,k_2,k_3) &= \int_{-\infty}^{\frac{x}{2}} \mathrm{d}y \; |\hat{\boldsymbol{\omega}}_n(y,k_2,k_3)| \;, \\ \hat{V}_n(x,k_2,k_3)| &= \int_{\frac{x}{2}}^{\infty} \mathrm{d}y \; \mathrm{e}^{-\sqrt{k_2^2+k_3^2}\,|x-y|} \; |\hat{\boldsymbol{\omega}}_n(y,k_2,k_3)| \;.\end{aligned}$$

Since

$$\sum_{n\in\mathbb{Z}} \sup_{z\leq 0} (1-z)^{\frac{3+p}{2}} \mathrm{M}_p[\boldsymbol{\omega}](z) \leq C_{1+p}(\boldsymbol{\omega}) \;,$$

we find for all $x \leq -4$ and $p \in \{1,2,3\}$ that

$$\sum_{n\in\mathbb{Z}} \sup_{x\leq -4} |x|^{\frac{1+p}{2}} \mathrm{M}_p[\hat{U}_n](x) \leq C\, C_{1+p}(\boldsymbol{\omega}) \sup_{x\leq -4} |x|^{\frac{1+p}{2}} \int_{-\infty}^{\frac{x}{2}} \frac{\mathrm{d}y}{(1-y)^{\frac{3+p}{2}}} \leq C\, C_{1+p}(\boldsymbol{\omega}) \;,$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\leq -4} |x|^{p-\frac{1}{2}} \mathrm{M}_p[\hat{V}_n](x) \leq C\, C_1(\boldsymbol{\omega}) \sup_{x\leq -4} |x|^{p-\frac{1}{2}} \int_{\frac{x}{2}}^{\infty} \frac{\mathrm{d}y}{|x-y|^p \sqrt{|y|}} \leq C\, C_1(\boldsymbol{\omega}) \ .$$

To complete the proof, we bound the $C_i(\boldsymbol{\omega})$ by first noting that

$$\|W_1(x)\boldsymbol{\omega}_n(x,\cdot)\|_p \leq \|W_\alpha(x,\cdot)\boldsymbol{\omega}_n(x,\cdot)\|_p \ ,$$

for all $\alpha \in [0,1]$, $n \in \mathbb{Z}$ and $1 \leq p \leq \infty$. We can then use Lemma 19 (see Appendix A) to get

$$\|W_1(x)\hat{\boldsymbol{\omega}}_n(x,\cdot)\|_1 \leq \|W_\alpha(x,\cdot)\boldsymbol{\omega}_n(x,\cdot)\|_2^{\frac{1}{2}} \cdot \|W_1(x)\triangle_{\mathrm{T}}\boldsymbol{\omega}_n(x,\cdot)\|_2^{\frac{1}{2}} \ ,$$
$$\|W_1(x)\nabla_{\mathrm{T}}\boldsymbol{\omega}_n(x,\cdot)\|_2 \leq \|W_\alpha(x,\cdot)\boldsymbol{\omega}_n(x,\cdot)\|_2^{\frac{1}{2}} \cdot \|W_1(x)\triangle_{\mathrm{T}}\boldsymbol{\omega}_n(x,\cdot)\|_2^{\frac{1}{2}} \ ,$$

which implies that

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^{\frac{3}{2}}}{(1+|x|-x)^{\frac{1}{2}}} \left(\|W_1(x)\hat{\boldsymbol{\omega}}_n(x,\cdot)\|_1 + \|W_1(x)\nabla_{\mathrm{T}}\boldsymbol{\omega}_n(x,\cdot)\|_2\right) \leq C\|\boldsymbol{\omega};\mathcal{W}_\alpha\| \ ,$$

giving $C_i(\boldsymbol{\omega}) \leq C\|\boldsymbol{\omega};\mathcal{W}_\alpha\|$. ■

In Appendix B, we prove additional properties of the velocity field, namely that it exhibits a *wake*, in the sense that it decays significantly slower inside cones extending in the downstream direction ($x \to \infty$) than outside of such cones.

4 The nonlinear map $\mathbf{u} \times \boldsymbol{\omega}$

In this section, we examine the nonlinear map $\mathcal{C}(\mathbf{u},\boldsymbol{\omega}) = \mathbf{u} \times \boldsymbol{\omega}$. We prove the

Lemma 8 *For all $\alpha \in [0,1]$, the bilinear map $\mathcal{C} : \mathcal{U} \times \mathcal{W}_\alpha \to \mathcal{Q}_\alpha$ is continuous.*

Proof. We first note that

$$\big(\mathbf{u} \times \boldsymbol{\omega}\big)_n(\mathbf{x}) = \sum_{m\in\mathbb{Z}} \mathbf{u}_{n-m}(\mathbf{x}) \times \boldsymbol{\omega}_m(\mathbf{x}) \ .$$

We also note that straightforward L_p-spaces interpolation gives

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^{\frac{3}{2}-\frac{1}{p}}}{(1+|x|-x)^{\frac{1}{2}}} \|W_\alpha(x,\cdot)\boldsymbol{\omega}_n(x,\cdot)\|_p \leq \|\boldsymbol{\omega};\mathcal{W}_\alpha\|$$

for all $1 \leq p \leq \infty$. We can then use

$$\|W_\alpha(x,\cdot)\mathbf{u}_{n-m}(x,\cdot) \times \boldsymbol{\omega}_m(x,\cdot)\|_p \leq \|\mathbf{u}_{n-m}(x,\cdot)\|_\infty \|W_\alpha(x,\cdot)\boldsymbol{\omega}_m(x,\cdot)\|_p \ ,$$

to conclude that

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{5}{2}-\frac{1}{p}} \|W_\alpha(x,\cdot)\big(\mathbf{u}\times\boldsymbol{\omega}\big)_n(x,\cdot)\|_p \leq C\|\mathbf{u};\mathcal{U}\| \cdot \|\boldsymbol{\omega};\mathcal{W}_\alpha\| \ ,$$

for all $1 \leq p \leq \infty$. On the other hand, using Lemma 19 (see Appendix A), we find

$$\|W_1(x)\triangle_{\mathrm{T}}(\mathbf{u}_{n-m}(x,\cdot) \times \boldsymbol{\omega}_m(x,\cdot))\|_2 \leq C(A_1(x) + A_2(x) + A_3(x)) \ ,$$

where

$$
\begin{aligned}
A_1(x) &\equiv \|\mathbf{u}_{n-m}(x,\cdot)\|_\infty \cdot \|W_1(x)\triangle_{\mathrm{T}}\boldsymbol{\omega}_m(x,\cdot)\|_2 \,, \\
A_2(x) &\equiv 2\|\nabla_{\mathrm{T}}\mathbf{u}_{n-m}(x,\cdot)\|_4 \cdot \|W_1(x)\nabla_{\mathrm{T}}\boldsymbol{\omega}_m(x,\cdot)\|_4 \\
&\leq \|\nabla_{\mathrm{T}}\mathbf{u}_{n-m}(x,\cdot)\|_2^{\frac{1}{2}} \cdot \|\triangle_{\mathrm{T}}\mathbf{u}_{n-m}(x,\cdot)\|_2^{\frac{1}{2}} \cdot \|W_\alpha(x,\cdot)\boldsymbol{\omega}_m(x,\cdot)\|_2^{\frac{1}{4}} \cdot \|W_1(x)\triangle_{\mathrm{T}}\boldsymbol{\omega}_m(x,\cdot)\|_2^{\frac{3}{4}} \,, \quad (21) \\
A_3(x) &\equiv \|\triangle_{\mathrm{T}}\mathbf{u}_{n-m}(x,\cdot)\|_2 \cdot \|W_1(x)\boldsymbol{\omega}_m(x,\cdot)\|_\infty \\
&\leq \|\triangle_{\mathrm{T}}\mathbf{u}_{n-m}(x,\cdot)\|_2 \cdot \|W_\alpha(x,\cdot)\boldsymbol{\omega}_m(x,\cdot)\|_2^{\frac{1}{2}} \cdot \|W_1(x)\triangle_{\mathrm{T}}\boldsymbol{\omega}_m(x,\cdot)\|_2^{\frac{1}{2}} \,, \quad (22)
\end{aligned}
$$

using Lemma 19 and $W_1(x) \leq W_\alpha(x,y,z)$ to get (21) and (22). ■

5 The $\mathcal{S}$ map

Our main purpose in this section is to find $\boldsymbol{\omega}$ solving

$$
(\partial_t + \partial_x - \triangle)\boldsymbol{\omega} = \nabla \times \mathbf{q}
$$

for a given $\mathbf{q} \in \mathcal{Q}_\alpha$. In terms of the Fourier coefficients with index $m \in \mathbb{Z}$, one can write

$$
\boldsymbol{\omega}_m(x,y,z) = \big(\mathcal{S}\mathbf{q}\big)_m(x,y,z) \equiv \int_{\mathbb{R}^3} \mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\; \nabla \times K_{\lambda m}(x-s,y-u,z-v)\,\mathbf{q}_m(s,u,v) \,, \tag{23}
$$

where $\nabla = (\partial_x, \partial_y, \partial_z)$ and

$$
K_n(x,y,z) = \frac{\mathrm{e}^{\frac{x - r\sqrt{1+4in}}{2}}}{r} \quad \text{for} \quad r = \sqrt{x^2+y^2+z^2} \,. \tag{24}
$$

We will prove the

Proposition 9 *For all $0 < \alpha \leq 1$, the map $\mathcal{S} : \mathcal{Q}_\alpha \to \mathcal{W}_\alpha$ is continuous.*

The proof of the proposition will be given in Section 5.2, once we have proved all the necessary estimates on the (family of) Kernels K_n in the next section. Note that we formally have $\nabla \cdot (\mathcal{S}\mathbf{q}) = 0$ since $\mathcal{S}\mathbf{q}$ is the curl of a function.

5.1 Estimates on Kernel

In this section, we establish estimates on the kernels K_n as defined in (24). For further reference, we note that the (partial) Fourier transform $\hat{K}_n(x,k_1,k_2)$ of $K_n(x,y,z)$ with respect to y, z is

$$
\hat{K}_n(x,k_1,k_2) = \frac{\mathrm{e}^{\frac{x-|x|\sqrt{1+4(in+k^2)}}{2}}}{\sqrt{1+4(in+k^2)}} \,,
$$

where $k = \sqrt{k_1^2 + k_2^2}$. Furthermore, we have

$$
\begin{aligned}
\partial_x K_n(x,y,z) &= \left(\frac{r - x\sqrt{1+4in}}{2r} - \frac{x}{r^2}\right)\frac{\mathrm{e}^{\frac{x-r\sqrt{1+4in}}{2}}}{r} \,, \\
\nabla_{\mathrm{T}} K_n(x,y,z) &= -\frac{\mathbf{x}_{\mathrm{T}}}{r}\left(\frac{(2 + r\sqrt{1+4in})}{r^2}\right)\mathrm{e}^{\frac{x-r\sqrt{1+4in}}{2}} \,,
\end{aligned}
$$

where $\mathbf{x}_\mathrm{T}=(0,y,z)$ and $r=\sqrt{x^2+y^2+z^2}$. In our estimates, we will use repeatedly the function

$$\Lambda(n)=\sqrt{\frac{\sqrt{1+16n^2}+1}{2}}\ .$$

We give below Sobolev estimates on K_n that are uniform in n. To improve readability, we will often use the shorthand notation Λ instead of $\Lambda(n)$, suprema over $n\in\lambda\mathbb{Z}$ becoming suprema over $\Lambda\geq 1$.

Lemma 10 *Let $1\leq p\leq 2$, $q>2$ and $0<\alpha\leq 1$. The following estimates hold:*

$$\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}|x|^{\frac{3}{2}-\frac{1}{p}}\mathrm{e}^{\frac{|x|(\Lambda-1)}{2}}\|W_\alpha(x,\cdot)\nabla_\mathrm{T}K_n(x,\cdot)\|_p\leq C(\alpha)\ ,$$
$$\sup_{n\in\lambda\mathbb{Z}}\sup_{|x|\geq\frac{1}{2}}|x|^{\frac{3}{2}-\frac{1}{q}}\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}\|W_\alpha(x,\cdot)\nabla_\mathrm{T}K_n(x,\cdot)\|_q\leq C(\alpha)\ ,$$

with $C(\alpha)\to\infty$ as $\alpha\to 0$.

Proof. Let $\rho=\sqrt{y^2+z^2}$. Straightforward computations give

$$|\nabla_\mathrm{T}K_n(x,y,z)|\leq\rho\sqrt{4+4r\Lambda+r^2(2\Lambda^2-1)}\frac{\mathrm{e}^{\frac{x-r\Lambda}{2}}}{r^3}\leq 2\frac{\rho(2+r\Lambda)}{r^3}\mathrm{e}^{\frac{x-r\Lambda}{2}}\ ,$$

from which we get that

$$\|W_\alpha(x,\cdot)\nabla_\mathrm{T}K_n(x,\cdot)\|_\infty\leq\sup_{\rho\geq 0}\frac{\rho(2+\sqrt{x^2+\rho^2}\Lambda)}{(x^2+\rho^2)^{\frac{3}{2}}}\mathrm{e}^{\frac{x-\sqrt{x^2+\rho^2}\Lambda}{2}+\frac{|x|-x}{2}\alpha+\frac{\sqrt{\rho^2+x^2}-x}{2}(1-\alpha)}\ .$$

Replacing the supremum over ρ by a supremum over $s=\Lambda(\sqrt{x^2+\rho^2}-|x|)$, we get

$$\begin{aligned}\|W_\alpha(x,\cdot)\nabla_\mathrm{T}K_n(x,\cdot)\|_\infty&\leq\mathrm{e}^{-\frac{|x|}{2}(\Lambda-1)}\sup_{s\geq 0}\mathrm{e}^{-\frac{s}{2\Lambda}(\Lambda-1+\alpha)}\frac{\Lambda^2\sqrt{s}(2+s+\Lambda|x|)}{(s+|x|\Lambda)^{\frac{5}{2}}}\\&\leq\mathrm{e}^{-\frac{|x|}{2}(\Lambda-1)}\sup_{s\geq 0}\frac{\mathrm{e}^{-\frac{s}{2\Lambda}(\Lambda-1+\alpha)}(2+s)}{|x|^2}+\frac{\mathrm{e}^{-\frac{s}{2\Lambda}(\Lambda-1+\alpha)}\sqrt{s\Lambda}}{|x|^{\frac{3}{2}}}\\&\leq C\frac{\Lambda\mathrm{e}^{-\frac{|x|}{2}(\Lambda-1)}}{|x|^{\frac{3}{2}}\sqrt{\Lambda-1+\alpha}}\left(1+\frac{1}{\sqrt{|x|}}\right)\end{aligned}\tag{25}$$

for all $0<\alpha\leq 1$. Using again the change of variables $s=\Lambda(\sqrt{x^2+\rho^2}-|x|)$, we find

$$\begin{aligned}\|\nabla_\mathrm{T}K_n(x,\cdot)\|_1&\leq\int_0^\infty\mathrm{d}\rho\ \frac{\rho^2(2+\sqrt{x^2+\rho^2}\Lambda)}{(x^2+\rho^2)^{\frac{3}{2}}}\mathrm{e}^{\frac{x-\sqrt{x^2+\rho^2}\Lambda}{2}}\\&\leq C\mathrm{e}^{\frac{x-|x|\Lambda}{2}}\int_0^\infty\mathrm{d}s\ \frac{\sqrt{s^2+2s|x|\Lambda}(2+s+|x|\Lambda)}{(s+|x|\Lambda)^2}\mathrm{e}^{-\frac{s}{2}}\\&\leq C\mathrm{e}^{\frac{x-|x|\Lambda}{2}}\int_0^\infty\mathrm{d}s\ \frac{\sqrt{s}(2+s+|x|\Lambda)}{(s+|x|\Lambda)^{\frac{3}{2}}}\mathrm{e}^{-\frac{s}{2}}\\&\leq C\mathrm{e}^{\frac{x-|x|\Lambda}{2}}\int_0^\infty\mathrm{d}s\ \frac{\sqrt{s}(2+s+|x|)}{(s+|x|)^{\frac{3}{2}}}\mathrm{e}^{-\frac{s}{2}}\leq C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}}{|x|^{\frac{1}{2}}}\ .\end{aligned}$$

Along the same lines, we find that

$$\|\nabla_{\mathrm{T}}K_n(x,\cdot)\|_2 \le C\left(\int_0^\infty \mathrm{d}\rho\ \frac{\rho^3(2+\sqrt{x^2+\rho^2}\Lambda)^2}{(x^2+\rho^2)^3}\mathrm{e}^{x-\sqrt{x^2+\rho^2}\Lambda}\right)^{\frac{1}{2}}$$
$$\le C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}}{|x|}\left(\int_0^\infty \mathrm{d}s\ \frac{|x|^2\Lambda^2 s(2+s+|x|\Lambda)^2}{(s+|x|\Lambda)^4}\mathrm{e}^{-s}\right)^{\frac{1}{2}} \le C\frac{\mathrm{e}^{\frac{x-\Lambda|x|}{2}}}{|x|}\ .$$

Using interpolation and straightforward modifications of the above to include the weight W_α, we find that

$$\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}|x|^{\frac{3}{2}-\frac{1}{p}}\mathrm{e}^{\frac{|x|(\Lambda-1)}{2}}\|W_\alpha(x,\cdot)\nabla_{\mathrm{T}}K_n(x,\cdot)\|_p \le C(\alpha) \quad \text{for all } 1\le p\le 2 \text{ and } 0<\alpha\le 1\ ,$$

with $C(\alpha)\to\infty$ as $\alpha\to 0$. Furthermore, from (25), we get

$$\sup_{n\in\lambda\mathbb{Z}}\sup_{|x|\ge\frac{1}{2}}|x|^{\frac{3}{2}-\frac{1}{q}}\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}\|W_\alpha(x,\cdot)\nabla_{\mathrm{T}}K_n(x,\cdot)\|_q \le C(\alpha) \quad \text{for all } q>2 \text{ and } 0<\alpha\le 1\ . \tag{26}$$

Note in (26), the restriction of the supremum over $|x|\ge\frac{1}{2}$ is essential to be able to use part of the exponential decay as $|x|\to\infty$ to compensate the growth of the algebraic pre-factor as $\Lambda\to\infty$ (or $|n|\to\infty$). ■

Lemma 11 *Let $\beta,\gamma\in\mathbb{N}$ with $\beta+\gamma\ge 1$. For any combination of i,j among y,z, it holds*

$$\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}\frac{|x|^3\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}}{1+|x|}\|W_1(x)\triangle_{\mathrm{T}}\nabla_{\mathrm{T}}K_n(x,\cdot)\|_2 \le C\ ,$$
$$\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}\frac{\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}|x|^{1+\beta+\gamma}}{(1+|x|)^{\frac{\beta+\gamma}{2}}}\|W_1(x)\partial_i^{1+\beta}\partial_j^{\gamma}K_n(x,\cdot)\|_2 \le C\ ,$$
$$\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}\frac{\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}|x|^{2+\beta+\gamma}}{(1+|x|)^{\frac{1+\beta+\gamma}{2}}}\|W_1(x)\partial_i^{1+\beta}\partial_j^{\gamma}K_n(x,\cdot)\|_\infty \le C\ .$$

Proof. In this proof, we work with the Fourier transform with respect to y,z. We find

$$\|\triangle_{\mathrm{T}}\nabla_{\mathrm{T}}K_n(x,\cdot)\|_2 \le C\left(\int_0^\infty \mathrm{d}k\ \exp\left(x-\sqrt{\frac{\sqrt{16n^2+(1+4k^2)^2}+1+4k^2}{2}}|x|\right)\frac{k^7}{\sqrt{16n^2+(1+4k^2)^2}}\right)^{\frac{1}{2}}\ .$$

Performing the change of variables

$$\sqrt{\frac{\sqrt{16n^2+(1+4k^2)^2}+1+4k^2}{2}} = \Lambda+\frac{s}{|x|}\ ,$$

we find that

$$\|\triangle_{\mathrm{T}}\nabla_{\mathrm{T}}K_n(x,\cdot)\|_2 \le C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}}{|x|^3}\left(\int_0^\infty \mathrm{d}s\ \frac{(s+2|x|\Lambda)^3(2|x|\Lambda^2+2s|x|\Lambda-|x|^2+s^2)^3 s^3\mathrm{e}^{-s}}{(s+|x|\Lambda)^7}\right)^{\frac{1}{2}}$$
$$\le C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}}{|x|^3}\left(\int_0^\infty \mathrm{d}s\ (s+|x|\Lambda)^2 s^3\mathrm{e}^{-s}\right)^{\frac{1}{2}} \le C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}(1+|x|\Lambda)}{|x|^3}\ .$$

We thus find

$$\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}\frac{\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}|x|^3}{1+|x|}\|W_1(x)\triangle_{\mathrm{T}}\nabla_{\mathrm{T}}K_n(x,\cdot)\|_2\leq C\ .$$

Proceeding similarly, we find for all $\beta+\gamma\geq 1$ and any combination of i,j among y,z that

$$\begin{aligned}\|\partial_i^{1+\beta}\partial_j^{\gamma}K_n(x,\cdot)\|_2 &\leq C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}}{|x|^{1+\beta+\gamma}}\left(\int_0^\infty \mathrm{d}s\ (s+|x|\Lambda)^{\beta+\gamma}s^{1+\beta+\gamma}\mathrm{e}^{-s}\right)^{\frac{1}{2}}\\ &\leq C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}(1+|x|\Lambda)^{\frac{\beta+\gamma}{2}}}{|x|^{1+\beta+\gamma}}\ ,\\ \|\partial_i^{1+\beta}\partial_j^{\gamma}K_n(x,\cdot)\|_\infty &\leq C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}}{|x|^{2+\beta+\gamma}}\int_0^\infty \mathrm{d}s\ (s+|x|\Lambda)^{\frac{1+\beta+\gamma}{2}}s^{\frac{1+\beta+\gamma}{2}}\mathrm{e}^{-s}\\ &\leq C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}(1+|x|\Lambda)^{\frac{\beta+\gamma}{2}}}{|x|^{1+\beta+\gamma}}\ ,\end{aligned}$$

where we used $\|F(x,\cdot)\|_\infty\leq\|\hat{F}(x,\cdot)\|_1$. ■

Lemma 12 *Let $1\leq p\leq 2$, $q>2$ and $0<\alpha\leq 1$. The following estimates hold:*

$$\begin{aligned}\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}\frac{|x|^{\frac{3}{2}-\frac{1}{p}}(1+|x|)^{\frac{1}{2}}}{1+|x|-x+(|x|+x)^{\frac{1}{2}}}\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}\|W_\alpha(x,\cdot)\partial_xK_n(x,\cdot)\|_p &\leq C(\alpha)\ ,\\ \sup_{n\in\lambda\mathbb{Z}}\sup_{|x|\geq\frac{1}{2}}\frac{|x|^{\frac{3}{2}-\frac{1}{q}}(1+|x|)^{\frac{1}{2}}}{1+|x|-x+(|x|+x)^{\frac{1}{2}}}\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}\|W_\alpha(x,\cdot)\partial_xK_n(x,\cdot)\|_q &\leq C(\alpha)\ ,\\ \sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}\frac{\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}|x|^3}{(1+|x|)^{\frac{1}{2}}(1+|x|-x)}\|W_1(x)\triangle_{\mathrm{T}}\partial_xK_n(x,\cdot)\|_2 &\leq C\ .\end{aligned}$$

Proof. We first note that

$$|\partial_xK_n(x,y,z)|\leq\frac{e^{\frac{x-\Lambda r}{2}}}{r^3}\sqrt{(r^2-xr\Lambda-2x)^2+x^2r^2(\Lambda^2-1)}\leq\mathrm{R}_{n,1}(x,y,z)+\mathrm{R}_{n,2}(x,y,z)\ ,$$

where

$$\mathrm{R}_{n,1}(x,y,z)=\frac{\mathrm{e}^{\frac{x-\Lambda r}{2}}}{r}\left|\frac{r-x}{2r}-\frac{x}{r^2}\right|\quad\text{and}\quad\mathrm{R}_{n,2}(x,y,z)=\frac{\mathrm{e}^{\frac{x-\Lambda r}{2}}}{r}\sqrt{\Lambda(\Lambda-1)}\ .$$

Using the change of variables $s=\Lambda(\sqrt{x^2+\rho^2}-|x|)$, we find that

$$\begin{aligned}\|W_\alpha(x,\cdot)\partial_xK_n(x,\cdot)\|_\infty &\leq\mathrm{e}^{-\frac{|x|}{2}(\Lambda-1)}\sup_{s\geq 0}\mathrm{e}^{-\frac{s}{2\Lambda}(\Lambda-1+\alpha)}\left(\frac{s}{2\Lambda|x|^2}+\frac{|x|-x}{|x|^2}+\frac{1}{|x|^2}+\frac{\sqrt{\Lambda(\Lambda-1)}}{|x|}\right)\\ &\leq\frac{\mathrm{e}^{-\frac{|x|}{2}(\Lambda-1)}}{|x|^2}\left(\frac{1}{\Lambda-1+\alpha}+|x|-x+\sqrt{\Lambda(\Lambda-1)}|x|\right)\ ,\end{aligned}$$

from which we deduce, for all $0<\alpha\leq 1$, that

$$\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}\frac{|x|^2\mathrm{e}^{\frac{|x|}{4}(\Lambda-1)}}{1+|x|-x+(|x|+x)^{\frac{1}{2}}}\|W_\alpha(x,\cdot)\partial_xK_n(x,\cdot)\|_\infty\leq C(\alpha)\ ,\tag{27}$$

with $C(\alpha) \to \infty$ as $\alpha \to 0$. For the other norms, we have

$$
\begin{aligned}
\|\mathrm{R}_{n,1}(x,\cdot)\|_1 &\leq C\mathrm{e}^{\frac{x-\Lambda|x|}{2}} \int_0^\infty \mathrm{d}s\ \frac{|(s+\Lambda|x|)(s+\Lambda(|x|-x))-2x\Lambda^2|}{\Lambda(s+\Lambda|x|)^2}\mathrm{e}^{-\frac{s}{2}} \\
&\leq C\frac{\mathrm{e}^{\frac{x-\Lambda|x|}{2}}(1+|x|-x)}{1+|x|}\ , \\
\|\mathrm{R}_{n,1}(x,\cdot)\|_2 &\leq C\mathrm{e}^{\frac{x-\Lambda|x|}{2}} \left(\int_0^\infty \mathrm{d}s\ \frac{((s+\Lambda|x|)(s+\Lambda(|x|-x))-2x\Lambda^2)^2}{(s+\Lambda|x|)^5}\mathrm{e}^{-s}\right)^{\frac{1}{2}} \\
&\leq C\frac{\mathrm{e}^{\frac{x-\Lambda|x|}{2}}(1+|x|-x)}{|x|(1+|x|)^{\frac{1}{2}}}\ , \\
\|\mathrm{R}_{n,2}(x,\cdot)\|_1 &\leq C\sqrt{1-\frac{1}{\Lambda}}\mathrm{e}^{\frac{x-\Lambda|x|}{2}} \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-\frac{s}{2}} \leq C\sqrt{1-\frac{1}{\Lambda}}\mathrm{e}^{\frac{x-\Lambda|x|}{2}}\ , \\
\|\mathrm{R}_{n,2}(x,\cdot)\|_2 &\leq C\sqrt{1-\frac{1}{\Lambda}}\mathrm{e}^{\frac{x-\Lambda|x|}{2}} \left(\int_0^\infty \mathrm{d}s\ \frac{\mathrm{e}^{-s}}{s+\Lambda|x|}\right)^{\frac{1}{2}} \\
&\leq C\sqrt{1-\frac{1}{\Lambda}}\frac{\mathrm{e}^{\frac{x-\Lambda|x|}{2}}}{|x|}\frac{|x|\Lambda}{\sqrt{1+|x|\Lambda}}\ .
\end{aligned}
$$

Modifying the above estimates to incorporate the weight W_α, we find for all $1 \leq p \leq 2$ and $0 < \alpha \leq 1$ that

$$
\|W_\alpha(x,\cdot)\mathrm{R}_{n,1}(x,\cdot)\|_p \leq C(\alpha)\frac{\mathrm{e}^{-\frac{|x|(\Lambda-1)}{2}}(1+|x|-x)}{|x|^{\frac{3}{2}-\frac{1}{p}}(1+|x|)^{\frac{1}{2}}}
$$

with $C(\alpha) \to \infty$ as $\alpha \to 0$, while

$$
\begin{aligned}
\|W_\alpha(x,\cdot)\mathrm{R}_{n,2}(x,\cdot)\|_1 &\leq C(\alpha)\sqrt{1-\frac{1}{\Lambda}}\mathrm{e}^{-\frac{|x|(\Lambda-1)}{2}}\ , \\
\|W_\alpha(x,\cdot)\mathrm{R}_{n,2}(x,\cdot)\|_2 &\leq C(\alpha)\sqrt{1-\frac{1}{\Lambda}}\frac{\mathrm{e}^{-\frac{|x|(\Lambda-1)}{2}}}{|x|}\frac{|x|\Lambda}{\sqrt{1+|x|\Lambda}}\ .
\end{aligned}
$$

We then get for all $1 \leq p \leq 2$ and $0 < \alpha \leq 1$ that

$$
\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}} \frac{|x|^{\frac{3}{2}-\frac{1}{p}}(1+|x|)^{\frac{1}{2}}}{1+|x|-x+(|x|+x)^{\frac{1}{2}}}\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}\|W_\alpha(x,\cdot)\partial_x K_n(x,\cdot)\|_p \leq C(\alpha)\ ,
$$

which, when combined with (27), gives

$$
\sup_{n\in\lambda\mathbb{Z}}\sup_{|x|\geq\frac{1}{2}} \frac{|x|^{\frac{3}{2}-\frac{1}{q}}(1+|x|)^{\frac{1}{2}}}{1+|x|-x+(|x|+x)^{\frac{1}{2}}}\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}\|W_\alpha(x,\cdot)\partial_x K_n(x,\cdot)\|_q \leq C(\alpha)
$$

for all $q > 2$ and $0 < \alpha \leq 1$.

Finally, we find

$$
\|\triangle_{\mathrm{T}}\partial_x K_n(x,\cdot)\|_2 \leq C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}}{|x|^3}\left(\int_0^\infty \mathrm{d}s\ \frac{(2|x|\Lambda^2+2s|x|\Lambda-|x|^2+s^2)^2 T(s,|x|,\Lambda)s^2\mathrm{e}^{-s}}{(s+|x|\Lambda)^7}\right)^{\frac{1}{2}}\ ,
$$

where

$$T(s,|x|,\Lambda) = (s+2|x|\Lambda)^4(s+|x|\Lambda - x)^2 + (s+2|x|\Lambda)^2(\Lambda^2-1)x^2(\Lambda|x|)^2 \,.$$

We thus get

$$\begin{aligned}\|\triangle_{\mathrm{T}}\partial_x K_n(x,\cdot)\|_2 &\leq C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}}{|x|^3}\left(\int_0^\infty\!\mathrm{d}s\,\,\left((s+|x|\Lambda)(s+|x|\Lambda-x)^2+(\Lambda^2-1)x^2|x|\Lambda\right)s^2\mathrm{e}^{-s}\right)^{\frac{1}{2}}\,,\\ &\leq C\frac{\mathrm{e}^{\frac{x-|x|\Lambda}{2}}\sqrt{1+|x|\Lambda}}{|x|^3}\begin{cases}1+|x|(\Lambda-1) & \text{if}\quad x\geq 0\\ 1+|x|\Lambda & \text{if}\quad x\leq 0\end{cases}\,,\end{aligned}$$

from which we deduce

$$\sup_{n\in\lambda\mathbb{Z}}\sup_{x\in\mathbb{R}}\frac{\mathrm{e}^{\frac{|x|(\Lambda-1)}{4}}|x|^3}{(1+|x|)^{\frac{1}{2}}(1+|x|-x)}\|W_1(x)\triangle_{\mathrm{T}}\partial_x K_n(x,\cdot)\|_2 \leq C\,,$$

using again part of the exponential decay to compensate for the growth of the pre-factor as $|x|\Lambda\to\infty$. ■

5.2 The map $\mathcal{S}$

It is practical to decompose the map $\mathcal{S}$ from (23) as $\mathcal{S} = \mathcal{S}_0^+ + \mathcal{S}_1^+ + \mathcal{S}_{0,\mathrm{T}}^- + \mathcal{S}_{1,\mathrm{T}}^- + \mathcal{S}_{0,\mathrm{L}}^- + \mathcal{S}_{1,\mathrm{L}}^-$, where, in terms of the Fourier coefficients with index $m\in\mathbb{Z}$, we set

$$\left(\mathcal{S}_0^+\mathbf{f}\right)_m(x,y,z) = \int_{-\infty}^{x-\frac{|x|+1}{2}}\mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}u\,\mathrm{d}v\;\nabla\times\left(K_{\lambda m}(x-s,y-u,z-v)\,\mathbf{f}_m(s,u,v)\right)\,, \tag{28}$$

$$\left(\mathcal{S}_1^+\mathbf{f}\right)_m(x,y,z) = \int_{x-\frac{|x|+1}{2}}^{x}\mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}u\,\mathrm{d}v\;\nabla\times\left(K_{\lambda m}(x-s,y-u,z-v)\,\mathbf{f}_m(s,u,v)\right)\,, \tag{29}$$

$$\left(\mathcal{S}_{0,\mathrm{T}}^-\mathbf{f}\right)_m(x,y,z) = \int_{x+\frac{|x|+1}{2}}^{\infty}\mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}u\,\mathrm{d}v\;\nabla_{\mathrm{T}}\times\left(K_{\lambda m}(x-s,y-u,z-v)\,\mathbf{f}_m(s,u,v)\right)\,, \tag{30}$$

$$\left(\mathcal{S}_{1,\mathrm{T}}^-\mathbf{f}\right)_m(x,y,z) = \int_{x}^{x+\frac{|x|+1}{2}}\mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}u\,\mathrm{d}v\;\nabla_{\mathrm{T}}\times\left(K_{\lambda m}(x-s,y-u,z-v)\,\mathbf{f}_m(s,u,v)\right)\,, \tag{31}$$

$$\left(\mathcal{S}_{0,\mathrm{L}}^-\mathbf{f}\right)_m(x,y,z) = \int_{x+\frac{|x|+1}{2}}^{\infty}\mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}u\,\mathrm{d}v\;\nabla_{\mathrm{L}}\times\left(K_{\lambda m}(x-s,y-u,z-v)\,\mathbf{f}_m(s,u,v)\right)\,, \tag{32}$$

$$\left(\mathcal{S}_{1,\mathrm{L}}^-\mathbf{f}\right)_m(x,y,z) = \int_{x}^{x+\frac{|x|+1}{2}}\mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}u\,\mathrm{d}v\;\nabla_{\mathrm{L}}\times\left(K_{\lambda m}(x-s,y-u,z-v)\,\mathbf{f}_m(s,u,v)\right)\,, \tag{33}$$

with $\nabla = (\partial_x,\partial_y,\partial_z)$, $\nabla_{\mathrm{T}} = (0,\partial_y,\partial_z)$ and $\nabla_{\mathrm{L}} = (\partial_x,0,0)$.

Note that the first argument of $K_{\lambda m}$ is positive in $\mathcal{S}_i^+$ and negative in $\mathcal{S}_{i,\mathrm{T}}^-$ and $\mathcal{S}_{i,\mathrm{L}}^-$. Also, in $\mathcal{S}_0^+$, $\mathcal{S}_{0,\mathrm{T}}^-$ and $\mathcal{S}_{0,\mathrm{L}}^-$, we avoid integrating close to the locations where some of the Kernel estimates of the previous section are singular. We can now break the proof of Proposition 9 into

Proposition 13 *For all $0<\alpha\leq 1$, the six linear maps defined in (28)–(33) are continuous from $\mathcal{Q}_\alpha$ to $\mathcal{W}_\alpha$.*

The proof follows immediately from the Lemmata 15, 16 and 17 below. We first quote an easy result:

Lemma 14 *For any* $\mathbf{f} \in \mathcal{Q}_\alpha$*, the quantities*

$$\begin{aligned}
\mathrm{A}_\alpha(\mathbf{f}) &= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{d}s \, \|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_1 \,, \\
\mathrm{B}_{p,\alpha}(\mathbf{f}) &= \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{5}{2}-\frac{1}{p}} \|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_p \,, \\
\mathrm{D}(\mathbf{f}) &= \sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^3 \|W_1(s)\triangle_{\mathrm{T}}\mathbf{f}_n(s,\cdot)\|_2 \,,
\end{aligned}$$

satisfy $\mathrm{A}_\alpha(\mathbf{f}) + \mathrm{B}_{p,\alpha}(\mathbf{f}) + \mathrm{D}(\mathbf{f}) \leq C\|\mathbf{f};\mathcal{Q}_\alpha\|$ *for all* $1 \leq p \leq \infty$*.*

We first estimate the $\mathcal{S}_i^+$ maps. Note that the estimates in (35) and (37) below are stronger than those needed for $\mathcal{S}_i^+\mathbf{f}$ to be in $\mathcal{W}_\alpha$. In particular, since $\mathcal{S}_1^+\mathbf{f}$ decays by a factor of $|x|^{-\frac{1}{2}}$ faster than $\mathcal{S}_0^+\mathbf{f}$ as $|x| \to \infty$, $\mathcal{S}_1^+\mathbf{f}$ can be neglected in a first order asymptotic expansion of $\mathcal{S}\mathbf{f}$ as $|x| \to \infty$.

Lemma 15 *Let* $0 < \alpha \leq 1$*. There exist a constant* $C(\alpha)$ *with* $C(\alpha) \to \infty$ *as* $\alpha \to 0$ *such that*

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{3}{2}-\frac{1}{p}} \|W_\alpha(x,\cdot)\big(\mathcal{S}_0^+\mathbf{f}\big)_n(x,\cdot)\|_p \leq C(\alpha)\, \mathrm{A}_\alpha(\mathbf{f}) \,, \tag{34}$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{2-\frac{1}{p}} \|W_\alpha(x,\cdot)\big(\mathcal{S}_1^+\mathbf{f}\big)_n(x,\cdot)\|_p \leq C(\alpha)\, \mathrm{B}_{p,\alpha}(\mathbf{f}) \,, \tag{35}$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^2 \|W_1(x)\triangle_{\mathrm{T}}\big(\mathcal{S}_0^+\mathbf{f}\big)_n(x,\cdot)\|_2 \leq C(\alpha)\, \mathrm{A}_\alpha(\mathbf{f}) \,, \tag{36}$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{5}{2}} \|W_1(x)\triangle_{\mathrm{T}}\big(\mathcal{S}_1^+\mathbf{f}\big)_n(x,\cdot)\|_2 \leq C(\alpha)\, \mathrm{D}(\mathbf{f}) \,, \tag{37}$$

for all $1 \leq p \leq \infty$*.*

Proof. Let $0 < \alpha < 1$, $\mathbf{r} = (x,y,z)$, $\mathbf{r}' = (s,u,v)$, $r = \sqrt{x^2+y^2+z^2}$ and $r' = \sqrt{s^2+u^2+v^2}$. By the triangle inequality, we have

$$\begin{aligned}
W_\alpha(x,y,z) &\leq \mathrm{e}^{\frac{|x-s|-(x-s)}{2}\alpha + \frac{|\mathbf{r}-\mathbf{r}'|-(x-s)}{2}(1-\alpha)} \mathrm{e}^{\frac{|s|-s}{2}\alpha + \frac{r'-s}{2}(1-\alpha)} \\
&= W_\alpha(x-s,y-u,z-v) W_\alpha(s,u,v) \,, \\
W_1(x) &\leq W_1(x-s) W_1(x) \,.
\end{aligned}$$

We find

$$\begin{aligned}
\|W_\alpha(x,\cdot)\big(\mathcal{S}_0^+\mathbf{f}\big)_n(x,\cdot)\|_p &\leq C(\alpha) \int_{-\infty}^{x-\frac{|x|+1}{2}} \mathrm{d}s \, \frac{\|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_1}{|x-s|^{\frac{3}{2}-\frac{1}{p}}} \,, \\
\|W_\alpha(x,\cdot)\big(\mathcal{S}_1^+\mathbf{f}\big)_n(x,\cdot)\|_p &\leq C(\alpha) \int_{x-\frac{|x|+1}{2}}^{x} \mathrm{d}s \, \frac{\|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_p}{|x-s|^{\frac{1}{2}}} \,, \\
\|W_1(x)\triangle_{\mathrm{T}}\big(\mathcal{S}_0^+\mathbf{f}\big)_n(x,\cdot)\|_2 &\leq C(\alpha) \int_{-\infty}^{x-\frac{|x|+1}{2}} \mathrm{d}s \, \frac{1+|x-s|}{|x-s|^3} \|W_1(s)\mathbf{f}_n(s,\cdot)\|_1 \,, \\
\|W_1(x)\triangle_{\mathrm{T}}\big(\mathcal{S}_1^+\mathbf{f}\big)_n(x,\cdot)\|_2 &\leq C(\alpha) \int_{x-\frac{|x|+1}{2}}^{x} \mathrm{d}s \, \frac{\|W_1(s)\triangle_{\mathrm{T}}\mathbf{f}_n(s,\cdot))\|_2}{|x-s|^{\frac{1}{2}}} \,,
\end{aligned}$$

for all $1 \leq p \leq \infty$, from which (34)–(37) follow at once, using

$$W_1(s) = \mathrm{e}^{\frac{|s|-s}{2}} \leq \mathrm{e}^{\frac{|s|-s}{2}\alpha + \frac{r'-s}{2}(1-\alpha)} = W_\alpha(s,u,v)$$

and Lemma 18 (see Appendix A). ■

We next estimate $\mathcal{S}^-_{i,\mathrm{T}}$. Note again the stronger estimates in (39) and (41), showing that $\mathcal{S}^-_{1,\mathrm{T}}\mathbf{f}$ can also be neglected in a first order expansion of $\mathcal{S}\mathbf{f}$ as $|x| \to \infty$.

Lemma 16 *Let* $0 < \alpha \leq 1$. *There exist a constant* $C(\alpha)$ *with* $C(\alpha) \to \infty$ *as* $\alpha \to 0$ *such that*

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{3}{2}-\frac{1}{p}} \|W_\alpha(x,\cdot)\big(\mathcal{S}^-_{0,\mathrm{T}}\mathbf{f}\big)_n(x,\cdot)\|_p \leq C(\alpha)\ \mathrm{A}_\alpha(\mathbf{f})\ , \tag{38}$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{2-\frac{1}{p}} \|W_\alpha(x,\cdot)\big(\mathcal{S}^-_{1,\mathrm{T}}\mathbf{f}\big)_n(x,\cdot)\|_p \leq C(\alpha)\ \mathrm{B}_{p,\alpha}(\mathbf{f})\ , \tag{39}$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{2} \|W_1(x)\triangle_\mathrm{T}\big(\mathcal{S}^-_{0,\mathrm{T}}\mathbf{f}\big)_n(x,\cdot)\|_2 \leq C(\alpha)\ \mathrm{A}_\alpha(\mathbf{f})\ , \tag{40}$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} (1+|x|)^{\frac{5}{2}} \|W_1(x)\triangle_\mathrm{T}\big(\mathcal{S}^-_{1,\mathrm{T}}\mathbf{f}\big)_n(x,\cdot)\|_2 \leq C(\alpha)\ \mathrm{D}(\mathbf{f})\ , \tag{41}$$

for all $1 \leq p \leq \infty$.

Proof. The estimates for $\mathcal{S}^-_{i,\mathrm{T}}$ are very similar to the ones for $\mathcal{S}^+_i$. Namely, we have

$$\|W_\alpha(x,\cdot)\big(\mathcal{S}^-_{0,\mathrm{T}}\mathbf{f}\big)_n(x,\cdot)\|_p \leq C(\alpha) \int_{x+\frac{|x|+1}{2}}^{\infty} \mathrm{d}s\ \frac{\|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_1}{|x-s|^{\frac{3}{2}-\frac{1}{p}}}\ ,$$

$$\|W_\alpha(x,\cdot)\big(\mathcal{S}^-_{1,\mathrm{T}}\mathbf{f}\big)_n(x,\cdot)\|_p \leq C(\alpha) \int_{x}^{x+\frac{|x|+1}{2}} \mathrm{d}s\ \frac{\|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_p}{|x-s|^{\frac{1}{2}}}\ ,$$

$$\|W_1(x)\triangle_\mathrm{T}\big(\mathcal{S}^-_{0,\mathrm{T}}\mathbf{f}\big)_n(x,\cdot)\|_2 \leq C(\alpha) \int_{x+\frac{|x|+1}{2}}^{\infty} \mathrm{d}s\ \frac{1+|x-s|}{|x-s|^3} \|W_1(s)\mathbf{f}_n(s,\cdot)\|_1\ ,$$

$$\|W_1(x)\triangle_\mathrm{T}\big(\mathcal{S}^-_{1,\mathrm{T}}\mathbf{f}\big)_n(x,\cdot)\|_2 \leq C(\alpha) \int_{x}^{x+\frac{|x|+1}{2}} \mathrm{d}s\ \frac{\|W_1(s)\triangle_\mathrm{T}\mathbf{f}_n(s,\cdot)\|_2}{|x-s|^{\frac{1}{2}}}\ ,$$

from which (38)–(41) follow at once, using Lemma 18 (see Appendix A). ■

We conclude this section with the estimates on $\mathcal{S}^-_{i,\mathrm{L}}$. All estimates below are stronger than what is needed for $\mathcal{S}^-_{i,\mathrm{L}}\mathbf{f}$ to be in $\mathcal{W}_\alpha$. In particular, $\mathcal{S}^-_{i,\mathrm{L}}\mathbf{f}$ decays by a factor of x^{-1} faster than, say, $\mathcal{S}^+_0\mathbf{f}$ as $x \to \infty$. This means that $\mathcal{S}^-_{0,\mathrm{L}}\mathbf{f}$ and $\mathcal{S}^-_{1,\mathrm{L}}\mathbf{f}$ can also both be neglected in an asymptotic expansion of $\mathcal{S}\mathbf{f}$ as $x \to \infty$. These functions are however relevant as $x \to -\infty$.

Lemma 17 *Let* $0 < \alpha \leq 1$. *There exist a constant* $C(\alpha)$ *with* $C(\alpha) \to \infty$ *as* $\alpha \to 0$ *such that*

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^{\frac{5}{2}-\frac{1}{p}}}{(1+|x|-x)^{\frac{3}{2}}} \|W_\alpha(x,\cdot)\big(\mathcal{S}^-_{0,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_p \leq C(\alpha) \max\big\{\mathrm{A}_\alpha(\mathbf{f})\ ,\ \mathrm{B}_{p,\alpha}(\mathbf{f})\big\}\ , \tag{42}$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^{\frac{5}{2}-\frac{1}{p}}}{(1+|x|-x)^{\frac{3}{2}}} \|W_\alpha(x,\cdot)\big(\mathcal{S}^-_{1,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_p \leq C(\alpha) \max\big\{\mathrm{A}_\alpha(\mathbf{f})\ ,\ \mathrm{B}_{p,\alpha}(\mathbf{f})\big\}\ , \tag{43}$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^{3}}{(1+|x|-x)^{\frac{3}{2}}} \|W_1(x)\triangle_\mathrm{T}\big(\mathcal{S}^-_{0,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_2 \leq C(\alpha) \max\big\{\mathrm{A}_\alpha(\mathbf{f})\ ,\ \mathrm{D}(\mathbf{f})\big\}\ , \tag{44}$$

$$\sum_{n\in\mathbb{Z}} \sup_{x\in\mathbb{R}} \frac{(1+|x|)^3}{(1+|x|-x)^{\frac{3}{2}}} \|W_1(x)\triangle_{\mathrm{T}}\big(\mathcal{S}^{-}_{1,\mathrm{L}}\mathbf{f}\big)(x,\cdot)\|_2 \leq C(\alpha)\max\left\{\mathrm{A}_\alpha(\mathbf{f})\ ,\ \mathrm{D}(\mathbf{f})\right\}\ , \tag{45}$$

for all $1 \leq p \leq \infty$, $n \in \mathbb{Z}$ *and* $x \in \mathbb{R}$.

Proof. We will consider separately the cases $x \geq 0$ and $x < 0$. We first note that the integration variable s always satisfy $s \geq x$. Furthermore, if $x \geq 0$, we have

$$\begin{aligned} W_\alpha(x,y,z) &= \mathrm{e}^{\frac{r-x}{2}(1-\alpha)} \leq \mathrm{e}^{\frac{|\mathbf{r}-\mathbf{r}'|-(x-s)}{2}(1-\alpha)}\mathrm{e}^{\frac{r'-s}{2}(1-\alpha)} \\ &= \mathrm{e}^{(x-s)\alpha}W_\alpha(x-s,y-u,z-v)W_\alpha(s,u,v)\ . \end{aligned}$$

We thus find the following estimates if $x \geq 0$:

$$\begin{aligned} \|W_\alpha(x,\cdot)\big(\mathcal{S}^{-}_{0,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_p &\leq C(\alpha)\int_{x+\frac{|x|+1}{2}}^{\infty}\mathrm{d}s\ \mathrm{e}^{(x-s)\alpha}\|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_p\ , \\ \|W_\alpha(x,\cdot)\big(\mathcal{S}^{-}_{1,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_p &\leq C(\alpha)\int_{x}^{x+\frac{|x|+1}{2}}\mathrm{d}s\ \mathrm{e}^{(x-s)\alpha}\|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_p\ , \\ \|W_1(x)\triangle_{\mathrm{T}}\big(\mathcal{S}^{-}_{0,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_2 &\leq C(\alpha)\int_{x+\frac{|x|+1}{2}}^{\infty}\mathrm{d}s\ \mathrm{e}^{(x-s)}\|W_1(s)\triangle_{\mathrm{T}}\mathbf{f}_n(s,\cdot)\|_2\ , \\ \|W_1(x)\triangle_{\mathrm{T}}\big(\mathcal{S}^{-}_{1,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_2 &\leq C(\alpha)\int_{x}^{x+\frac{|x|+1}{2}}\mathrm{d}s\ \mathrm{e}^{(x-s)}\|W_1(s)\triangle_{\mathrm{T}}\mathbf{f}_n(s,\cdot)\|_2\ , \end{aligned}$$

while, if $x < 0$, we have

$$\begin{aligned} \|W_\alpha(x,\cdot)\big(\mathcal{S}^{-}_{0,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_p &\leq C(\alpha)\int_{x+\frac{|x|+1}{2}}^{\infty}\mathrm{d}s\ \frac{\sqrt{1+|x-s|}}{|x-s|^{\frac{3}{2}-\frac{1}{p}}}\|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_1\ , \\ \|W_\alpha(x,\cdot)\big(\mathcal{S}^{-}_{1,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_p &\leq C(\alpha)\int_{x}^{x+\frac{|x|+1}{2}}\mathrm{d}s\ \|W_\alpha(s,\cdot)\mathbf{f}_n(s,\cdot)\|_p\ , \\ \|W_1(x)\triangle_{\mathrm{T}}\big(\mathcal{S}^{-}_{0,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_2 &\leq C(\alpha)\int_{x+\frac{|x|+1}{2}}^{\infty}\mathrm{d}s\ \frac{(1+|x-s|)^{\frac{3}{2}}}{|x-s|^3}\|W_1(s)\mathbf{f}_n(s,\cdot)\|_1\ , \\ \|W_1(x)\triangle_{\mathrm{T}}\big(\mathcal{S}^{-}_{1,\mathrm{L}}\mathbf{f}\big)_n(x,\cdot)\|_2 &\leq C(\alpha)\int_{x}^{x+\frac{|x|+1}{2}}\mathrm{d}s\ \|W_1(s)\triangle_{\mathrm{T}}\mathbf{f}(s,\cdot)\|_2\ , \end{aligned}$$

from which (42)–(45) follow at once for $x < 0$, using again Lemma 18 (see Appendix A). ■

A Useful estimates

In this section, we collect useful estimates. The first one is straightforward:

Lemma 18 *Let* $\beta > 0$ *and* $x \in \mathbb{R}$. *Then*

$$\sup_{s\in[x-\frac{|x|+1}{2},x+\frac{|x|+1}{2}]}\frac{1}{(1+|s|)^\beta} \leq \frac{2^\beta}{(1+|x|)^\beta}\ .$$

The second one is a collection of classical estimates:

Lemma 19 *Let $f \in \mathrm{H}^2(\mathbb{R}^2)$, then*

$$\begin{aligned}\|\nabla f\|_2 &\leq \sqrt{\|f\|_2 \cdot \|\triangle f\|_2}\,,\\ \|f\|_\infty \leq \|\hat{f}\|_1 &\leq \pi\sqrt{\|f\|_2 \cdot \|\triangle f\|_2}\,,\\ \|\nabla f\|_4 &\leq 2^{\frac{3}{4}}\sqrt{\pi}\sqrt{\min\left\{\|\nabla f\|_2 \cdot \|\triangle f\|_2\,,\ \|f\|_2^{\frac{1}{2}} \cdot \|\triangle f\|_2^{\frac{3}{2}}\right\}}\,.\end{aligned}$$

Proof. Let $a = \frac{\|f\|_2}{\|\triangle f\|_2}$, $b = \frac{\|\nabla f\|_2}{\|\triangle f\|_2}$ and $k = \sqrt{k_1^2 + k_2^2}$. We have

$$\begin{aligned}\|\nabla f\|_2 &\leq \left(\int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{k}\ k^2 |f(\mathbf{k})|^2\right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{k}\ \left(\frac{1}{2a} + \frac{ak^4}{2}\right)|f(\mathbf{k})|^2\right)^{\frac{1}{2}} = \sqrt{\|f\|_2 \|\triangle f\|_2}\,,\\ \|\hat{f}\|_1 &\leq \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{k}\ \frac{\sqrt{1+a^2k^4}|\hat{f}(\mathbf{k})|}{\sqrt{1+a^2k^4}} \leq \left(\|f\|_2^2 \left(\int_0^\infty \frac{4\pi k\ \mathrm{d}k}{(1+a^2k^4)}\right)\right)^{\frac{1}{2}} \leq \pi\sqrt{\|f\|_2 \|\triangle f\|_2}\,,\\ \|\nabla f\|_4 &\leq \left(\int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{k}\ \frac{|\sqrt{1+(bk)^2}k\hat{f}(\mathbf{k})|^{\frac{4}{3}}}{(1+(bk)^2)^{\frac{2}{3}}}\right)^{\frac{3}{4}}\\ &\leq \left(\int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{k}\ (1+(bk)^2)k^2|\hat{f}(\mathbf{k})|^2 \left(\int_0^\infty \frac{2\pi k\ \mathrm{d}k}{(1+(bk)^2)^2}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \leq 2^{\frac{3}{4}}\sqrt{\pi \|\nabla f\|_2 \|\triangle f\|_2}\,.\end{aligned}$$

The proof is completed using the classical inequality $\|f\|_\infty \leq \|\hat{f}\|_1$. ■

B Additional properties on the solution of $\nabla \times \mathbf{u} = \boldsymbol{\omega}$

We now prove some decay properties of the velocity field. To do so, we first introduce some notation: $\mathbf{r} = (x, y, z)$, $|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$, $\theta(x, y, z) = \arccos(\frac{x}{r})$ and

$$\chi_\sigma(\theta) = \begin{cases} 1 & \text{if} \quad \theta \geq \sigma \\ 0 & \text{if} \quad \theta < \sigma \end{cases}\,.$$

Proposition 20 *Let $\boldsymbol{\omega} \in \mathcal{W}_\alpha$, then for all $\frac{4}{3} < p < \frac{3}{2}$ and $0 < \sigma < \pi/2$, there exists a constant $C(\epsilon, \sigma)$ such that the (divergence-free) solution of $\nabla \times \mathbf{u} = \boldsymbol{\omega}$ satisfies*

$$\sup_{t\in\mathbb{R}} \sup_{\mathbf{r}\in\mathbb{R}^3} (1 + |\mathbf{r}|)(1 + \chi_\sigma(\theta(\mathbf{r}))|\mathbf{r}|^{\frac{3}{p}-2})|\mathbf{u}(\mathbf{r}, t)| \leq C(p, \sigma)\ \|\boldsymbol{\omega}; \mathcal{W}_\alpha\|\,, \tag{46}$$

with $C(p, \sigma) \to \infty$ if $p \to \frac{4}{3}$ or $\sigma \to 0$ or $\sigma \to \pi/2$.

In other words, we prove an upper bound $|\mathbf{r}|^{-1}$ on $\mathbf{u}$ inside any cone of positive aperture σ extending in the downstream direction, and a stronger one of at least $|\mathbf{r}|^{1-\frac{3}{p}}$ outside of such cones. The optimal rate $\mathbf{u}(\mathbf{r}) \sim |\mathbf{r}|^{-2}$ outside downward extending cones (corresponding to $p = 1$) can only be proved once a downstream asymptotic expansion of the vorticity is established, see the remark after the proof.

Proof. Fix $0 < \sigma < \frac{\pi}{2}$. Note first that the supremum over t in (46) is bounded by the ℓ_1-sum over the Fourier index $n \in \mathbb{Z}$. We then recall that by Proposition 7,

$$\sum_{n\in\mathbb{Z}} \sup_{(x,y,z)\in\mathbb{R}^3} (1 + |x|)|\mathbf{u}_n(x, y, z)| \leq C\|\boldsymbol{\omega}; \mathcal{W}_\alpha\|\,.$$

Also, we note for future reference that

$$\sum_{n\in\mathbb{Z}} |\boldsymbol{\omega}_n(x,y,z)| \leq \|\boldsymbol{\omega};\mathcal{W}_\alpha\| \frac{\sqrt{1+|x|-x}}{(1+|x|)^{\frac{3}{2}}} \mathrm{e}^{\frac{x-|x|}{2}\alpha+\frac{x-r}{2}(1-\alpha)} \leq \|\boldsymbol{\omega};\mathcal{W}_\alpha\| \mathrm{e}^{-|\mathbf{r}|(1-\cos(\theta(\mathbf{r})))\frac{1-\alpha}{2}} \ .$$

Inside the cone of aperture σ, we thus find

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \sup_{\mathbf{r}\in\mathbb{R}^3,\theta(\mathbf{r})\leq\sigma} (1+|\mathbf{r}|)|\mathbf{u}_n(\mathbf{r})| &\leq \sum_{n\in\mathbb{Z}} \sup_{\mathbf{r}\in\mathbb{R}^3,\theta(\mathbf{r})\leq\sigma} (1+|x|\sqrt{1+\tan(\theta)^2})|\mathbf{u}_n(\mathbf{r})| \\ &\leq C(1+\sqrt{1+\tan(\sigma)^2})\ \|\boldsymbol{\omega};\mathcal{W}_\alpha\| \ .\end{aligned}$$

Outside of the cone, we will use the following bound on $\mathbf{u}$:

$$|\mathbf{u}(\mathbf{r},t)| \leq \mathcal{I}_1(\mathbf{r},t) + \mathcal{I}_2(\mathbf{r},t) + \mathcal{I}_3(\mathbf{r},t) \ ,$$

where, for $i \in \{1,2,3\}$, we have

$$\mathcal{I}_i(x,y,z,t) = \int_{\mathcal{R}_i(x,y,z)} \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\ \frac{|\boldsymbol{\omega}(a,b,c,t)|}{(x-a)^2+(y-b)^2+(z-c)^2}$$

with, for $\mathbf{a} = (a,b,c)$,

$$\begin{aligned}\mathcal{R}_1(x,y,z) &= \left\{\mathbf{a}\in\mathbb{R}^3 \text{ s.t. } |\mathbf{r}-\mathbf{a}| \leq |\mathbf{r}|\sin(\sigma/2)\right\} \ , \\ \mathcal{R}_2(x,y,z) &= \left\{\mathbf{a}\in\mathbb{R}^3 \text{ s.t. } \theta(\mathbf{a}) > \sigma/2 \text{ and } |\mathbf{r}-\mathbf{a}| > |\mathbf{r}|\sin(\sigma/2)\right\} \ , \\ \mathcal{R}_3(x,y,z) &= \left\{\mathbf{a}\in\mathbb{R}^3 \text{ s.t. } \theta(\mathbf{a}) \leq \sigma/2 \text{ and } |\mathbf{r}-\mathbf{a}| > |\mathbf{r}|\sin(\sigma/2)\right\} \ .\end{aligned}$$

We then note that if $\theta(\mathbf{r}) > \sigma$, $\mathcal{R}_1(\mathbf{r})$ lies outside the cone of aperture $\sigma/2$, and outside the ball of radius $|\mathbf{r}|(1-\sin(\sigma/2))$. Hence, for all $\mathbf{r}$ with $\theta(\mathbf{r}) > \sigma$, we have

$$\begin{aligned}\mathcal{I}_1(\mathbf{r},t) &\leq |\mathbf{r}|\sin\left(\frac{\sigma}{2}\right) \sum_{n\in\mathbb{Z}} \sup_{(a,b,c)\in\mathcal{R}_1(\mathbf{r})} |\boldsymbol{\omega}_n(a,b,c)| \\ &\leq \|\boldsymbol{\omega};\mathcal{W}_\alpha\| \sin\left(\frac{\sigma}{2}\right) |\mathbf{r}| \exp\big(-|\mathbf{r}|(\tfrac{1-\alpha}{2})(1-\cos(\sigma/2))(1-\sin(\sigma/2))\big) \\ &= \|\boldsymbol{\omega};\mathcal{W}_\alpha\| \sin\left(\frac{\sigma}{2}\right) |\mathbf{r}| \mathrm{e}^{-|\mathbf{r}|c(\alpha,\sigma)}\end{aligned}$$

with $c(\alpha,\sigma) > 0$ since $0 < \sigma < \frac{\pi}{2}$. Therefore, $\mathcal{I}_1(x,y,z,t)$ decays exponentially outside the cone of aperture σ.

For the $\mathcal{I}_2$ integral, we find

$$\begin{aligned}\mathcal{I}_2(\mathbf{r},t) &\leq \frac{1}{|\mathbf{r}|^2\sin(\sigma/2)^2} \int_{\mathcal{R}_2(\mathbf{r})} \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\ \sum_{n\in\mathbb{Z}} |\boldsymbol{\omega}_n(a,b,c)| \\ &\leq C(\sigma)\frac{\|\boldsymbol{\omega};\mathcal{W}_\alpha\|}{|\mathbf{r}|^2} \int_0^\infty \mathrm{d}\rho\,\rho^2 \exp\big(-\rho(\tfrac{1-\alpha}{2})(1-\cos(\sigma/2))\big) \leq C(\sigma)\frac{\|\boldsymbol{\omega};\mathcal{W}_\alpha\|}{|\mathbf{r}|^2} \ .\end{aligned}$$

Finally, for the $\mathcal{I}_3$ integral, we note that for all $\frac{4}{3} < p < \frac{3}{2}$, we have

$$\mathcal{I}_3(\mathbf{r},t) \leq \left(\int_0^\infty \mathrm{d}a \int_{\mathbb{R}^2} \mathrm{d}b\,\mathrm{d}c\ (1-\chi_{\sigma/2}(\theta(a,b,c)))\ \sum_{n\in\mathbb{Z}} |\boldsymbol{\omega}_n(a,b,c)|^p\right)^{1/p} \left(4\pi \int_{|\mathbf{r}|\sin\left(\frac{\sigma}{2}\right)}^\infty \mathrm{d}r\ r^{-\frac{2}{p-1}}\right)^{1-\frac{1}{p}}$$

$$\leq C(p,\sigma)\,|\mathbf{r}|^{1-\frac{3}{p}}\|\boldsymbol{\omega};\mathcal{W}_\alpha\|\left(\int_0^\infty \mathrm{d}x\,(1+x)^{-\frac{3p}{2}+1}\right)^{1/p} \leq C(p,\sigma)\,|\mathbf{r}|^{1-\frac{3}{p}}\|\boldsymbol{\omega};\mathcal{W}_\alpha\|\;, \tag{47}$$

since $\sum_{n\in\mathbb{Z}}\|\boldsymbol{\omega}_n(x,\cdot)\|_p \leq \|\boldsymbol{\omega};\mathcal{W}_\alpha\|(1+|x|)^{-\frac{3}{2}+\frac{1}{p}}$ for $x\geq 0$. ■

As is apparent from the above proof, the failure of $\mathbf{u}$ to decay like $|\mathbf{r}|^{-2}$ outside downstream extending cones is due to the vorticity $\boldsymbol{\omega}$ not being in $\mathrm{L}^1(\{\mathbf{r}\in\mathbb{R}^3 \text{ s.t. } \theta(\mathbf{r})<\sigma/2\})$ due to its (relatively) slow decay in the wake (see the $\mathcal{I}_3$ term above). The optimal decay rate of $|\mathbf{r}|^{-2}$ outside the wake can only be obtained for the velocity field corresponding to a vorticity decaying faster than $1/x$ inside the wake, for instance once an asymptotic expansion for the vorticity is obtained. One would then write

$$\boldsymbol{\omega}=\boldsymbol{\omega}-\boldsymbol{\omega}_e+\boldsymbol{\omega}_e \quad\text{and}\quad u=u-u_e+u_e$$

where $\boldsymbol{\omega}_e$ and u_e are the first few explicit terms in the asymptotic development of the wake, see e.g. [10, 8]. In particular, we expect $\mathbf{u}_e\sim|\mathbf{r}|^{-2}$ outside downward extending cones, and $\boldsymbol{\omega}-\boldsymbol{\omega}_e$ to decay faster in the wake. One would thus get improved estimates on $u-u_e$ by pushing the estimate (47) above down to $p=1$.

C Divergence free extensions in exterior domains

Let Ω be a 3D exterior domain. We denote by $\delta(\Omega^c)$ the diameter of the smallest (closed) sphere containing Ω^c, and choose as origin of the coordinate system the center of that sphere. For $1<a<b$, we also denote by $\chi_{a,b}$ a smooth function interpolating between $\chi_{a,b}(r)=0$ if $r\leq a\delta(\Omega^c)$ and $\chi_{a,b}(r)=1$ if $r\geq b\delta(\Omega^c)$.

We now construct a divergence-free extension to $\mathbb{R}^3$ of a vector field $\mathbf{u}:\Omega\to\mathbb{R}^3$, for a particular class of divergence-free vector fields vanishing on $\partial\Omega$ and having non-zero limits as $|\mathbf{x}|\to\infty$.

Proposition 21 *Let $1<a<b$ and $\mathbf{u}$ satisfy*

$$\nabla\mathbf{u}\in\mathrm{L}^2(\Omega)\;,\quad \nabla\cdot\mathbf{u}=0\;,\quad \mathbf{u}|_{\partial\Omega}=0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty}\mathbf{u}(\mathbf{x})=\mathbf{u}_\infty\neq 0\;, \tag{48}$$

then

$$\begin{aligned}
\mathrm{E}_{a,b}[\mathbf{u}](\mathbf{x}) &= \chi_{a,b}(|\mathbf{x}|)\mathbf{u}(\mathbf{x})+\mathrm{T}_{a,b}[\mathbf{u}_\infty](\mathbf{x})(1-\chi_{a,b}(|\mathbf{x}|))+\nabla\chi_{a,b}(|\mathbf{x}|)\times\boldsymbol{\psi}(\mathbf{x})\\
\textit{with}\quad \boldsymbol{\psi}(\mathbf{x}) &= \frac{1}{4\pi}\int_\Omega \nabla\times\big(\mathbf{u}(\mathbf{y})-\mathrm{T}_{a,b}[\mathbf{u}_\infty](\mathbf{y})\big)\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}-\frac{1}{|\mathbf{y}|}\right)\mathrm{d}^3\mathbf{y}\\
\textit{and}\quad \mathrm{T}_{a,b}[\mathbf{u}_\infty](\mathbf{x}) &= \mathbf{u}_\infty\left(\chi_{a,b}(|\mathbf{x}|)+\frac{1}{2}|\mathbf{x}|\chi'_{a,b}(|\mathbf{x}|)\right)-\mathbf{x}\left(\frac{(\mathbf{u}_\infty\cdot\mathbf{x})\,\chi'_{a,b}(|\mathbf{x}|)}{2|\mathbf{x}|}\right)
\end{aligned}$$

is well defined $\forall\mathbf{x}\in\mathbb{R}^3$, satisfies

$$\mathrm{E}_{a,b}[\mathbf{u}](\mathbf{x})=\begin{cases}\mathbf{u}(\mathbf{x}) & \textit{if}\quad |\mathbf{x}|\geq b\delta(\Omega^c)\\ 0 & \textit{if}\quad |\mathbf{x}|\leq a\delta(\Omega^c)\end{cases}\;. \tag{49}$$

and is divergence-free, i.e. $\nabla\cdot\mathrm{E}_{a,b}[\mathbf{u}]=0$.

Note that both $\mathrm{T}_{a,b}[\mathbf{u}_\infty]$ and $\mathrm{E}_{a,b}[\mathbf{u}]$ are divergence-free functions over $\mathbb{R}^3$, vanishing identically inside a sphere containing Ω^c and identically equal to their argument outside of a bigger sphere containing Ω^c. While $\mathrm{E}_{a,b}[\mathbf{u}]$ is the main object of interest here, $\mathrm{T}_{a,b}[\mathbf{u}_\infty]$ is a convenient way of avoiding boundary terms when using the divergence theorem to study $\boldsymbol{\psi}$ below.

Proof. That (49) is satisfied is trivial. In particular, $\mathrm{E}_{a,b}[\mathbf{u}](\mathbf{x})$ is well defined outside the annulus $\mathcal{A}_{a,b} = \{\mathbf{x} \in \mathbb{R}^3 \text{ s.t. } a\delta(\Omega^c) \leq |\mathbf{x}| \leq b\delta(\Omega^c)\}$. We next show that $\boldsymbol{\psi}(\mathbf{x})$ is well defined for $\mathbf{x} \in \mathcal{A}_{a,b}$. For convenience, we define $\mathbf{v} = \mathbf{u} - \mathrm{T}_{a,b}[\mathbf{u}_\infty]$, and note that $\|\nabla \times \mathbf{v}\|_2 < \infty$. Then, there exists a compactly supported function $\rho : \mathbb{R}^+ \to \mathbb{R}$ with $\rho(0) = 1$, and a constant C such that for all $\mathbf{x} \in \mathcal{A}_{a,b}$, there holds

$$\sup_{\mathbf{x}\in\mathcal{A}_{a,b}} |\boldsymbol{\psi}(\mathbf{x})| \leq C \sup_{\mathbf{x}\in\mathcal{A}_{a,b}} \int_\Omega \mathrm{d}^3\mathbf{y} \left(\frac{\rho(|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|} + \frac{|\mathbf{x}|}{|\mathbf{y}|^2} \right) |\nabla \times \mathbf{v}(\mathbf{y})| \leq C\|\nabla \times \mathbf{v}\|_2 \, .$$

This shows that $\mathrm{E}_{a,b}[\mathbf{u}](\mathbf{x})$ is well defined for all $\mathbf{x} \in \mathbb{R}^3$.

It only remains to show that $\nabla \cdot \mathrm{E}_{a,b}[\mathbf{u}] = 0$. Since $\nabla \cdot \mathrm{T}_{a,b}[\mathbf{u}_\infty] = 0$ and $\nabla \cdot \mathbf{u} = 0$, we find

$$\nabla \cdot \mathrm{E}_{a,b}[\mathbf{u}](|\mathbf{x}|) = \nabla\chi_{a,b}(|\mathbf{x}|) \cdot \big(\mathbf{v}(\mathbf{x}) - \nabla \times \boldsymbol{\psi}(\mathbf{x})\big) \, .$$

We thus only need to show that $\nabla \times \boldsymbol{\psi}(\mathbf{x}) = \mathbf{v}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{A}_{a,b}$ since $\nabla\chi_{a,b}(|\mathbf{x}|) = 0$ if $\mathbf{x} \notin \mathcal{A}_{a,b}$. Using $\epsilon_{i,j,k}$, the completely antisymmetric tensor with $\epsilon_{1,2,3} = 1$ so that $[\nabla \times \mathbf{u}]_i = \sum_{j,k} \epsilon_{i,j,k}\partial_{x_j}\mathbf{u}_k$, we find

$$\begin{aligned}
[\nabla \times \boldsymbol{\psi}(\mathbf{x})]_i &= \sum_{j,k,l,m=1}^{3} \frac{\epsilon_{i,j,k}\epsilon_{k,l,m}}{4\pi} \int_\Omega \left(\partial_{y_l}\mathbf{v}_m(\mathbf{y})\right) \partial_{x_j}\left(|\mathbf{x}-\mathbf{y}|^{-1}\right) \mathrm{d}^3\mathbf{y} \\
&= \lim_{R\to\infty} \left(\mathbf{I}_i(R,\mathbf{x}) + \mathbf{J}_i(R,\mathbf{x})\right)
\end{aligned}$$

in the sense of distributions, where

$$\begin{aligned}
\mathbf{I}_i(R,\mathbf{x}) &= \int_{\Omega_R} \sum_{l=1}^{3} \partial_{y_l}\Bigg(\sum_{j,k,m=1}^{3} \frac{\epsilon_{i,j,k}\epsilon_{k,l,m}}{4\pi} \mathbf{v}_m(\mathbf{y})\partial_{x_j}\left(|\mathbf{x}-\mathbf{y}|^{-1}\right) \Bigg)\mathrm{d}^3\mathbf{y} \, , \\
\mathbf{J}_i(R,\mathbf{x}) &= \sum_{j,k,l,m=1}^{3} \frac{\epsilon_{i,j,k}\epsilon_{k,l,m}}{4\pi} \int_{\Omega_R} \mathbf{v}_m(\mathbf{y})\partial_{x_l}\partial_{x_j}\left(|\mathbf{x}-\mathbf{y}|^{-1}\right) \mathrm{d}^3\mathbf{y} \, ,
\end{aligned}$$

with $\Omega_R = B_R \cap \Omega$. We now claim that for all $\mathbf{x} \in \mathcal{A}_{a,b}$, we have

$$\lim_{R\to\infty} \mathbf{I}(R,\mathbf{x}) = 0 \quad \text{and} \quad \lim_{R\to\infty} \mathbf{J}(R,\mathbf{x}) = \mathbf{v}(\mathbf{x}) \, .$$

We first consider the function $\mathbf{J}$. Since $\nabla \cdot \mathbf{v} = 0$, using the symbols $\nabla_\mathbf{x}$, respectively $\nabla_\mathbf{y}$ to denote the Nabla operators in the variables $\mathbf{x}$ resp. $\mathbf{y}$, we find

$$\begin{aligned}
\mathbf{J}(R,\mathbf{x}) &= \int_{\Omega_R} \nabla_\mathbf{x} \times \nabla_\mathbf{x} \times \left(\frac{\mathbf{v}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}^3\mathbf{y} = \mathbf{v}(\mathbf{x}) + \int_{\Omega_R} \nabla_\mathbf{x}\nabla_\mathbf{x} \cdot \left(\frac{\mathbf{v}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}^3\mathbf{y} \\
&= \mathbf{v}(\mathbf{x}) - \int_{\Omega_R} \nabla_\mathbf{x}\nabla_\mathbf{y} \cdot \left(\frac{\mathbf{v}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}^3\mathbf{y} \equiv \mathbf{v}(\mathbf{x}) + \mathbf{K}(R,\mathbf{x}) \, ,
\end{aligned}$$

since

$$\triangle\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \quad \text{and} \quad \triangle\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\right) = \delta(\mathbf{x}-\mathbf{y}) \, . \tag{50}$$

We now assume that $R > b\delta(\Omega^c)$, and apply the divergence theorem to $\mathbf{K}$ and $\mathbf{I}_i$. Since $\mathcal{A}_{a,b} \cap \partial\Omega_R = \emptyset$, $|\mathbf{x}-\mathbf{y}|^{-1}$ is nonsingular for $\mathbf{y} \in \partial\Omega_R$ and $\mathbf{x} \in \mathcal{A}_{a,b}$. Using $\mathrm{d}\sigma = R^2\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$ to denote the surface element on ∂B_R, we thus get

$$\mathbf{K}(R,\mathbf{x}) = \int_{\partial B_R} \frac{(\mathbf{x}-\mathbf{y})(\mathbf{v}(\mathbf{y})\cdot\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^3\,|\mathbf{y}|}\mathrm{d}\sigma \, ,$$

$$\mathbf{I}_i(R,\mathbf{x}) = \int_{\partial B_R} \sum_{l=1}^{3} \left(\sum_{j,k,m=1}^{3} \epsilon_{i,j,k}\epsilon_{k,l,m}\mathbf{v}_m(\mathbf{y}) \frac{\mathbf{x}_j - \mathbf{y}_j}{|\mathbf{x}-\mathbf{y}|^3} \right) \frac{\mathbf{y}_l}{|\mathbf{y}|} \mathrm{d}\sigma \ ,$$

since $\mathbf{v}(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$. The proof is complete, since

$$|\mathbf{K}(R,\mathbf{x})|^2 + |\mathbf{I}(R,\mathbf{x})|^2 \le C \left(\int_{\partial B_R} \frac{|\mathbf{v}(\mathbf{y})|}{|\mathbf{y}|^2} \mathrm{d}\sigma \right)^2 \le C \int_{\partial B_R} \frac{|\mathbf{v}(\mathbf{y})|^2}{|\mathbf{y}|^2} \mathrm{d}\sigma \ .$$

which tends to 0 as $R \to \infty$ by Lemma 22 below. ■

To complete the proof of Proposition 21, we need the following result, essentially due to J. Leray [6]:

Lemma 22 *Let Ω be an exterior domain, with a boundary $\partial\Omega$ of finite area $\Sigma(\partial\Omega)$. Let $\mathbf{u} : \Omega \to \mathbb{R}^3$ have a finite Dirichlet integral, i.e. be such that $\nabla\mathbf{u} \in \mathrm{L}^2(\Omega)$. Then there exist a constant vector $\mathbf{u}_\infty$ such that*

$$\int_\Omega \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty|^2}{|\mathbf{x}|^2} \mathrm{d}^3\mathbf{x} \le \int_\Omega |\nabla\mathbf{u}(\mathbf{x})|^2 \mathrm{d}^3\mathbf{x} < \infty \ . \tag{51}$$

$$\lim_{R\to\infty} \int_{\partial B_R} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty|^2}{|\mathbf{x}|^2} \mathrm{d}\sigma = 0 \ , \tag{52}$$

where $\partial B_R = \{\mathbf{x} \in \mathbb{R}^3 \ s.t. \ |\mathbf{x}| = R\}$ and $\mathrm{d}\sigma = R^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$.

Proof. The existence of $\mathbf{u}_\infty$ and the estimate (51) is a classical result of Leray [6] (see also Galdi [1]). In particular, it implies that the vector function

$$\mathbf{f}(\mathbf{x}) = \frac{\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty}{|\mathbf{x}|}$$

is in $\mathrm{H}^1(\Omega)$. Namely (51) gives $\|\mathbf{f}\|_2 \le \|\nabla\mathbf{u}\|_2$ and $\|\nabla\mathbf{f}\|_2 \le \rho^{-1}(\|\mathbf{f}\|_2 + \|\nabla\mathbf{u}\|_2) \le 2\rho^{-1}\|\nabla\mathbf{u}\|_2$, where ρ is the diameter of the largest sphere contained in Ω^c. Using again $\Omega_R = B_R \cap \Omega$, we next consider

$$F(R) = \int_{\Omega_R} \nabla \cdot \left(\frac{|\mathbf{f}(\mathbf{x})|^2 \, \mathbf{x}}{|\mathbf{x}|} \right) \mathrm{d}^3\mathbf{x} - \int_{\partial\Omega} \frac{|\mathbf{u}_\infty|^2 \, \mathbf{x}\cdot\mathbf{n}(\mathbf{x})}{|\mathbf{x}|^3} \, \mathrm{d}\Sigma \ , \tag{53}$$

where $\mathbf{n}(\mathbf{x})$ is the outward normal on $\partial\Omega$ and $\mathrm{d}\Sigma$ its surface element. Straightforward arguments give

$$\sup_{R\ge\delta(\Omega^c)} |F(R)| \le \frac{2}{\rho}\|\mathbf{f}\|_2^2 + \|\mathbf{f}\|_2\|\nabla\mathbf{f}\|_2 + \frac{|\mathbf{u}_\infty|^2}{\rho^2}\Sigma(\partial\Omega) \ , \tag{54}$$

$$\lim_{R\to\infty} F(R) = \int_\Omega \nabla \cdot \left(\frac{|\mathbf{f}(\mathbf{x})|^2 \, \mathbf{x}}{|\mathbf{x}|} \right) \mathrm{d}^3\mathbf{x} - \int_{\partial\Omega} \frac{|\mathbf{u}_\infty|^2 \, \mathbf{x}\cdot\mathbf{n}(\mathbf{x})}{|\mathbf{x}|^3} \, \mathrm{d}\Sigma \ . \tag{55}$$

Note that (54) and (55) imply that $F(R)$ has a finite limit as $R \to \infty$. Now, by the divergence theorem, we have

$$F(R) = \int_{\partial B_R} |\mathbf{f}(\mathbf{x})|^2 \, \mathrm{d}\sigma + \int_{\partial\Omega} \frac{|\mathbf{f}(\mathbf{x})|^2 \, \mathbf{x}\cdot\mathbf{n}(\mathbf{x})}{|\mathbf{x}|} \, \mathrm{d}\Sigma - \int_{\partial\Omega} \frac{|\mathbf{u}_\infty|^2 \, \mathbf{x}\cdot\mathbf{n}(\mathbf{x})}{|\mathbf{x}|^3} \, \mathrm{d}\Sigma = \int_{\partial B_R} |\mathbf{f}(\mathbf{x})|^2 \, \mathrm{d}\sigma \ .$$

Since for any finite $r \ge \delta(\Omega^c)$, we have

$$\int_r^\infty F(R) \, \mathrm{d}R = \int_r^\infty \int_{\partial B_R} |\mathbf{f}(\mathbf{x})|^2 \, \mathrm{d}\sigma \, \mathrm{d}R = \int_{\mathbb{R}^3 \setminus B_r} |\mathbf{f}(\mathbf{x})|^2 \, \mathrm{d}^3\mathbf{x} \le \|\mathbf{f}\|_2^2 < \infty \ ,$$

we get $F(R) \to 0$ as $R \to \infty$. ■

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