Leading order asymptotics of stationary Navier-Stokes flows in the presence of a wall

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Abstract

We consider the problem of a body moving within an incompressible fluid at constant speed parallel to a wall, in an otherwise unbounded domain. We give a detailed description of the asymptotic behavior on the fluid flow in a half-space using as a starting point the theory of the existence of solutions which uses the coordinate perpendicularly to the wall as time variable.

Contents

Mathematics Subject Classification (2000): 35Q30, 76D05, 35B40, 35B42.

Keywords: Navier-Stokes equations; Stationary solutions; Asymptotics.

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1 Introduction

We consider solutions of the three-dimensional Navier-Stokes equations in a half space which are stationary in a frame moving with speed one from right to left along the x-direction, i.e., solutions of

$$-\partial_x \mathbf{u} + \Delta \mathbf{u} = \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \ , \tag{1}$$

in the domain $\Omega_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z > 1\}$, subject to the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0 \ , \tag{2}$$

the boundary conditions

$$\mathbf{u}(x, y, 1) = 0 \ , \qquad (x, y) \in \mathbb{R}^2 \ , \tag{3}$$

$$\lim_{\mathbf{x}\to\infty} \mathbf{u}(\mathbf{x}) = 0 \ , \qquad \mathbf{x} \in \Omega_+ \ , \tag{4}$$

and with $\mathbf{F}$ a smooth vector field with compact support in Ω_+, *i.e.*, $\mathbf{F} \in C_c^\infty(\Omega_+)$.

This model can be used to describe the motion of a body moving within an incompressible fluid at constant speed parallel to a wall, in an otherwise unbounded domain. A very important practical application of such a situation is the description of the motion of bubbles rising in a liquid parallel to a nearby wall. Interesting recent experimental work is described in [19, 20]. Numerical studies can be found in [4, 6, 12, 17].

In what follows we consider the situation of a single bubble of fixed shape which rises with constant velocity in a regime of Reynolds numbers less than about fifty. The resulting fluid flow is then laminar. The Stokes equations provide a good quantitative description (forces determined within an error of one percent) only for Reynolds numbers less than one. For the larger Reynolds numbers under consideration the Navier-Stokes equations need to be solved in order to obtain precise results. The vertical speed of the bubble depends on the drag, and the distance from the wall at which the bubble rises requires one to find the position relative to the wall where the transverse force is zero. Since at low Reynolds numbers the transverse forces are orders of magnitude smaller than the forces along the flow, this turns out to be a very delicate problem which needs to be solved numerically with the help of high precision computations. But, if done by brute force, such computations are excessively costly even with today's computers. In [1, 2, 14, 15], the second author and his collaborators have developed techniques that lead for similar problems to an overall gain of computational efficiency of typically several orders of magnitude. These techniques use as an input a precise asymptotic description of the flow. The present work is an important step towards the extension of this technique to the case of motions close to a wall.

The present paper is concerned with the asymptotic behavior of solutions, this asymptotic description is important in view of the formulation of artificial boundary conditions. The techniques used here take as the starting point existence results which have been obtained in [10]. Our main result is a detailed description of the asymptotic behavior of the velocity components. Each of these components is treated in the subsequent sections, separately.

The following theorems are our main results. They provide the leading order asymptotic behavior of solutions whose existence has been shown in [10]. We use the notation and definitions of [10] throughout this paper.

Theorem 1 *Let $\mathbf{F} \in C_c^\infty(\Omega_+)$, and let $\hat{\mathbf{F}}$ be the Fourier transform of $\mathbf{F}$ with respect to the variables x and y. If $\hat{\mathbf{F}}$ is sufficiently small in a sense to be defined below, then there exist a* *divergence free vector field* $\mathbf{u} = (u_1, u_2, u_3) \in H^1(\Omega_+)$ *and a function* p *satisfying the Navier-Stokes equations (1), (2) in* Ω_+ *subject to the boundary conditions (3), (4). In addition, there exist constants* C_1 *and* C_2 *such that*

$$\lim_{z \to \infty} z^{\frac{5}{2}} \sup_{(x,y) \in \mathbb{R}^2} |u_1(x,y,z) - u_1^{as}(x,y,z)| = 0 \ , \tag{5}$$

with

$$u_1^{as}(x,y,z) = -\frac{3C_1}{16\pi^{\frac{3}{2}}} \int_0^{2\pi} \frac{\cos\phi\sqrt{i\cos\phi}}{\psi(x,y,z,\phi)} d\phi - \frac{3C_2}{16\pi^{\frac{3}{2}}} \int_0^{2\pi} \frac{\sin\phi\sqrt{i\cos\phi}}{\psi(x,y,z,\phi)} d\phi \ ,$$

where $r = \sqrt{x^2+y^2+z^2}$, $\psi(x,y,z,\phi) = [z + i(x\cos\phi + y\sin\phi)]^{\frac{5}{2}}$.

Theorem 2 *Under the same assumptions as for Theorem 1, we have the following asymptotic description for velocity field* $\mathbf{u}$*:*

$$\lim_{z \to \infty} z^2 \sup_{(x,y) \in \mathbb{R}^2} |u_1(x,y,z)| = 0 \ , \tag{6}$$

$$\lim_{z \to \infty} z^2 \sup_{(x,y) \in \mathbb{R}^2} |u_2(x,y,z) - u_2^{as}(x,y,z)| = 0 \ , \tag{7}$$

$$\lim_{z \to \infty} z^2 \sup_{(x,y) \in \mathbb{R}^2} |u_3(x,y,z) - u_3^{as}(x,y,z)| = 0 \ , \tag{8}$$

with

$$u_2^{as}(x,y,z) = C_2 \int_{\mathbb{R}^2} e^{-ik_1x} e^{-ik_2y} \frac{i\,(k+\kappa)\,k}{k_1} (e^{-\kappa z} - e^{-kz}) d^2\mathbf{k} + C_2 \frac{z\,(r+2)}{4\pi r^3} \exp\left(\frac{x-r}{2}\right) \ ,$$

$$u_3^{as}(x,y,z) = C_2 \int_{\mathbb{R}^2} e^{-ik_1x} e^{-ik_2y} \frac{(k+\kappa)\,k_2}{k_1} (e^{-\kappa z} - e^{-kz}) d^2\mathbf{k} \ ,$$

and

$$\nabla \cdot (0, u_2^{as}, u_3^{as}) = 0 \ .$$

Remark 3 *Existence of solutions has been proved in [10] where we showed that* $|u_i(x,y,z)| \leq C/z^2$*. Theorem 2 provides an explicit description of the dominant behavior of the velocity field and Theorem 1 implies that the bound in [10] is sharp for* u_2 *and* u_3*, but not for* u_1*.*

The proofs of the above theorems will be given by extracting the leading asymptotic terms from each component of the velocity in the following sections. For the convenience of the reader we recollect the main result and some expressions which have been proved in [10] and which will be used again in this paper.

2 Preliminaries

As in [10] we use throughout this paper a hat to indicate functions in Fourier space, i.e., we express functions f by:

$$f(x,y,z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-ik_1x} e^{-ik_2y} \hat{f}(k_1,k_2,z) dk_1 dk_2 \ .$$

In the existence paper [10], we have introduced the following function space. Let $\alpha, r \geq 0$ and $\mathbf{k} = (k_1,k_2) \in \mathbb{R}^2$, and let

$$\bar{\mu}_\alpha(\mathbf{k},t) = \frac{1}{1 + (|\mathbf{k}|t)^\alpha} \ .$$

Definition 4 *Let $\mathbb{R}_0^2 = \mathbb{R}^2 \backslash \{0\}$. We define, for fixed $\alpha \geq 0$ and $p \geq 0$, $\mathcal{B}_{\alpha,p}$ to be the Banach space of functions $f \in C(\mathbb{R}_0^2 \times [1,\infty), \mathbb{C})$, for which the norm*

$$||f;\mathcal{B}_{\alpha,p}|| = \sup_{t\geq 1} \sup_{\mathbf{k}\in\mathbb{R}_0^2} \frac{|f(\mathbf{k},t)|}{\frac{1}{t^p}\bar{\mu}_\alpha(\mathbf{k},t)}$$

is finite. Furthermore, we set $\mathcal{B}_{\alpha,p}^n = \underbrace{\mathcal{B}_{\alpha,p} \times ... \times \mathcal{B}_{\alpha,p}}_{n \text{ times}}$, $\mathcal{W}_\alpha = \mathcal{B}_{\alpha,3}^3$, and $\mathcal{V}_\alpha = \mathcal{B}_{\alpha,1}^3 \times \mathcal{B}_{\alpha,0}^3$.

The result in [10] shows that $\hat{\mathbf{u}} \in \mathcal{B}_{\alpha,0}^3$ and $\hat{\mathbf{Q}} \in \mathcal{B}_{\alpha,3}^3$, where

$$\hat{\mathbf{Q}} = \hat{\mathbf{q}} - \hat{\mathbf{F}} = \left(\hat{Q}_1, \hat{Q}_2, \hat{Q}_3\right) .$$

with $\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2, \hat{q}_3)$,

$$\begin{aligned}
\hat{q}_1 &= \frac{1}{4\pi^2} \left(\hat{\omega}_3 * \hat{u}_2 - \hat{\omega}_2 * \hat{u}_3\right) \ , \\
\hat{q}_2 &= \frac{1}{4\pi^2} \left(\hat{\omega}_1 * \hat{u}_3 - \hat{\omega}_3 * \hat{u}_1\right) \ , \\
\hat{q}_3 &= \frac{1}{4\pi^2} \left(\hat{\omega}_2 * \hat{u}_1 - \hat{\omega}_1 * \hat{u}_2\right) \ .
\end{aligned}$$

Remark 5 *The constants in Theorem 1 and Theorem 2 have the following representation*

$$C_1 = \int_1^\infty (s-1)\hat{Q}_1(0,s)ds, \ C_2 = \int_1^\infty (s-1)\hat{Q}_2(0,s)ds \ . \tag{9}$$

We furthermore define

$$k = \sqrt{k_1^2 + k_2^2}, \ \kappa = \sqrt{k^2 - ik_1} \ ,$$

and

$$\Lambda_- = -\operatorname{Re}(\kappa) = -\frac{1}{2}\sqrt{2\sqrt{k_1^2 + k^4} + 2k^2} \ , \tag{10}$$

$$\Lambda_*^- = -\operatorname{Re}\left(\sqrt{k_2^2 - ik_1}\right) = -\frac{1}{2}\sqrt{2\sqrt{k_1^2 + k_2^4} + 2k_2^2} \ . \tag{11}$$

We have that

$$|\kappa| = (k_1^2 + k^4)^{1/4} \leq |k_1|^{1/2} + k \leq 2^{3/4}|\kappa| \leq 2^{3/4}\left(1+k\right) \ ,$$

and that

$$k \leq |\Lambda_-| \leq |\kappa| \leq \sqrt{2}|\Lambda_-| \ .$$

Therefore, we have in particular that for $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$,

$$e^{\Lambda_- \sigma} \leq e^{-k\sigma} \ .$$

We will also need the following inequalities. For all $N \in \mathbb{N} = \{0, 1, 2, 3, ...\}$, we have for $z \in \mathbb{C}$ with $\operatorname{Re}(z) \leq 0$,

$$\left| \frac{e^z - \sum_{n=0}^N \frac{1}{n!} z^n}{z^{N+1}} \right| \leq \text{const.} \ , \tag{12}$$

and for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$,

$$\left| \frac{e^z - \sum_{n=0}^{N} \frac{1}{n!} z^n}{z^{N+1}} \right| \leq \mathrm{const.} e^{\mathrm{Re}(z)} \ . \tag{13}$$

In the following, we will routinely use (12) and (13) without mention. In what follows, we systematically use the notation introduced above, but, for simplicity, we set

$$\mu(\mathbf{k}, s) = \frac{1}{s^3} \bar{\mu}_\alpha(\mathbf{k}, s) \ ,$$

and $\|Q\| = C \ \|\hat{\mathbf{Q}}; \mathcal{W}_\alpha\|$ with $\alpha > 2$, and C a constant. This constant may be different from instance to instance, changing even within the same line. The following is the basic result of [10] concerning the asymptotic behavior.

Lemma 6 *[10] For all $\mathbf{F} \in C_c^\infty(\Omega_+)$ with $||\hat{\mathbf{F}}; \mathcal{W}_\alpha||$ sufficiently small, there exist a vector field $\mathbf{u} = (u_1, u_2, u_3) \in H^1(\Omega_+)$ and a function p satisfying the Navier-Stokes equations (1), (2) in Ω_+ subject to the boundary conditions (3), (4). Moreover, there exists a constant C such that, uniformly in $(x, y, z) \in \Omega_+$, $|u_i(x, y, z)| \leq C/z^2$, for $i = 1, 2, 3$.*

Let us recall the structure of $\mathbf{u}$ in [10]:

$$\hat{u}_i = \sum_{m=1,2,3} \sum_{n=1,2,3} \hat{u}_{i,n,m}, \ \ i = 1, 2, 3 \ ,$$

for certain functions $\hat{u}_{i,n,m}$ which we recall below.

3 Asymptotic behavior of $\hat{u}_1$

We have already shown in [10] that $\hat{u}_{1,1,1} \in \mathcal{B}_{\alpha,0}$, $\hat{u}_{1,1,2} \in \mathcal{B}_{\alpha,\frac{1}{2}}$, and that the remaining $\hat{u}_{1,i,j}$ are in $\mathcal{B}_{\alpha,1}$. Therefore, $\hat{u}_{1,1,1}$ and $\hat{u}_{1,1,2}$ are the leading order terms of $\hat{u}_1$ in Fourier space. We first prove (5), which is an immediate consequence of:

Proposition 7 *Let $\hat{u}_1(\mathbf{k}, t)$ be as above. Then we have*

$$\lim_{t \to \infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} |\hat{u}_1(\mathbf{k}, t) - \hat{u}_1^{as}(\mathbf{k}, t)| \, d^2\mathbf{k} = 0 \ ,$$

where

$$\hat{u}_1^{as}(\mathbf{k}, t) = -C_1 \frac{ik_1\sqrt{-ik_1}}{k} e^{-kt} - C_2 \frac{ik_2\sqrt{-ik_1}}{k} e^{-kt} \ ,$$

with C_1, C_2 as defined in (9).

To prove this proposition we analyze $\hat{u}_{1,1,1}$ and $\hat{u}_{1,1,2}$ in detail.

(i) *Discussion of* $\hat{u}_{1,1,1}$

We have the following expression for $\hat{u}_{1,1,1}(\mathbf{k}, t)$:

$$\hat{u}_{1,1,1}(\mathbf{k},t) = \hat{u}_{1,\kappa,1}(\mathbf{k},t) + \hat{u}_{1,k,1}(\mathbf{k},t) \ , \tag{14}$$

with

$$\hat{u}_{1,\kappa,1}(\mathbf{k},t) = \frac{1}{2} e^{-\kappa(t-1)} \int_1^t f_{1,1,1}(\mathbf{k},s-1) \hat{Q}_1(\mathbf{k},s) ds \ ,$$

$$\hat{u}_{1,k,1}(\mathbf{k},t) = \frac{1}{2} e^{-k(t-1)} \int_1^t g_{1,1,1}(\mathbf{k},s-1) \hat{Q}_1(\mathbf{k},s) ds \ ,$$

with $f_{1,1,1}(\mathbf{k},s)$ and $g_{1,1,1}(\mathbf{k},s)$ as in [10]. In [10] we have shown that

$$\hat{u}_{1,\kappa,1}(\mathbf{k},t) \in \mathcal{B}_{\alpha,0} \text{ and } \hat{u}_{1,k,1}(\mathbf{k},t) \in \mathcal{B}_{\alpha,\frac{1}{2}} \ .$$

We first consider $\hat{u}_{1,\kappa,1}(\mathbf{k},t)$. Let

$$\hat{H}_{1,\kappa,1}(\mathbf{k},t) = e^{-\kappa(t-1)} \int_1^t (s-1) \hat{Q}_1(\mathbf{k},s) ds \ .$$

Using the triangle inequality we get that

$$\begin{aligned} |\hat{u}_{1,\kappa,1}(\mathbf{k},t)| &= \left| \hat{u}_{1,\kappa,1}(\mathbf{k},t) - \hat{H}_{1,\kappa,1}(\mathbf{k},t) + \hat{H}_{1,\kappa,1}(\mathbf{k},t) \right| \\ &\leq \left| \hat{u}_{1,\kappa,1}(\mathbf{k},t) - \hat{H}_{1,\kappa,1}(\mathbf{k},t) \right| + \left| \hat{H}_{1,\kappa,1}(\mathbf{k},t) \right| \ . \end{aligned} \tag{15}$$

We bound each term in (15) separately. By definition

$$\hat{u}_{1,\kappa,1}(\mathbf{k},t) - \hat{H}_{1,\kappa,1}(\mathbf{k},t) = \frac{1}{2} e^{-\kappa(t-1)} \int_1^t \left[f_{1,1,1}(\mathbf{k},s-1) - 2(s-1) \right] \hat{Q}_1(\mathbf{k},s) ds \ .$$

For the expression

$$f_{1,1,1}(k,\sigma) - 2\sigma = \frac{ik_1+1}{\kappa} \left(e^{\kappa\sigma} - e^{-\kappa\sigma} \right) + \frac{2\kappa(\kappa+k)}{k} \left(e^{-\kappa\sigma} - e^{-k\sigma} \right) - 2\sigma \ ,$$

a straightforward bound may be derived using that

$$\begin{aligned} f_{1,1,1}(k,\sigma) - 2\sigma = &- 2\left(ik_1+1\right) e^{\kappa\sigma} \sigma \frac{\left(1-e^{-2\kappa\sigma}\right)}{-2\kappa\sigma} - 2\sigma \\ &+ \frac{2i\kappa k_1}{k} \sigma e^{-k\sigma} \frac{\left(e^{(k-\kappa)\sigma} - 1\right)}{(k-\kappa)\sigma} \ , \end{aligned}$$

from which we get that

$$|f_{1,1,1}(k,\sigma) - 2\sigma| \leq \text{const.} \left(1 + |\Lambda_-|\right) \sigma e^{|\Lambda_-|\sigma} \ .$$

This bound is however not sufficient to bound all contributions in (15), and we therefore need an additional representation of the integral kernel:

$$\begin{aligned} f_{1,1,1}(k,\sigma) - 2\sigma &= -2ik_1\sigma e^{\kappa\sigma} \frac{\left(1-e^{-2\kappa\sigma}\right)}{-2\kappa\sigma} + \frac{2i\kappa k_1}{k} \sigma e^{-k\sigma} \frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\sigma} \\ &\quad - 2e^{\kappa\sigma}\sigma \frac{\left(1-e^{-2\kappa\sigma}\right)}{-2\kappa\sigma} - 2\sigma \\ &= -2ik_1\sigma e^{\kappa\sigma} \frac{\left(1-e^{-2\kappa\sigma}\right)}{-2\kappa\sigma} + \frac{2i\kappa k_1}{k} \sigma e^{-k\sigma} \frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\sigma} \\ &\quad - 2\kappa\sigma^2 \frac{\left(e^{\kappa\sigma}-1\right)}{\kappa\sigma} \frac{\left(1-e^{-2\kappa\sigma}\right)}{-2\kappa\sigma} - 4\kappa\sigma^2 \left(\frac{e^{-2\kappa\sigma} - 1 + 2\kappa\sigma}{(-2\kappa\sigma)^2} \right) \ . \end{aligned}$$

Consequently, we also have the bound

$$|f_{1,1,1}(k,\sigma)-2\sigma|\le \text{const.}\min\{(1+|\Lambda_-|)\,,(|\Lambda_-|+|\Lambda_-|\sigma)\}\sigma e^{|\Lambda_-|\sigma}\ .$$

Then, Proposition 20 and Proposition 21 (see Appendix A) yield

$$\left|\hat{u}_{1,\kappa,1}(\mathbf{k},t)-\hat{H}_{1,\kappa,1}(\mathbf{k},t)\right|\le \|Q\|\left(\frac{1}{t^{1-\varepsilon}}\bar{\mu}_\alpha(\Lambda_*^-,t)+\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)\right)\ ,$$

where ε can be taken arbitrarily small. Therefore,

$$\begin{aligned}
&\lim_{t\to\infty} t^{3-\varepsilon}\int_{\mathbb{R}^2}\left|\hat{u}_{1,\kappa,1}(\mathbf{k},t)-\hat{H}_{1,\kappa,1}(\mathbf{k},t)\right|d^2\mathbf{k}\\
&\le \lim_{t\to\infty}\|Q\|\frac{t^{3-\varepsilon}}{t^{1-\varepsilon}}\int_{\mathbb{R}^2}\frac{1}{1+\left(\frac{1}{2}\sqrt{2\sqrt{k_1^2+k_2^4}+2k_2^2}\,t\right)^\alpha}d^2\mathbf{k}\\
&\quad+\lim_{t\to\infty}\|Q\|t^{2-\varepsilon}\int_{\mathbb{R}^2}\frac{1}{1+(kt)^\alpha}d^2\mathbf{k}\\
&=\lim_{t\to\infty}\|Q\|\frac{1}{t}\int_{\mathbb{R}^2}\frac{1}{1+\left(\frac{1}{2}\sqrt{2\sqrt{k_1^2+k_2^4}+2k_2^2}\right)^\alpha}d^2\mathbf{k}\\
&\quad+\lim_{t\to\infty}\|Q\|\frac{1}{t^\varepsilon}\int_{\mathbb{R}^2}\frac{1}{1+k^\alpha}d^2\mathbf{k}=0\ .
\end{aligned}\tag{16}$$

For the second term in (15) we have

$$\lim_{t\to\infty} t^{3-\varepsilon}\int_{\mathbb{R}^2}\left|\hat{H}_{1,\kappa,1}(\mathbf{k},t)\right|d^2\mathbf{k}=0\ .\tag{17}$$

In fact,

$$\begin{aligned}
&\lim_{t\to\infty} t^{3-\varepsilon}\int_{\mathbb{R}^2}\left|\hat{H}_{1,\kappa,1}(\mathbf{k},t)\right|d^2\mathbf{k}\\
&=\lim_{t\to\infty} t^{3-\varepsilon}\int_{\mathbb{R}^2}e^{-\kappa(t-1)}\int_1^t(s-1)\hat{Q}_1(\mathbf{k},s)ds\,d^2\mathbf{k}\\
&\le \lim_{t\to\infty} t^{3-\varepsilon}\int_{\mathbb{R}^2}e^{-\kappa(t-1)}d^2\mathbf{k}=\lim_{t\to\infty}\frac{t^{3-\varepsilon}}{t^3}\int_{\mathbb{R}^2}e^{-\sqrt{\frac{k_1^2}{t^4}+\frac{k_2^2}{t^2}-i\frac{k_1}{t^2}}(t-1)}d^2\mathbf{k}\\
&\le \lim_{t\to\infty}\frac{1}{t^\varepsilon}\int_{\mathbb{R}^2}e^{\frac{1}{2}\Lambda_*^-}d^2\mathbf{k}=0\ ,
\end{aligned}$$

where we have used the integrability of $e^{\frac{1}{2}\Lambda_*^-}$ in $\mathbb{R}^2$. The combination of (16) and (17) yields

$$\lim_{t\to\infty} t^{3-\varepsilon}\int_{\mathbb{R}^2}|\hat{u}_{1,\kappa,1}(\mathbf{k},t)|\,d^2\mathbf{k}=0\ .$$

We now consider $\hat{u}_{1,k,1}(\mathbf{k},t)$. Let

$$\hat{u}_{1,k,1}^{as}(\mathbf{k},t)=-e^{-kt}\frac{iC_1k_1\sqrt{-ik_1}}{k}\ ,$$

and

$$\hat{G}_{1,k,1}(\mathbf{k},t)=-e^{-k(t-1)}\int_1^t\frac{ik_1(\kappa+k)}{k}(s-1)\hat{Q}_1(\mathbf{k},s)ds\ .$$

By the triangle inequality we obtain that

$$\left|\hat{u}_{1,k,1}(\mathbf{k},t)-\hat{u}^{as}_{1,k,1}(\mathbf{k},t)\right| \leq \left|\hat{G}_{1,k,1}(\mathbf{k},t)-\hat{u}^{as}_{1,k,1}(\mathbf{k},t)\right| + \left|\hat{u}_{1,k,1}(\mathbf{k},t)-\hat{G}_{1,k,1}(\mathbf{k},t)\right| \ . \tag{18}$$

We bound each term in (18) separately. We prove

Proposition 8 *For $\mathbf{k}\in\mathbb{R}^2$ we have*

$$\lim_{t\to\infty} t^{\frac{1}{2}}\hat{G}^t_{1,k,1}(\mathbf{k},t) = -\frac{iC_1k_1\sqrt{-ik_1}}{k}e^{-k} \ , \tag{19}$$

where

$$\hat{G}^t_{1,k,1}(\mathbf{k},t) = \hat{G}_{1,k,1}(\frac{k_1}{t},\frac{k_2}{t},t) \ .$$

Proof. Let $\hat{Q}^t_1(\mathbf{k},t) = \hat{Q}_1(\frac{k_1}{t},\frac{k_2}{t},t)$. It is trivial that for $\mathbf{k}\in\mathbb{R}^2$

$$-\lim_{t\to\infty} t^{\frac{1}{2}}e^{-\frac{k}{t}(t-1)}\frac{ik_1\left(\sqrt{\frac{k^2}{t^2}-\frac{ik_1}{t}}+\frac{k}{t}\right)}{k} = -\frac{ik_1\sqrt{-ik_1}}{k}e^{-k} \ .$$

We therefore only need to show that

$$\lim_{t\to\infty}\int_1^t (s-1)\hat{Q}_1(\frac{k_1}{t},\frac{k_2}{t},s)ds = C_1 \ ,$$

i.e., that

$$\lim_{t\to\infty}\left[\int_1^\infty (s-1)\left(\hat{Q}^t_1-\hat{Q}_1(0,s)\right)ds - \int_t^\infty (s-1)\hat{Q}^t_1 ds\right] = 0 \ .$$

On one hand, using that $\hat{Q}_1\in\mathcal{B}_{\alpha,3}$ we obtain

$$\begin{aligned}\left|\int_t^\infty (s-1)\hat{Q}_1(\frac{k_1}{t},\frac{k_2}{t},s)ds\right| &\leq \int_t^\infty (s-1)\left|\hat{Q}_1(\frac{k_1}{t},\frac{k_2}{t},s)\right|ds \\ &\leq \|Q\|\int_t^\infty \frac{1}{s^2}ds \leq \|Q\|\frac{1}{t} \ ,\end{aligned}$$

from which we get that

$$\lim_{t\to\infty}\int_t^\infty (s-1)\hat{Q}_1(\frac{k_1}{t},\frac{k_2}{t},s)ds = 0 \ .$$

On the other hand, it follows from the continuity of the nonlinear term $\mathbf{Q}$, that

$$\lim_{t\to\infty}(s-1)\hat{Q}_1(\frac{k_1}{t},\frac{k_2}{t},s) = (s-1)\hat{Q}_1(0,s) \ .$$

Furthermore, we have

$$\begin{aligned}(s-1)\left|\hat{Q}_1(\frac{k_1}{t},\frac{k_2}{t},s)\right| &\leq (s-1)\|Q\|\frac{1}{s^3}\bar{\mu}(\sqrt{\frac{k_1^2}{t^2}+\frac{k_2^2}{t^2}},s) \\ &\leq \|Q\|\frac{1}{s^2} \ ,\end{aligned}$$

which shows that for $s \in [1,\infty)$, $(s-1)\hat{Q}_1^t$ can be controlled by the L^1 function $\|Q\| \frac{1}{s^2}$. Therefore, we obtain by the Lebesgue dominated convergence theorem

$$\lim_{t\to\infty} \int_1^\infty \left|(s-1)\hat{Q}_1^t(k_1,k_2,s)ds - (s-1)\hat{Q}_1(0,s)\right| ds = 0 \ .$$

This completes the proof of our proposition. ■

Next we have that for all $t > 2$

$$\begin{aligned}
\left|t^{\frac{1}{2}} G_{1,k,1}^t\right| &= \left| e^{-\frac{k}{t}(t-1)} \frac{ik_1\left(\sqrt{\frac{k^2}{t} - ik_1} + \frac{k}{\sqrt{t}}\right)}{k} \int_1^t (s-1)\,\hat{Q}_1^t ds \right| \\
&< \text{const.}(k^{\frac{1}{2}} + k)e^{-\frac{k}{2}} \int_1^\infty (s-1)\,\hat{Q}_1^t ds \\
&< \text{const.}(k^{\frac{1}{2}} + k)e^{-\frac{k}{2}} \ ,
\end{aligned} \tag{20}$$

where we used the fact that for $t > 1$,

$$\begin{aligned}
\mathrm{Re}\left(\sqrt{\frac{k^2}{t} - ik_1}\right) &= \frac{1}{2}\sqrt{2\sqrt{k_1^2 + \frac{k^4}{t^2}} + 2\frac{k^2}{t}} \\
&< \frac{1}{2}\sqrt{2\sqrt{k_1^2 + k^4} + 2k^2} \ .
\end{aligned}$$

Note that $(k^{\frac{1}{2}} + k)e^{-\frac{k}{2}}$ is integrable over $\mathbb{R}^2$. Therefore, it follow from (19) and (20) by the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} \left|\hat{G}_{1,k,1} - \hat{u}_{1,k,1}^{as}\right| d^2\mathbf{k} = \lim_{t\to\infty} \int_{\mathbb{R}^2} \left| t^{\frac{1}{2}}\hat{G}_{1,k,1}^t - t^{\frac{1}{2}}\hat{u}_{1,k,1}^{as}(\frac{k_1}{t}, \frac{k_2}{t}, t)\right| d^2\mathbf{k} = 0 \ .$$

Next we estimate the second term on the right-hand side of (18). It is sufficient to analyze

$$\hat{u}_{1,k,1}(\mathbf{k},t) - \hat{G}_{1,k,1}(\mathbf{k},t) = \frac{1}{2}e^{-k(t-1)} \int_1^t \left[g_{1,1,1}(\mathbf{k}, s-1) + \frac{2ik_1(\kappa + k)}{k}(s-1)\right] \hat{Q}_1(\mathbf{k},s)ds \ .$$

We consider the integral kernel (see [10]),

$$\begin{aligned}
&g_{1,1,1}(\mathbf{k},\sigma) + \frac{2ik_1(\kappa+k)}{k}\sigma \\
&= -i(\kappa+k)k_1\sigma^2 \frac{\left(e^{-k\sigma} - 1\right)}{-k\sigma} \frac{\left(1 - e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma} + \frac{k_1^2\sigma^2}{k} \frac{e^{(k-\kappa)\sigma} - 1 - (k-\kappa)\,\sigma}{(k-\kappa)^2\,\sigma^2} \\
&\quad - ik_1\,(k+\kappa)\,\sigma^2 \frac{\left(e^{k\sigma} - 1\right)}{k\sigma} \frac{\left(e^{-(k+\kappa)\sigma} - 1\right)}{-(k+\kappa)\,\sigma} + \frac{i\,(k+\kappa)^2\,\sigma^2 k_1}{k} \frac{e^{-(k+\kappa)\sigma} - 1 + (k+\kappa)\,\sigma}{(k+\kappa)^2\,\sigma^2} \ ,
\end{aligned}$$

and we get the bound

$$\left| g_{1,1,1}(\mathbf{k},\sigma) + \frac{2ik_1(\kappa+k)}{k}\sigma \right| \le \text{const.}\min\{(k^{\frac{1}{2}} + k), (k + k^2)\sigma\}\sigma e^{k\sigma} \ . \tag{21}$$

Using Proposition 22 and Proposition 23 (see Appendix A), we conclude that

$$\left|\hat{u}_{1,k,1}(\mathbf{k},t) - \hat{G}_{1,k,1}(\mathbf{k},t)\right| \le \|Q\| \frac{1}{t^{1-\varepsilon}} \bar{\mu}_\alpha(\mathbf{k},t) \ ,$$

for all $\varepsilon > 0$. Therefore we have, for ε sufficiently small,

$$\lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} \left| \hat{u}_{1,k,1}(\mathbf{k},t) - \hat{G}_{1,k,1}(\mathbf{k},t) \right| d^2\mathbf{k}$$
$$\leq \lim_{t\to\infty} \frac{t^{\frac{1}{2}}}{t^{1-\varepsilon}} \|Q\| \int_{\mathbb{R}^2} \frac{1}{1+k^{\alpha}} d^2\mathbf{k} = 0 \ .$$

We have therefore proved the following proposition for $\hat{u}_{1,1,1}(\mathbf{k},t)$:

Proposition 9 *Let $\hat{u}_{1,1,1}(\mathbf{k},t)$ be defined as in (14), then*

$$\lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} \left| \hat{u}_{1,1,1}(\mathbf{k},t) - \hat{u}^{as}_{1,k,1}(\mathbf{k},t) \right| d^2\mathbf{k} = 0 \ ,$$

where

$$\hat{u}^{as}_{1,k,1}(\mathbf{k},t) = -C_1 \frac{ik_1\sqrt{-ik_1}}{k} e^{-kt} \ .$$

(ii) *Discussion of* $\hat{u}_{1,1,2}$

For $\hat{u}_{1,1,2}(\mathbf{k},t)$ we have the expression

$$\hat{u}_{1,1,2}(\mathbf{k},t) = \hat{u}_{1,\kappa,2} + \hat{u}_{1,k,2} \ , \tag{22}$$

where

$$\hat{u}_{1,\kappa,2}(\mathbf{k},t) = \frac{1}{2} e^{-\kappa(t-1)} \int_1^t f_{1,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds \ ,$$
$$\hat{u}_{1,k,2}(\mathbf{k},t) = \frac{1}{2} e^{-k(t-1)} \int_1^t g_{1,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds \ ,$$

with

$$f_{1,1,2}(\mathbf{k},\sigma) = \frac{ik_2}{\kappa}\left(e^{\kappa\sigma} - e^{-\kappa\sigma}\right) + \frac{2\kappa k_2(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right) \ ,$$
$$g_{1,1,2}(\mathbf{k},\sigma) = -\frac{ik_2}{k} e^{k\sigma} + \frac{k_2(\kappa+k)^2}{k_1 k} e^{-k\sigma} - \frac{2\kappa k_2(\kappa+k)}{kk_1} e^{-\kappa\sigma} \ .$$

In [10] we have shown that

$$\hat{u}_{1,\kappa,2}(\mathbf{k},t) \in \mathcal{B}_{\alpha,1} \text{ and } \hat{u}_{1,k,2}(\mathbf{k},t) \in \mathcal{B}_{\alpha,\frac{1}{2}} \ .$$

It is therefore sufficient to analyze $\hat{u}_{1,k,2}(\mathbf{k},t)$. Let

$$\hat{u}^{as}_{1,k,2}(\mathbf{k},t) = -C_2 \frac{ik_2\sqrt{-ik_1}}{k} e^{-kt} \ ,$$

and

$$\hat{B}_{1,k,2}(\mathbf{k},t) = -\frac{1}{2} e^{-k(t-1)} \int_1^t \frac{2i(\kappa+k)k_2}{k}(s-1)\hat{Q}_2(\mathbf{k},s)ds \ .$$

By the triangle inequality we have

$$\left|\hat{u}_{1,k,2}(\mathbf{k},t) - \hat{u}^{as}_{1,k,2}(\mathbf{k},t)\right| \leq \left|\hat{B}_{1,k,2}(\mathbf{k},t) - \hat{u}^{as}_{1,k,2}(\mathbf{k},t)\right| + \left|\hat{u}_{1,k,2}(\mathbf{k},t) - \hat{B}_{1,k,2}(\mathbf{k},t)\right| \ . \tag{23}$$

We bound each term in (23) separately. For the first term we note that

$$\lim_{t\to\infty} t^{\frac{1}{2}} \hat{B}^t_{1,k,2} = -C_2 \frac{ik_2\sqrt{-ik_1}}{k} e^{-k} \ ,$$

where

$$\hat{B}^t_{1,k,2}(\mathbf{k},t) = \hat{B}_{1,k,2}(\frac{k_1}{t},\frac{k_2}{t},t) \ .$$

Next we have for all $t > 2$

$$\begin{aligned}\left| t^{\frac{1}{2}} \hat{B}^t_{1,k,2} \right| &= \left| e^{-\frac{k}{t}(t-1)} \frac{k_2 \left(\sqrt{\frac{k^2}{t} - ik_1} + \frac{k}{\sqrt{t}} \right)}{k} \int_1^t (s-1)\, \hat{Q}^t_2 ds \right| \\ &< \text{const.}(k^{\frac{1}{2}} + k) e^{-\frac{k}{2}} \ ,\end{aligned}$$

where

$$\hat{Q}^t_2(\mathbf{k},s) = \hat{Q}_2(\frac{k_1}{t},\frac{k_2}{t},s) \ ,$$

which shows that for $t \in (2,\infty)$, $t^{\frac{1}{2}} \hat{B}^t_{1,k,2}$ is controlled by an L^1 function independent of t. Therefore, we get by the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} t^{\frac{5}{2}} \int \left| \hat{B}_{1,k,2}(\mathbf{k},t) - \hat{u}^{as}_{1,k,2}(\mathbf{k},t) \right| d^2\mathbf{k} = \lim_{t\to\infty} \int \left| t^{\frac{1}{2}} \hat{B}^t_{1,k,2} - t^{\frac{1}{2}} \hat{u}^{as}_{1,k,2}(\frac{k_1}{t},\frac{k_2}{t},t) \right| d^2\mathbf{k} = 0 \ .$$

For the second term, we follow the same method as for $\hat{u}_{1,k,1}$ in order to analyze

$$\hat{u}_{1,k,2}(\mathbf{k},t) - \hat{B}_{1,k,2}(\mathbf{k},t) = \frac{1}{2} e^{-k(t-1)} \int_1^t \left[g_{1,1,2}(\mathbf{k},s-1) + \frac{2i(\kappa+k)k_2}{k}(s-1) \right] \hat{Q}_2(\mathbf{k},s) ds \ .$$

For the integral kernel we have (see [10]),

$$\begin{aligned} g_{1,1,2}(\mathbf{k},\sigma) + \frac{2i(\kappa+k)k_2}{k}\sigma =& \frac{ik_2(\kappa+k)\sigma}{k} e^{-k\sigma} \frac{\left(1 - e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\sigma} + \frac{i(\kappa+k)k_2}{k}\sigma \\ &- \frac{ik_2\,(k+\kappa)}{k} \sigma e^{k\sigma} \frac{\left(e^{-(k+\kappa)\sigma} - 1\right)}{-(k+\kappa)\sigma} + \frac{i(\kappa+k)k_2}{k}\sigma \ . \end{aligned}$$

Proceeding as for the proof of (21) we find that a bound for this kernel is

$$\left| g_{1,1,2}(\mathbf{k},\sigma) + \frac{2i(\kappa+k)k_2}{k}\sigma \right| \leq \text{const.} \min\{(k^{\frac{1}{2}} + k), (k+k^2)\sigma\} \sigma e^{k\sigma} \ .$$

We conclude using Proposition 22 and Proposition 23 (see Appendix A) that

$$\left| \hat{u}_{1,k,2} - \hat{B}_{1,k,2} \right| \leq ||Q|| \frac{1}{t^{1-\varepsilon}} \bar{\mu}_\alpha(\mathbf{k},t) \ ,$$

for all $\varepsilon > 0$ sufficiently small. Finally we have

$$\lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} \left| \hat{u}_{1,k,1}(\mathbf{k},t) - \hat{B}_{1,k,2}(\mathbf{k},t) \right| d^2\mathbf{k} \leq \lim_{t\to\infty} \frac{t^{\frac{1}{2}}}{t^{1-\varepsilon}} \|Q\| \int_{\mathbb{R}^2} \frac{1}{1+k^\alpha} d^2\mathbf{k} = 0 \ .$$

Therefore the leading asymptotic behavior of $\hat{u}_{1,k,2}$ is given by $\hat{u}^{as}_{1,k,2}$ in the following sense:

Proposition 10 *Let $\hat{u}_{1,k,2}(\mathbf{k},t)$ be defined as in (22), then we have that*

$$\lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} \left|\hat{u}_{1,k,2}(\mathbf{k},t) - \hat{u}^{as}_{1,k,2}(\mathbf{k},t)\right| d^2\mathbf{k} = 0 \ ,$$

where

$$\hat{u}^{as}_{1,k,2}(\mathbf{k},t) = -C_2 \frac{ik_2\sqrt{-ik_1}}{k} e^{-kt} \ .$$

It is trivial that

$$\hat{u}_1(\mathbf{k},t) = \hat{u}_{1,1,1}(\mathbf{k},t) + \hat{u}_{1,k,2}(\mathbf{k},t) + \tilde{u}_1(\mathbf{k},t) \ ,$$

with

$$\hat{u}_1(\mathbf{k},t) - (\hat{u}_{1,1,1} + \hat{u}_{1,k,2})(\mathbf{k},t) := \tilde{u}_1(\mathbf{k},t) \in \mathcal{B}_{\alpha,1} \ .$$

Therefore

$$\begin{aligned}
&\lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} |\hat{u}_1(\mathbf{k},t) - \hat{u}^{as}_1(\mathbf{k},t)|\, d^2\mathbf{k} \\
&\leq \lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} \left|\hat{u}_{1,1,1}(\mathbf{k},t) - \hat{u}^{as}_{1,k,1}(\mathbf{k},t)\right| d^2\mathbf{k} \\
&+ \lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} \left|\hat{u}_{1,k,2}(\mathbf{k},t) - \hat{u}^{as}_{1,k,2}(\mathbf{k},t)\right| d^2\mathbf{k} + \lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} |\tilde{u}_1(\mathbf{k},t)|\, d^2\mathbf{k} \ .
\end{aligned}$$

Consequently, Proposition 7 follows from Proposition 9, Proposition 10 and the fact that

$$\begin{aligned}
\lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} |\tilde{u}_1(\mathbf{k},t)|\, d^2\mathbf{k} &\leq \lim_{t\to\infty} t^{\frac{5}{2}} \int_{\mathbb{R}^2} \frac{1}{t} \frac{1}{1+(kt)^\alpha} d^2\mathbf{k} \\
&= \lim_{t\to\infty} \frac{1}{t^{\frac{1}{2}}} \int_{\mathbb{R}^2} \frac{1}{1+k^\alpha} d^2\mathbf{k} = 0 \ .
\end{aligned}$$

Finally, by definition of the inverse Fourier transform we have

$$u_1(x,y,z) - u^{as}_1(x,y,z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-ik_1x} e^{-ik_2y} \left[\hat{u}_1(\mathbf{k},z) - \hat{u}^{as}_1(\mathbf{k},z)\right] d^2\mathbf{k} \ . \tag{24}$$

Rearranging the terms, taking the supremum over x, y and then the limit as $z \to \infty$ of the absolute value of (24) we get

Proposition 11 *Let u_1 be the first component of velocity field of (1). Then we have*

$$\lim_{z\to\infty} z^{\frac{5}{2}} \sup_{(x,y)\in\mathbb{R}^2} |u_1(x,y,z) - u^{as}_1(x,y,z)| = 0 \ .$$

Proof. The proof follows immediately from (24) and Proposition 7. ■

This completes the proof of (5), and (6) is an immediate consequence of (5), since we have

$$z^2 |u_1(x,y,z)| = z^2 |u_1(x,y,z) - u^{as}_1(x,y,z)| + z^2 |u^{as}_1(x,y,z)| \ ,$$

and since, using the explicit expression of u^{as}_1 of Proposition 7

$$\lim_{z\to\infty} z^2 |u^{as}_1(x,y,z)| \leq \lim_{z\to\infty} \frac{z^2}{4\pi^2} \int_{\mathbb{R}^2} |\hat{u}^{as}_1(\mathbf{k},z)|\, d^2\mathbf{k} \leq \lim_{z\to\infty} \frac{\text{const.}}{z^{\frac{1}{2}}} = 0 \ . \tag{25}$$

The expression of $u^{as}_1(x,y,z)$ in direct space is given in Appendix B.1.

4 Asymptotic behavior of $\hat{u}_2$

In [10], we have shown that $\hat{u}_{2,1,2} \in \mathcal{B}_{\alpha,0}$ and that the remaining $\hat{u}_{2,i,j}$ are in $\mathcal{B}_{\alpha,\frac{1}{2}-\varepsilon}$, for all $\varepsilon > 0$ sufficiently small. Therefore $\hat{u}_{2,1,2}$ is the leading order term of $\hat{u}_2$ in Fourier space.

Proposition 12 *Let $\hat{u}_2(\mathbf{k},t)$ be as above, then we have*

$$\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} |\hat{u}_2(\mathbf{k},t) - \hat{u}_2^{as}(\mathbf{k},t)|\, d^2\mathbf{k} = 0 \ ,$$

where

$$\hat{u}_2^{as}(\mathbf{k},t) = C_2 e^{-\kappa t} + C_2 \frac{ik\,(k+\kappa)}{k_1}(e^{-\kappa t} - e^{-kt}) \ .$$

To prove this proposition, we extract the leading asymptotic terms from $\hat{u}_{2,1,2}$.
The representation of $\hat{u}_{2,1,2}(\mathbf{k},t)$ is

$$\hat{u}_{2,1,2}(\mathbf{k},t) = \hat{u}_{2,\kappa,2}(\mathbf{k},t) + \hat{u}_{2,H,2}(\mathbf{k},t) \ , \tag{26}$$

where

$$\hat{u}_{2,\kappa,2}(\mathbf{k},t) = \frac{1}{2} e^{-\kappa(t-1)} \int_1^t f_{2,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds \ ,$$
$$\hat{u}_{2,H,2}(\mathbf{k},t) = H_1(\mathbf{k},t-1) \int_1^t h_{2,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds \ ,$$

with (see [10])

$$f_{2,1,2}(\mathbf{k},\sigma) = \frac{k_1 + ik_2^2}{\kappa k_1}\left(e^{\kappa\sigma} - e^{-\kappa\sigma}\right) + \frac{ik_2^2}{kk_1}\left(e^{-k\sigma} - e^{k\sigma}\right) \ ,$$
$$h_{2,1,2}(\mathbf{k},\sigma) = \frac{ik_2^2}{k\,(\kappa+k)} e^{k\sigma} + \frac{2\kappa k_2^2}{kk_1} e^{-\kappa\sigma} - \frac{k_2^2(\kappa+k)}{kk_1} e^{-k\sigma} \ ,$$
$$H_1(\mathbf{k},t-1) = \frac{(k+\kappa)}{k_1}\left(\frac{1}{2}e^{-\kappa(t-1)} - \frac{1}{2}e^{-k(t-1)}\right) \ .$$

The results in [10] show that

$$\hat{u}_{2,\kappa,2}(\mathbf{k},t) \in \mathcal{B}_{\alpha,0}, \ \ \hat{u}_{2,H,2}(\mathbf{k},t) \in \mathcal{B}_{\alpha,0} \ .$$

We now analyze $\hat{u}_{2,\kappa,2}(\mathbf{k},t)$ and $\hat{u}_{2,H,2}(\mathbf{k},t)$ in detail. For $\hat{u}_{2,\kappa,2}(\mathbf{k},t)$ we have

Proposition 13 *Let $\hat{u}_{2,\kappa,2}(\mathbf{k},t)$ be defined as in (26). Then we have*

$$\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left|\hat{u}_{2,\kappa,2}(\mathbf{k},t) - \hat{u}_{2,\kappa,2}^{as}(\mathbf{k},t)\right| d^2\mathbf{k} = 0 \ ,$$

where

$$\hat{u}_{2,\kappa,2}^{as}(\mathbf{k},t) = C_2 e^{-\kappa t} \ .$$

Proof. By the triangle inequality we have

$$
\begin{aligned}
&\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left|\hat{u}_{2,\kappa,2}(\mathbf{k},t) - \hat{u}^{as}_{2,\kappa,2}(\mathbf{k},t)\right| d^2\mathbf{k} \\
&\leq \lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left|\frac{1}{2}e^{-\kappa(t-1)} \int_1^t f_{2,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds - \frac{1}{2}e^{-\kappa t}\int_1^t f_{2,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds\right| d^2\mathbf{k} \\
&\quad + \lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left|\frac{1}{2}e^{-\kappa t} \int_1^t f_{2,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds - \frac{1}{2}e^{-\kappa t}\int_1^t 2(s-1)\hat{Q}_2(\mathbf{k},s)ds\right| d^2\mathbf{k} \\
&\quad + \lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left|\frac{1}{2}e^{-\kappa t} \int_1^t 2(s-1)\hat{Q}_2(\mathbf{k},s)ds - \frac{1}{2}e^{-\kappa t}\int_1^\infty 2(s-1)\hat{Q}_2(0,s)ds\right| d^2\mathbf{k}\ . \qquad (27)
\end{aligned}
$$

We now show that each term in (27) is zero. For the first term, we use that

$$
\begin{aligned}
\left|e^{-\kappa(t-1)} - e^{-\kappa t}\right| = \left|\kappa \int_{t-1}^t e^{-\kappa s}ds\right| &\leq |\kappa| \int_{t-1}^t e^{\Lambda_- s}ds \\
&\leq \text{const.}\,|\Lambda_-|\, e^{\Lambda_-(t-1)}\ .
\end{aligned}
$$

and also that

$$
\left|e^{-\kappa(t-1)} - e^{-\kappa t}\right| \leq \text{const.}e^{\Lambda_-(t-1)}\ .
$$

Therefore

$$
\left|e^{-\kappa(t-1)} - e^{-\kappa t}\right| \leq \text{const.}\min\{1,|\Lambda_-|\}e^{\Lambda_-(t-1)}\ .
$$

Using the bound of $f_{2,1,2}(\mathbf{k},\sigma)$ (see [10]), we get

$$
|f_{2,1,2}(\mathbf{k},\sigma)| \leq \text{const.}\,(1+|\Lambda_-|\,\sigma)\,\sigma e^{|\Lambda_-|\sigma}\ .
$$

It is easy to see that

$$
\begin{aligned}
&\left|\frac{1}{2}e^{-\kappa(t-1)} \int_1^t f_{2,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds - \frac{1}{2}e^{-\kappa t}\int_1^t f_{2,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds\right| \\
&\leq \text{const.}e^{\Lambda_-(t-1)} \int_1^t \min\{1,|\Lambda_-|\}\,(1+|\Lambda_-|\,(s-1))\,(s-1)e^{|\Lambda_-|(s-1)}ds \\
&\leq \text{const.}\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)\ .
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left|\frac{1}{2}e^{-\kappa(t-1)} \int_1^t f_{2,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds - \frac{1}{2}e^{-\kappa t}\int_1^t f_{2,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds\right| d^2\mathbf{k} \\
&\leq \text{const.}\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \frac{1}{t}\frac{1}{1+(kt)^\alpha} d^2\mathbf{k} = \text{const.}\lim_{t\to\infty}\frac{1}{t}\int_{\mathbb{R}^2}\frac{1}{1+k^\alpha}d^2\mathbf{k} = 0\ .
\end{aligned}
$$

For the second term, it is sufficient to analyze the difference

$$
\begin{aligned}
f_{2,1,2}(\mathbf{k},\sigma) - 2\sigma = &-\frac{k_2^2}{(k+\kappa)}\int_{-\sigma}^\sigma s e^{\kappa s}\frac{1-e^{(k-\kappa)s}}{(k-\kappa)s}ds \\
&+ 2\sigma e^{\kappa\sigma}\frac{\left(e^{-2\kappa\sigma}-1\right)}{-2\kappa\sigma} - 2\sigma\ ,
\end{aligned}
$$

where $\sigma > 0$. This structure is very similar to the one of $f_{2,1,2}(\mathbf{k}, \sigma)$, but we need a better bound to estimate the second limit in (27). We have

$$
\begin{aligned}
f_{2,1,2}(\mathbf{k}, \sigma) - 2\sigma = & - \frac{k_2^2}{(k+\kappa)} \int_{-\sigma}^{\sigma} s e^{\kappa s} \frac{1 - e^{(k-\kappa)s}}{(k-\kappa)s} ds \\
& + 2\kappa\sigma^2 \frac{(e^{\kappa\sigma} - 1)}{\kappa\sigma} \frac{\left(e^{-2\kappa\sigma} - 1\right)}{-2\kappa\sigma} - 4\kappa\sigma^2 \left[\frac{e^{-2\kappa\sigma} - 1 + 2\kappa\sigma}{(-2\kappa\sigma)^2} \right],
\end{aligned}
$$

and therefore

$$
|f_{2,1,2}(\mathbf{k}, \sigma) - 2\sigma| \leq \text{const.} |\Lambda_-| \sigma^2 e^{|\Lambda_-|\sigma} .
$$

We conclude using Proposition 20 and Proposition 21 (see Appendix A) that

$$
e^{\Lambda_-(t-1)} \left| \int_1^t \left[f_{2,1,2}(\mathbf{k}, s-1) - 2(s-1) \right] \hat{Q}_2(\mathbf{k}, s) ds \right| \leq \|Q\| \left(\frac{1}{t^{1-\varepsilon}} \bar{\mu}_\alpha(\Lambda, t) + \frac{1}{t} \bar{\mu}_\alpha(\mathbf{k}, t) \right) ,
$$

for $\varepsilon > 0$ sufficiently small. Therefore,

$$
\begin{aligned}
& \lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left| \frac{1}{2} e^{-\kappa t} \int_1^t f_{2,1,2}(\mathbf{k}, s-1) \hat{Q}_2(\mathbf{k}, s) ds - \frac{1}{2} e^{-\kappa t} \int_1^t 2(s-1) \hat{Q}_2(\mathbf{k}, s) ds \right| d^2\mathbf{k} \\
\leq & \|Q\| \lim_{t\to\infty} \frac{1}{t^{2-\varepsilon}} \int_{\mathbb{R}^2} \frac{1}{1 + \left(\frac{1}{2} \sqrt{2\sqrt{k_1^2 + k_2^4} + 2k_2^2} \right)^\alpha} d^2\mathbf{k} \\
& + \|Q\| \lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{R}^2} \frac{1}{1 + k^\alpha} d^2\mathbf{k} = 0 .
\end{aligned}
$$

We now discuss the third term in (27). We show that $\int_1^\infty 2(s-1)\hat{Q}_2(0, s)ds$ is the pointwise limit of $\int_1^t 2(s-1)\hat{Q}_2^t ds$ as t goes to infinity, with $\hat{Q}_2^t(\mathbf{k}, s) = \hat{Q}_2(\frac{k_1}{t}, \frac{k_2}{t}, s)$, by the same method as for Proposition 8. Moreover,

$$
\left| e^{-\Lambda} \int_1^t 2(s-1) \hat{Q}_2(\mathbf{k}, s) ds \right| \leq \text{const.} e^{-\Lambda} \int_1^\infty \frac{1}{s^2} ds \leq \text{const.} e^{-\Lambda} .
$$

Therefore, it follows from the Lebesgue dominated convergence theorem that

$$
\begin{aligned}
& \lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left| \frac{1}{2} e^{-\kappa t} \int_1^t 2(s-1) \hat{Q}_2(\mathbf{k}, s) ds - \frac{1}{2} e^{-\kappa t} \int_1^\infty 2(s-1) \hat{Q}_2(0, s) ds \right| d^2\mathbf{k} \\
\leq & \text{const.} \lim_{t\to\infty} \int_{\mathbb{R}^2} e^{-\Lambda} \left| \int_1^t 2(s-1) \hat{Q}_2^t ds - \int_1^\infty 2(s-1) \hat{Q}_2(0, s) ds \right| d^2\mathbf{k} = 0 .
\end{aligned}
$$

This completes the proof. ■

We now discuss the second term $\hat{u}_{2,H,2}(\mathbf{k}, t)$ of $\hat{u}_{2,1,2}(\mathbf{k}, t)$. The technique used to extract the asymptotic term is slightly different form what we have done before because the kernel $H_1(\mathbf{k}, t)$ behaves differently from what we have seen up to now. We have:

Proposition 14 *Let $\hat{u}_{2,H,2}(\mathbf{k}, t)$ be defined as in (26). Then we have*

$$
\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left| \hat{u}_{2,H,2}(\mathbf{k}, t) - \hat{u}_{2,H,2}^{as}(\mathbf{k}, t) \right| d^2\mathbf{k} = 0 .
$$

where

$$\hat{u}^{as}_{2,H,2}(\mathbf{k},t) = 2iC_2 k\tilde{H}_1(\mathbf{k},t) \ ,$$

with

$$\tilde{H}_1(\mathbf{k},t) = \frac{(k+\kappa)}{k_1}(\frac{1}{2}e^{-\kappa t} - \frac{1}{2}e^{-kt}) \ .$$

Proof. Using the triangle inequality we get

$$\begin{aligned}
&\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left|\hat{u}_{2,H,2}(\mathbf{k},t) - \hat{u}^{as}_{2,H,2}(\mathbf{k},t)\right| d^2\mathbf{k} \\
\leq &\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left| kH_1 \int_1^t \frac{h_{2,1,2}(\mathbf{k},s-1)}{k}\hat{Q}_2(\mathbf{k},s)ds - kH_1 \int_1^t 2i(s-1)\hat{Q}_2(\mathbf{k},s)ds \right| d^2\mathbf{k} \\
&+ \lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left| kH_1 \int_1^t 2i(s-1)\hat{Q}_2(\mathbf{k},s)ds - k\tilde{H}_1 \int_1^t 2i(s-1)\hat{Q}_2(\mathbf{k},s)ds \right| d^2\mathbf{k} \\
&+ \lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left| k\tilde{H}_1 \int_1^t 2i(s-1)\hat{Q}_2(\mathbf{k},s)ds - k\tilde{H}_1 \int_1^\infty 2i(s-1)\hat{Q}_2(0,s)ds \right| d^2\mathbf{k} \ . \qquad (28)
\end{aligned}$$

Consequently, it is sufficient to show that each term in (28) is zero. For the first term on the right-hand side of (28), we show that

$$\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} |kH_1| \left| \int_1^t \left[\frac{h_{2,1,2}(\mathbf{k},s-1)}{k} - 2i(s-1)\right] \hat{Q}_2(\mathbf{k},s)ds \right| d^2\mathbf{k} = 0 \ . \qquad (29)$$

An immediate bound is

$$\left|\frac{h_{2,1,2}(\mathbf{k},\sigma)}{k} - 2i\sigma\right| \leq \text{const.}\sigma e^{k\sigma} \ , \qquad (30)$$

but this bound is not sufficient to prove (29), and we need to bound the integral kernel in a second way. Since

$$\begin{aligned}
&\frac{h_{2,1,2}(\mathbf{k},\sigma)}{k} - \frac{2ik_2^2}{k^2}\sigma \\
=&\frac{ik_2^2\sigma}{k^2}\left[-k\sigma\frac{e^{k\sigma}-1}{k\sigma}\frac{1-e^{-(k+\kappa)\sigma}}{-(k+\kappa)\sigma} + \frac{e^{-(k+\kappa)\sigma}-1+(k+\kappa)\sigma}{-(k+\kappa)^2\sigma^2}(k+\kappa)\sigma \right. \\
&\left. -(-k\sigma)\frac{e^{-k\sigma}-1}{-k\sigma}\frac{1-e^{(k-\kappa)\sigma}}{(k-\kappa)\sigma} + \frac{e^{(k-\kappa)\sigma}-1-(k-\kappa)\sigma}{(k-\kappa)^2\sigma^2}(k-\kappa)\sigma \right] \ ,
\end{aligned}$$

and using (30) we get the bound

$$\left|\frac{h_{2,1,2}(\mathbf{k},\sigma)}{k} - \frac{2ik_2^2}{k^2}\sigma\right| \leq \text{const.}\min\{1, (k+k^{\frac{1}{2}})\sigma\}\sigma e^{k\sigma} \ .$$

Therefore we get using Proposition 22 and Proposition 23 (see Appendix A), that for all $\varepsilon > 0$,

$$\begin{aligned}
&t^2\,|kH_1| \left| \left[\int_1^t \frac{h_{2,1,2}(\mathbf{k},s-1)}{k} - \frac{2ik_2^2}{k^2}(s-1)\right] \hat{Q}_2(\mathbf{k},s)ds \right| \\
&\leq \frac{t^2}{t^{\frac{1}{2}-\varepsilon}}\frac{1}{1+(kt)^\alpha} \ ,
\end{aligned}$$

and

$$t^2\frac{k\,(k+\kappa)}{k_1}(e^{-\kappa(t-1)}-e^{-k(t-1)})\left|\left[\int_1^t \frac{ik_1^2}{k^2}(s-1)\right]\hat{Q}_2(\mathbf{k},s)ds\right|$$
$$\leq \text{const.}t^2e^{-k(t-1)})\left|\left[\int_1^t \left(k^{\frac{1}{2}}+k\right)(s-1)e^{k(s-1)}\right]\hat{Q}_2(\mathbf{k},s)ds\right|$$
$$\leq \text{const.}\frac{t^2}{t^{\frac{1}{2}}}\frac{1}{1+(kt)^{\alpha}}\ .$$

Therefore

$$\lim_{t\to\infty}\int_{\mathbb{R}^2}\frac{t^2}{t^{\frac{1}{2}-\varepsilon}}\frac{1}{1+(kt)^{\alpha}}d^2\mathbf{k}+\lim_{t\to\infty}\int_{\mathbb{R}^2}\frac{t^2}{t^{\frac{1}{2}}}\frac{1}{1+(kt)^{\alpha}}d^2\mathbf{k}$$
$$=\lim_{t\to\infty}\left(\frac{1}{t^{\frac{1}{2}-\varepsilon}}+\frac{1}{t^{\frac{1}{2}}}\right)\int_{\mathbb{R}^2}\frac{1}{1+k^{\alpha}}d^2\mathbf{k}=0\ ,$$

which implies (29). For the second term on the right-hand side of (28), we show that

$$\lim_{t\to\infty}t^2\int_{\mathbb{R}^2}\left|\left(kH_1-k\tilde{H}_1\right)\int_1^t 2i(s-1)\hat{Q}_2(\mathbf{k},s)ds\right|d^2\mathbf{k}=0\ .$$

Namely

$$\begin{aligned}
kH_1-k\tilde{H}_1 &= \frac{k(k+\kappa)}{2k_1}\left[e^{-\kappa(t-1)}-e^{-\kappa t}-e^{-k(t-1)}+e^{-kt}\right]\\
&= \frac{k(k+\kappa)}{2k_1}\left[\int_{t-1}^t\left(\kappa e^{-\kappa s}-ke^{-ks}\right)ds\right]\\
&= \frac{k(k+\kappa)}{2k_1}\left[\int_{t-1}^t\left((\kappa-k)\,e^{-\kappa s}+k\left(e^{-\kappa s}-e^{-ks}\right)\right)ds\right]\\
&= \frac{k(k+\kappa)}{2k_1}\int_{t-1}^t(\kappa-k)\,e^{-\kappa s}+\frac{k(k+\kappa)}{2k_1}\int_{t-1}^t k\left(e^{-\kappa s}-e^{-ks}\right)ds\\
&= \frac{-ik}{2}\int_{t-1}^t e^{-\kappa s}ds+\frac{ik^2}{2}\int_{t-1}^t se^{-ks}\frac{(e^{(k-\kappa)s}-1)}{(k-\kappa)\,s}ds\ ,
\end{aligned}$$

and therefore

$$\left|kH_1-k\tilde{H}_1\right|\leq \text{const.}\left(ke^{-k(t-1)}+k^2te^{-k(t-1)}\right).$$

On the other hand, we have

$$\begin{aligned}
\left|kH_1-k\tilde{H}_1\right| &\leq \text{const.}\left(k(t-1)e^{-k(t-1)}+kte^{-kt}\right)\\
&\leq \text{const.}kte^{-k(t-1)}\ .
\end{aligned}$$

We therefore have

$$\left|kH_1-k\tilde{H}_1\right|\leq \text{const.}\min\{t,1+kt\}ke^{-k(t-1)}\ .$$

Consequently we get using Proposition 22 and Proposition 23 (see Appendix A)

$$\left|\left(kH_1-k\tilde{H}_1\right)\int_1^t 2i(s-1)\hat{Q}_2(\mathbf{k},s)ds\right|$$
$$\leq \text{const.}\min\{t,1+kt\}e^{-k(t-1)}\int_1^t k(s-1)e^{k(s-1)}\hat{Q}_2(\mathbf{k},s)ds$$
$$\leq \text{const.}\frac{1}{t}\frac{1}{1+(kt)^{\alpha}}\ .$$

We therefore have

$$\begin{aligned}&\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left| \left(kH_1 - k\tilde{H}_1\right) \int_1^t 2i(s-1)\hat{Q}_2(\mathbf{k},s)ds \right| d^2\mathbf{k} \\ &\leq \text{const.} \lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{R}^2} \frac{1}{1+k^\alpha} d^2\mathbf{k} = 0 \ .\end{aligned}$$

We now prove that the third term on the right-hand side of (28) is zero. It is trivial that

$$\begin{aligned}&\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left| k\tilde{H}_1 \int_1^t 2i(s-1)\hat{Q}_2(\mathbf{k},s)ds - k\tilde{H}_1 \int_1^\infty 2i(s-1)\hat{Q}_2(0,s)ds \right| d^2\mathbf{k} \\ \leq&\text{const.} \lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left| kte^{-kt} \right| \left| \int_1^t 2i(s-1)\hat{Q}_2(\mathbf{k},s)ds - \int_1^\infty 2i(s-1)\hat{Q}_2(0,s)ds \right| d^2\mathbf{k} \\ \leq&\text{const.} \lim_{t\to\infty} \int_{\mathbb{R}^2} \left| ke^{-k} \int_1^t 2i(s-1)\hat{Q}_2^t ds - ke^{-k} \int_1^\infty 2i(s-1)\hat{Q}_2(0,s)ds \right| d^2\mathbf{k} \ ,\end{aligned}$$

where $\hat{Q}_2^t(\mathbf{k},s) = \hat{Q}_2(\frac{k_1}{t}, \frac{k_2}{t}, s)$. Consequently, we have

$$\lim_{t\to\infty} \int_{\mathbb{R}^2} ke^{-k} \left| \int_1^t 2i(s-1)\hat{Q}_2^t ds - ke^{-k} \int_1^\infty 2i(s-1)\hat{Q}_2(0,s)ds \right| d^2\mathbf{k} = 0 \ . \tag{31}$$

Next, it is easy to show that $ke^{-k}\int_1^\infty 2i(s-1)\hat{Q}_2(0,s)ds$ is the pointwise limit of $ke^{-k}\int_1^t 2i(s-1)\hat{Q}_2^t ds$, and furthermore that

$$ke^{-k} \left| \int_1^t 2i(s-1)\hat{Q}_2^t ds \right| \leq \text{const.} ke^{-k} \ ,$$

which shows that $ke^{-k}\int_1^t 2i(s-1)\hat{Q}_2^t ds$ can be controlled by an L^1 function independent of t. Therefore (31) follows immediately using the Lebesgue dominated convergence theorem. This completes the proof. ■

We can now obtain the leading asymptotic terms of $\hat{u}_2(\mathbf{k},t)$ in Fourier space. It is trivial that

$$\begin{aligned}\hat{u}_2 &= \hat{u}_{2,\kappa,2} + \hat{u}_{2,H,2} + \tilde{u}_2 \\ &= \hat{u}^{as}_{2,\kappa,2} + \hat{u}_{2,\kappa,2} - \hat{u}^{as}_{2,\kappa,2} + \hat{u}^{as}_{2,H,2} + \hat{u}_{2,H,2} - \hat{u}^{as}_{2,H,2} + \tilde{u}_2 \ ,\end{aligned} \tag{32}$$

with

$$\tilde{u}_2 = \hat{u}_2 - (\hat{u}_{2,\kappa,2} + \hat{u}_{2,H,2}) \in \mathcal{B}_{\alpha,\frac{1}{2}-\varepsilon} \ .$$

This decomposition, however, leads to a more profound understanding than just $\hat{u}_2 \in \mathcal{B}_{\alpha,0}$. Namely,

$$\hat{u}^{as}_2(\mathbf{k},t) = \hat{u}^{as}_{2,\kappa,2}(\mathbf{k},t) + \hat{u}^{as}_{2,H,2}(\mathbf{k},t) \ .$$

By the triangle inequality we have

$$|\hat{u}_2 - \hat{u}^{as}_2| \leq \left|\hat{u}_{2,\kappa,2} - \hat{u}^{as}_{2,\kappa,2}\right| + \left|\hat{u}_{2,H,2} - \hat{u}^{as}_{2,H,2}\right| + |\tilde{u}_2| \ .$$

Since we have already shown Proposition 13 and Proposition 14, it is sufficient to show that

$$\begin{aligned}&\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} |\tilde{u}_2| \, d^2\mathbf{k} \leq \text{const.} \lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \frac{1}{t^{\frac{1}{2}-\varepsilon}} \frac{1}{1+(kt)^\alpha} d^2\mathbf{k} \\ &\leq \text{const.} \lim_{t\to\infty} \frac{1}{t^{\frac{1}{2}-\varepsilon}} \int_{\mathbb{R}^2} \frac{1}{1+k^\alpha} d^2\mathbf{k} = 0 \ .\end{aligned}$$

We now complete the proof of Proposition 12. By definition of the inverse Fourier transform we have

$$u_2(x,y,z) - u_2^{as}(x,y,z) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}\left[\hat{u}_2(\mathbf{k},z) - \hat{u}_2^{as}(\mathbf{k},z)\right] d^2\mathbf{k}\ . \tag{33}$$

Rearranging the terms, taking the supremum over x, y and then the limit as $z \to \infty$ of the absolute value of (33) we get the main result of this section.

Proposition 15 *Let u_2 be the second component of velocity field of (1), then we have*

$$\lim_{z\to\infty} z^2 \sup_{(x,y)\in\mathbb{R}^2} |u_2(x,y,z) - u_2^{as}(x,y,z)| = 0\ ,$$

Proof. The proof follows immediately from (33) and Proposition 12. This completes the proof of (7). ■

The expression of $u_2^{as}(x,y,z)$ in direct space is given in the Appendix B.2.

5 Asymptotic behavior of $\hat{u}_3$

We know from [10] that $\hat{u}_{3,1,2} \in \mathcal{B}_{\alpha,0}$ is the leading order term of $\hat{u}_3(\mathbf{k},t)$ since the remaining $\hat{u}_{3,i,j}$ are in $\mathcal{B}_{\alpha,\frac{1}{2}-\varepsilon}$, for arbitrarily small $\varepsilon > 0$. Therefore the leading order asymptotic term of $\hat{u}_3(\mathbf{k},t)$ should stem from $\hat{u}_{3,1,2}(\mathbf{k},t)$.

The representation of $\hat{u}_{3,1,2}(\mathbf{k},t)$ is

$$\hat{u}_{3,1,2}(\mathbf{k},t) = \hat{u}_{3,\kappa,2}(\mathbf{k},t) + \hat{u}_{3,H,2}(\mathbf{k},t)\ , \tag{34}$$

where

$$\hat{u}_{3,\kappa,2}(\mathbf{k},t) = \frac{1}{2}e^{-\kappa(t-1)}\int_1^t f_{3,1,2}(\mathbf{k},s-1)\hat{Q}_2(\mathbf{k},s)ds\ ,$$
$$\hat{u}_{3,H,2}(\mathbf{k},t) = H_1(\mathbf{k},t-1)\int_1^t \frac{h_{3,1,2}(\mathbf{k},s-1)}{k}\hat{Q}_2(\mathbf{k},s)ds\ ,$$

with (see [10]),

$$f_{3,1,2}(\mathbf{k},\sigma) = \frac{k_2}{k_1}\left(e^{\kappa\sigma} - e^{k\sigma}\right) + \frac{k_2}{k_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right)\ ,$$
$$h_{3,1,2}(\mathbf{k},\sigma) = \frac{k_2}{\kappa+k}e^{k\sigma} - \frac{2i\kappa k_2}{k_1}e^{-\kappa\sigma} + \frac{ik_2(\kappa+k)}{k_1}e^{-k\sigma}\ .$$

In [10] we have shown that

$$\hat{u}_{3,\kappa,2}(\mathbf{k},t) \in \mathcal{B}_{\alpha,1-\varepsilon},\ \ \hat{u}_{3,H,2}(\mathbf{k},t) \in \mathcal{B}_{\alpha,0}\ .$$

As for the case of $\hat{u}_{2,H,2}(\mathbf{k},t)$ we have

Proposition 16 *Let $\hat{u}_{3,H,2}(\mathbf{k},t)$ be defined as in (34), then*

$$\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} \left|\hat{u}_{3,H,2}(\mathbf{k},t) - \hat{u}_{3,H,2}^{as}\right| d^2\mathbf{k} = 0\ ,$$

where

$$\hat{u}_{3,H,2}^{as} = 2C_2k_2\tilde{H}_1\ .$$

Proof. It is easy to prove Proposition 16 by following the steps in the proof of Proposition 14. We omit the details. ■

We therefore have

Proposition 17 *Let $\hat{u}_3(\mathbf{k},t)$ be the third component of the velocity field of (1) in Fourier space, then we have*

$$\lim_{t\to\infty} t^2 \int_{\mathbb{R}^2} |\hat{u}_3(\mathbf{k},t) - \hat{u}_3^{as}(\mathbf{k},t)|\, d^2\mathbf{k} = 0 \ ,$$

where

$$\hat{u}_3^{as}(\mathbf{k},t) = 2C_2 k_2 \tilde{H}_1 \ .$$

Proof. The proof is similar to proof of Proposition 12. ■

Finally, we get

Proposition 18 *Let u_3 be the third component of velocity field of (1), then we have*

$$\lim_{z\to\infty} z^2 \sup_{(x,y)\in\mathbb{R}^2} |u_3(x,y,z) - u_3^{as}(x,y,z)| = 0 \ .$$

A Main technical lemma

Proposition 19 *Let α', β', $\gamma' \geq 0$ with $\alpha' - \beta' + \gamma' \geq 0$, and let $\mu > 0$. Then, we have the bound*

$$\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{\mu\Lambda_-(t-1)} |\Lambda_-|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.}\ \frac{1}{t^{\beta'}} \frac{1}{1+\left(\left|\Lambda_*^-\right| t\right)^{\alpha'-\beta'+\gamma'}} \ , \tag{35}$$

uniformly in $\mathbf{k} \in \mathbb{R}^2$ and $t \geq 1$. Similarly, for positive α', β', $\gamma'$with $\alpha' - \beta' + \gamma' \geq 0$ and $\mu > 0$ we have the bound

$$\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{-\mu k(t-1)} k^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.}\ \frac{1}{t^{\beta'}} \frac{1}{1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}} \ , \tag{36}$$

uniformly in $\mathbf{k} \in \mathbb{R}^2$, $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$ and $t \geq 1$.

Proof. We first prove (35). For $1 \leq t \leq 2$ and $|\mathbf{k}| \leq 1$ we have that

$$\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{\mu\Lambda_-(t-1)} |\Lambda_-|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.} \leq \text{const.}\ \frac{1}{t^{\beta'}} \frac{1}{1+\left(\left|\Lambda_*^-\right| t\right)^{\alpha'-\beta'+\gamma'}} \ ,$$

Next, for $1 \leq t \leq 2$ and $|\mathbf{k}| > 1$ we have that

$$\begin{aligned}
&\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{\mu\Lambda_-(t-1)} |\Lambda_-|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.}\ \frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{\mu\Lambda_-(t-1)} \left(|\Lambda_-|\,(t-1)\right)^{\gamma'} \ \ |\Lambda_-|^{\beta'-\gamma'} \\
&\leq \text{const.}\ \frac{1}{1+|\mathbf{k}|^{\alpha'}}\ k^{\beta'-\gamma'} \leq \text{const.}\ \frac{1}{1+\left|\Lambda_*^-\right|^{\alpha'-\beta'+\gamma'}} \\
&\leq \text{const.}\ \frac{1}{t^{\beta'}} \frac{1}{1+\left(\left|\Lambda_*^-\right| t\right)^{\alpha'-\beta'+\gamma'}} \ .
\end{aligned}$$

as claimed, and for $t > 2$ and $\mathbf{k} \in \mathbb{R}^2$ we use that

$$\begin{aligned}
&\left(1 + \left(\left|\Lambda_*^-\right| t\right)^{\alpha'-\beta'+\gamma'}\right) e^{\mu\Lambda_-(t-1)} \left|\Lambda_- t\right|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.} \left(1 + \left(\left|\Lambda_*^-\right| t\right)^{\alpha'-\beta'+\gamma'}\right) e^{\frac{1}{2}\mu\Lambda_- t} \left|\Lambda_- t\right|^{\beta'} \\
&\leq \text{const.} \left(1 + \left(\left|\Lambda_*^-\right| t\right)^{\alpha'-\beta'+\gamma'} e^{\frac{1}{2}\mu\Lambda_- t} \left|\Lambda_- t\right|^{\beta'}\right) \\
&\leq \text{const.} \left(1 + \frac{\left|\Lambda_*^-\right|^{\alpha'-\beta'+\gamma'}}{\left|\Lambda_-\right|^{\alpha'-\beta'+\gamma'}} \left|\Lambda_- t\right|^{\alpha'-\beta'+\gamma'} \left|\Lambda_- t\right|^{\beta'} e^{\frac{1}{2}\mu\Lambda_- t}\right) \\
&\leq \text{const.} \left(1 + \frac{\left|\Lambda_*^-\right|^{\alpha'-\beta'+\gamma'}}{\left|\Lambda_-\right|^{\alpha'-\beta'+\gamma'}}\right) \leq \text{const.}\ ,
\end{aligned}$$

and (35) follows. We now prove (36). For $1 \leq t \leq 2$ and $|\mathbf{k}| \leq 1$ we have that

$$\frac{1}{1 + |\mathbf{k}|^{\alpha'}} e^{-\mu k(t-1)} k^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.} \leq \text{const.}\ \frac{1}{t^{\beta'}} \frac{1}{1 + (|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}}\ ,$$

and for $1 \leq t \leq 2$ and $|\mathbf{k}| > 1$ we have that

$$\begin{aligned}
&\frac{1}{1 + |\mathbf{k}|^{\alpha'}} e^{-\mu k(t-1)} k^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.}\ \frac{1}{1 + |\mathbf{k}|^{\alpha'}} e^{-\mu k(t-1)} \left(k\,(t-1)\right)^{\gamma'}\ k^{\beta'-\gamma'} \\
&\leq \text{const.}\ \frac{1}{1 + |\mathbf{k}|^{\alpha'}}\ k^{\beta'-\gamma'} \leq \text{const.}\ \frac{1}{1 + |\mathbf{k}|^{\alpha'-\beta'+\gamma'}} \\
&\leq \text{const.}\ \frac{1}{t^{\beta'}} \frac{1}{1 + (|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}}\ .
\end{aligned}$$

Finally, for $t > 2$ we use that

$$\begin{aligned}
&\left(1 + (|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}\right) e^{-\mu k(t-1)} (kt)^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.} \left(1 + (|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}\right) e^{-\frac{1}{2}\mu k t} (kt)^{\beta'} \leq \text{const.} \leq \text{const.} \left(1 + |\mathbf{k}|^{\alpha'}\right)\ ,
\end{aligned}$$

and (36) follows. ■

Proposition 20 *Let $\alpha \geq 0$, $r \geq 0$ and $\delta \geq 0$ and $\gamma + 1 \geq \beta \geq 0$. Then,*

$$\begin{aligned}
&e^{\Lambda_-(t-1)} \int_1^{\frac{t+1}{2}} e^{|\Lambda_-|(s-1)} |\Lambda_-|^{\beta} \frac{(s-1)^{\gamma}}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k}, s) ds \\
&\leq \begin{cases}
\text{const.} \dfrac{1}{t^{\beta}} \bar{\mu}_{\alpha}(\Lambda_*^-, t), & \text{if } \delta > \gamma + 1 \\[2ex]
\text{const.} \dfrac{\log(1+t)}{t^{\beta}} \bar{\mu}_{\alpha}(\Lambda_*^-, t), & \text{if } \delta = \gamma + 1 \\[2ex]
\text{const.} \dfrac{t^{\gamma+1-\delta}}{t^{\beta}} \bar{\mu}_{\alpha}(\Lambda_*^-, t), & \text{if } \delta < \gamma + 1
\end{cases}
\end{aligned} \tag{37}$$

uniformly in $t \geq 1$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. We have that

$$
\begin{aligned}
&e^{\Lambda_-(t-1)}\int_1^{\frac{t+1}{2}} e^{|\Lambda_-|(s-1)}|\Lambda_-|^{\beta}\frac{(s-1)^{\gamma}}{s^{\delta}}\mu_{\alpha,r}(\mathbf{k},s)ds \\
&\leq e^{\Lambda_-(t-1)}e^{|\Lambda_-|\frac{t-1}{2}}|\Lambda_-|^{\beta}\mu_{\alpha,r}(\mathbf{k},1)\int_1^{\frac{t+1}{2}}\frac{(s-1)^{\gamma}}{s^{\delta}}ds \\
&\leq \text{const.}\left(\frac{t-1}{t}\right)^{\gamma+1} e^{\Lambda_-\frac{t-1}{2}}|\Lambda_-|^{\beta}\mu_{\alpha,1}(\mathbf{k},1)\begin{cases} 1, \text{ if } \delta>\gamma+1 \\ \log(1+t), \text{ if } \delta=\gamma+1 \\ t^{\gamma+1-\delta}, \text{ if } \delta<\gamma+1\end{cases}
\end{aligned}
$$

The bounds in (37) now follow using Proposition 35. ∎

Proposition 21 *Let $\alpha\geq 0$, $r\geq 0$, $\delta\in\mathbb{R}$, and $\beta\in\{0,1\}$. Then,*

$$
e^{\Lambda_-(t-1)}\int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)}|\Lambda_-|^{\beta}\frac{1}{s^{\delta}}\mu_{\alpha,r}(\mathbf{k},s)ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}}\mu_{\alpha,r}(\mathbf{k},t)\ , \tag{38}
$$

uniformly in $t\geq 1$ and $\mathbf{k}\in\mathbb{R}^2$.

Proof. If $\beta=0$ we have that

$$
e^{\Lambda_-(t-1)}\int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)}\frac{1}{s^{\delta}}\mu_{\alpha,r}(\mathbf{k},s)ds \leq \frac{\text{const.}}{t^{\delta}}\mu_{\alpha,r}(\mathbf{k},t)\int_{\frac{t+1}{2}}^{t} ds\ ,
$$

and (38) follows, and if $\beta=1$ we have that

$$
\begin{aligned}
e^{\Lambda_-(t-1)}\int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)}|\Lambda_-|^{\beta}\frac{1}{s^{\delta}}\mu_{\alpha,r}(\mathbf{k},s)ds &\leq \frac{\text{const.}}{t^{\delta}}\mu_{\alpha,r}(\mathbf{k},t)e^{\Lambda_-(t-1)}\int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)}|\Lambda_-|ds \\
&\leq \frac{\text{const.}}{t^{\delta}}\mu_{\alpha,r}(\mathbf{k},t)\ ,
\end{aligned}
$$

and (38) follows. Using Hölder's inequality the proposition can also be proved for intermediate values of β. ∎

Proposition 22 *Let $\alpha\geq 0$, $r\geq 0$ and $\delta\geq 0$ and $\gamma+1\geq\beta\geq 0$. Then,*

$$
\begin{aligned}
&e^{-k(t-1)}\int_1^{\frac{t+1}{2}} e^{k(s-1)}k^{\beta}\frac{(s-1)^{\gamma}}{s^{\delta}}\mu_{\alpha,r}(\mathbf{k},s)ds \\
&\leq \begin{cases} \text{const.}\frac{1}{t^{\beta}}\bar{\mu}_{\alpha}(\mathbf{k},t),\ \textit{if } \delta>\gamma+1 \\ \text{const.}\frac{\log(1+t)}{t^{\beta}}\bar{\mu}_{\alpha}(\mathbf{k},t),\ \textit{if } \delta=\gamma+1 \\ \text{const.}\frac{t^{\gamma+1-\delta}}{t^{\beta}}\bar{\mu}_{\alpha}(\mathbf{k},t),\ \textit{if } \delta<\gamma+1 \end{cases}
\end{aligned}
$$

uniformly in $t\geq 1$ and $k=|\mathbf{k}|=\sqrt{k_1^2+k_2^2}$, $\mathbf{k}\in\mathbb{R}^2$.

Proof. The proof is identical to the one of Proposition 20. ■

Next we have:

Proposition 23 *Let $\alpha \geq 0$, $r \geq 0$, $\delta \in \mathbb{R}$, and $\beta \in \{0,1\}$. Then,*

$$e^{-k(t-1)} \int_{\frac{t+1}{2}}^{t} e^{k(s-1)} k^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k},s) ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}} \mu_{\alpha,r}(\mathbf{k},t) \ ,$$

uniformly in $t \geq 1$ and $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$, $\mathbf{k} \in \mathbb{R}^2$.

Proof. The proof is as for Proposition 21. ■

B Representation in direct space

B.1 Expression on $u_1^{as}(x,y,z)$

The definition of inverse Fourier transform gives

$$\begin{aligned} u_1^{as}(x,y,z) &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-ik_1 x} e^{-ik_2 y} \hat{u}_1^{as}(\mathbf{k},z) d^2\mathbf{k} \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-ik_1 x} e^{-ik_2 y} \left(\hat{u}_{1,k,1}^{as} + \hat{u}_{1,k,2}^{as} \right) (\mathbf{k},z) d^2\mathbf{k} \ . \end{aligned}$$

By changing variables we get that

$$\begin{aligned} &\frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-ik_1 x} e^{-ik_2 y} \hat{u}_{1,k,1}^{as}(\mathbf{k},z) d^2\mathbf{k} \\ =& \frac{C_1}{4\pi^2} \int_{\mathbb{R}^2} e^{-ik_1 x} e^{-ik_2 y} \frac{-ik_1 \sqrt{-ik_1}}{k} e^{-kz} d^2\mathbf{k} \\ =& \frac{C_1}{4\pi^2} \frac{1}{z^{\frac{3}{2}}} \int_{\mathbb{R}^2} e^{-i\frac{k_1}{z} x} e^{-i\frac{k_2}{z} y} \frac{-ik_1 \sqrt{-ik_1}}{k} e^{-k} d^2\mathbf{k} \\ =& \frac{C_1}{4\pi^2} \frac{1}{z^{\frac{3}{2}}} \partial_x \left(\int_{\mathbb{R}^2} e^{-i\frac{k_1}{z} x} e^{-i\frac{k_2}{z} y} \frac{\sqrt{-ik_1}}{k} e^{-k} d^2\mathbf{k} \right) \ , \end{aligned}$$

where we have used that

$$\partial_x \mathcal{F}^{-1}\left(\hat{f}\right)(\mathbf{x}) = \mathcal{F}^{-1}\left(-ik_1 \hat{f}\right)(\mathbf{x}) \ ,$$

whenever the expressions are well defined. Therefore it is sufficient to consider the integral

$$\int_{\mathbb{R}^2} e^{-ik_1 x} e^{-ik_2 y} \frac{\sqrt{-ik_1}}{k} e^{-k} d^2\mathbf{k} \ .$$

In polar coordinates we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} e^{-ik_1 x} e^{-ik_2 y} e^{-k} \frac{\sqrt{-ik_1}}{k} d^2\mathbf{k} \\ =& \int_0^{\infty} \int_0^{2\pi} e^{-k-ik(x\cos\phi + y\sin\phi)} \sqrt{-ik\cos\phi} \, dk d\phi \\ =& \int_0^{2\pi} \frac{\sqrt{-i\cos\phi}}{[1+i(x\cos\phi + y\sin\phi)]^{\frac{3}{2}}} d\phi \int_0^{\infty} e^{-k} \sqrt{k} dk \\ =& \frac{\sqrt{\pi}}{2} \int_0^{2\pi} \frac{\sqrt{-i\cos\phi}}{[1+i(x\cos\phi + y\sin\phi)]^{\frac{3}{2}}} d\phi \ . \end{aligned}$$

Therefore,

$$\begin{aligned}
u^{as}_{1,k,1}(x,y,z) &= \frac{\sqrt{\pi}C_1}{8\pi^2}\partial_x\left(\int_0^{2\pi}\frac{\sqrt{-i\cos\phi}}{[z+i(x\cos\phi+y\sin\phi)]^{\frac{3}{2}}}d\phi\right)\\
&= -\frac{3C_1}{16\pi^{\frac{3}{2}}}\int_0^{2\pi}\frac{\cos\phi\sqrt{i\cos\phi}}{[z+i(x\cos\phi+y\sin\phi)]^{\frac{5}{2}}}d\phi\ .
\end{aligned}$$

As for $u^{as}_{1,k,1}(x,y,z)$ we get

$$\begin{aligned}
u^{as}_{1,k,2}(x,y,z) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}\hat{u}^{as}_{1,k,2}(\mathbf{k},z)\ d^2\mathbf{k}\\
&= \frac{C_2}{4\pi^2}\frac{1}{z^{\frac{3}{2}}}\ \partial_y\left(\int_{\mathbb{R}^2} e^{-i\frac{k_1}{z}x}e^{-i\frac{k_2}{z}y}e^{-k}\frac{\sqrt{-ik_1}}{k}\ d^2\mathbf{k}\right)\\
&= -\frac{3C_2}{16\pi^{\frac{3}{2}}}\int_0^{2\pi}\frac{\sin\phi\sqrt{i\cos\phi}}{[z+i(x\cos\phi+y\sin\phi)]^{\frac{5}{2}}}d\phi\ .
\end{aligned}$$

B.2 Expression on $u^{as}_2(x,y,z)$

By definition

$$\begin{aligned}
u^{as}_2(x,y,z) =& \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}\hat{u}^{as}_2(\mathbf{k},z)d^2\mathbf{k}\\
=& \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}\left(\hat{u}^{as}_{2,\kappa,2}+\hat{u}^{as}_{2,H,2}\right)(\mathbf{k},z)d^2\mathbf{k}\ .
\end{aligned}\tag{39}$$

We now compute an explicit expression for the first term of the right-hand side of (39). We have

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}\hat{u}^{as}_{2,\kappa,2}(\mathbf{k},z)d^2\mathbf{k} = \frac{C_2}{4\pi^2}\int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}e^{-\kappa z}d^2\mathbf{k}\ .$$

It is not very easy to calculate this integral directly so that we have to change our strategy to obtain $\mathcal{F}^{-1}\left(e^{-\kappa z}\right)(x,y)$. Let

$$e^{-\kappa z} = \hat{f}(\mathbf{k},z)\ ,\tag{40}$$

then

$$\begin{aligned}
f(x,y,z) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}e^{-\kappa z}d^2\mathbf{k}\\
&= \frac{1}{4\pi^2}\frac{1}{z^3}\int_{\mathbb{R}^2} e^{-ik_1\frac{x}{z^2}}e^{-ik_2\frac{y}{z}}e^{-\sqrt{\frac{k_1^2}{z^4}+\frac{k_2^2}{z^2}-i\frac{k_1}{z^2}}z}d^2\mathbf{k}\ ,
\end{aligned}$$

and we have

$$\begin{aligned}
4\pi^2\lim_{z\to\infty} z^3 f(xz^2,yz,z) &= \lim_{z\to\infty}\int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}e^{-\sqrt{\frac{k_1^2}{z^4}+\frac{k_2^2}{z^2}-i\frac{k_1}{z^2}}z}d^2\mathbf{k}\\
&= \int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}e^{-\sqrt{k_2^2-ik_1}}d^2\mathbf{k}\ ,
\end{aligned}$$

where we use the Lebesgue dominated convergence theorem to exchange the limit and the integral. Taking derivative on the both sides of (40) with respect to z twice, we get

$$\left(\partial_z^2-\kappa^2\right)\hat{f}(\mathbf{k},z) = 0\ .$$

Then we have by inverse Fourier transform

$$(\Delta - \partial_x) f(x,y,z) = 0 \ . \tag{41}$$

In what follows, we set $f(x,y,z) = e^{\frac{1}{2}x} g(x,y,z)$. Then (41) becomes

$$\left(-\Delta + \frac{1}{4}\right) g(x,y,z) = 0 \ .$$

Therefore, we have the solution in half-space

$$g(x,y,z) = \frac{z\,(r+2)}{4\pi r^3} \exp\left(-\frac{r}{2}\right) \ ,$$

where $r = \sqrt{x^2+y^2+z^2}$. It follows that

$$f(x,y,z) = \frac{z\,(r+2)}{4\pi r^3} \exp\left(\frac{x-r}{2}\right) \ . \tag{42}$$

Therefore we obtain

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-ik_1 x} e^{-ik_2 y} \hat{u}^{as}_{2,\kappa,2}(\mathbf{k}, z) d^2\mathbf{k} = \frac{C_2 z\,(r+2)}{4\pi r^3} \exp\left(\frac{x-r}{2}\right) \ .$$

Acknowledgement

This work is partially supported by the SSSTC Project (No. EG19-032009) and the Swiss National Science Foundation, grant No. 200021-124403. Zhengguang Guo thanks for the support of overseas research grant from East China Normal University in 2009.

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