On the existence of solutions to the planar exterior Navier Stokes system

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Abstract

We consider the stationary incompressible Navier Stokes equation in the exterior of a disk $B \subset \mathbb{R}^2$ with non-zero Dirichlet boundary conditions on the disk and zero boundary conditions at infinity. We prove the existence of solutions for an open set of boundary conditions without symmetry.

Contents

1 Introduction

In this paper we consider the incompressible Navier Stokes equations in an exterior domain:

$$\left\{ \begin{array}{r} \Delta \mathbf{u} - \nabla p = \mathbf{u} \cdot \nabla \mathbf{u} \ , \\ \operatorname{div} \mathbf{u} = 0 \ , \end{array} \right. \quad \text{in } \mathbb{R}^2 \setminus \overline{B} \ , \tag{1}$$

with B a smooth bounded domain, with non-zero Dirichlet boundary conditions on ∂B, and zero boundary conditions at infinity:

$$\mathbf{u}_{|\partial B} = \mathbf{u}^* \ , \quad \lim_{|\mathbf{x}| \to \infty} \mathbf{u}(\mathbf{x}) = 0 \ . \tag{2}$$

Of particular interest is the case of boundary data $\mathbf{u}^*$ with zero flux:

$$\int_{\partial B} \mathbf{u}^* \cdot \mathbf{n} \, \mathrm{d}\sigma = 0 \ . \tag{3}$$

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We note that, since the size of B is arbitrary, we have set without restriction of generality all the physical constants in (1) equal to one.

The above system is a special case of the exterior Navier Stokes problem:

$$\left\{ \begin{array}{r} -\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} - \lambda\partial_1\mathbf{u} + \Delta\mathbf{u} - \nabla p = 0 \ , \\ \nabla\cdot\mathbf{u} = 0 \ , \end{array} \right. \quad \text{in } \mathbb{R}^n\setminus\overline{B} \ , \tag{4}$$

with $n = 2$ or 3, with B a smooth bounded domain in $\mathbb{R}^n$, with boundary conditions (2), and with $\lambda \in \{0, 1\}$ distinguishing between the case of a flow "around" B ($\lambda = 0$) and a flow "past" B ($\lambda = 1$), respectively. The system (1)-(3) corresponds to $n = 2$ and $\lambda = 0$. The case $\lambda = 0$ is in many respects more complicated than the case $\lambda = 1$, and, whereas the picture is rather complete for $n = 3$, the case $n = 2$, $\lambda = 0$, presents particular difficulties. The difficulty with the classical method for solving the Navier Stokes equations consists in the fact that the linearization around $\mathbf{u} = \mathbf{0}$ is given by the Stokes system, which, for $n = 2$, does not admit a solution satisfying (2), unless the domain B and the boundary data $\mathbf{u}^*$ satisfy certain symmetry conditions. This fact is known as the Stokes paradox. For completeness we note that if one relaxes the no flux condition (3), there exists a two parameter family of solutions to (1)-(3), the so called Hamel solutions, see [7]. These examples emphasize that the decay of solutions can be arbitrary slow and that uniqueness might be lost for some boundary data. However, these solutions have flux larger than one, and are far from the regime which we will consider here.

In what follows we construct a new class of solutions to (1)-(3), by linearizing not around $\mathbf{u} = \mathbf{0}$, but around $\mathbf{u} = \mu\mathbf{x}^{\perp}/\left|\mathbf{x}\right|^2$, with $|\mu| > \sqrt{48}$. This improves the decay of the solutions to the vorticity equation, yielding vorticities decaying at infinity generically faster than $|\mathbf{x}|^{-2}$, instead of like $|\mathbf{x}|^{-1}$ as would be the case for the Stokes equation, thus avoiding the Stokes paradox when reconstructing $\mathbf{u}$ *via* the Bio-Savart law.

To put our problem into a wider context, we briefly recall the concept of weak solutions for (4), (2) (also known as generalized solutions or D–solutions), and the method of J. Leray [16] for proving the existence of such weak solutions.

Definition 1 *Given $\mathbf{u}^* \in H^{1/2}(\partial B)$ satisfying (3), a function $\mathbf{u}$ which satisfies the following conditions is called a weak solution to (4), (2):*

1. *$\mathbf{u} \in D^{1,2}(\mathbb{R}^n\setminus\overline{B})$, where $D^{1,2}(\mathbb{R}^n\setminus\overline{B})$ is the subset of $L^1_{loc}(\mathbb{R}^2\setminus\overline{B})$ containing functions with gradient in $L^2(\mathbb{R}^n\setminus\overline{B})$,*
2. *$\mathbf{u}$ is divergence-free and $\mathbf{u} = \mathbf{u}^*$ on ∂B,*
3. *for all divergence-free vector fields $\mathbf{w} \in C_c^\infty(\mathbb{R}^n\setminus B)$, there holds:*

$$\int_{\mathbb{R}^n\setminus\overline{B}} \nabla\mathbf{u} \colon \nabla\mathbf{w} + \int_{\mathbb{R}^n\setminus\overline{B}} \left(\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} + \lambda\partial_1\mathbf{u}\right)\cdot\mathbf{w} = 0 \ .$$

The method of J. Leray to prove the existence of solutions according to this definition, and a posteriori to (4), (2), in the sense of distributions, consists in the following steps:

- First, one introduces a sequence of approximate problems by restricting (4) to bounded subsets $\Omega \subset \mathbb{R}^n$ containing B, with zero Dirichlet boundary conditions on $\partial\Omega\setminus\partial B$.
- Second, one proves the existence of (weak) solutions to all these approximate problems.
- Third, one shows that for any sequence of bounded subsets exhausting $\mathbb{R}^n\setminus\overline{B}$, there exists a subsequence, such that the corresponding approximate solutions converge to a weak solution of (4), (2).
- Finally, given a weak solution $\mathbf{u}$, a pressure p an be constructed *via* De Rham's theory, such that the equations (4) are satisfied in $\mathcal{D}'(\mathbb{R}^n\setminus\overline{B})$.

See also [12, 13, 20, 23, 24], where this method has been adapted to a similar system with more general boundary conditions. Note that if B has a smooth boundary, the ellipticity of the Stokes operator (see [5, Section IX.1]) and the smoothness of $\mathbf{u}^*$ imply that weak solutions are smooth. Therefore, for smooth data, the only possible shortcoming of weak solutions is that they may not satisfy the boundary condition at infinity in a point-wise sense. Much work has been devoted to clarify the situation in various cases (see [7] for more details):

For $n = 3$, the condition $\mathbf{u} \in D^{1,2}(\mathbb{R}^3 \setminus \overline{B})$ implies that weak solutions tend to zero at infinity. The exact decay can be obtained by various methods yielding the following results:

- for $\lambda = 1$, there exists a solution that decays like the fundamental solution of the Oseen equation (the linear system obtained from (4) by deleting the nonlinear convective terms) [2, 3, 4]. This result can be obtain by a detailed analysis of the Oseen equation with a source term in the usual Sobolev spaces [2, 4], and also in weighted Sobolev spaces [3].
- for $\lambda = 0$ and sufficiently small boundary data, there exists a unique weak solution, and this solution decays like a Landau solution [15], a special solution of the nonlinear system which decays like $1/|\mathbf{x}|$. This result can been obtained by constructing first a strong solution to (2), (4), which is asymptotic to the Landau solution, by perturbative techniques. Using the known decay of this particular solution as an input [7, Section IX.9], one then proves a weak-strong uniqueness result for small data.

For $n = 2$, the situation is more delicate since the condition $\mathbf{u} \in D^{1,2}(\mathbb{R}^2 \setminus \overline{B})$ does not guarantee that the boundary condition at infinity is satisfied:

- For $\lambda = 1$, the relevant linear system is again the Oseen equation, but the results concerning the decay are limited to small data, since, as for the case $n = 3$, $\lambda = 0$, perturbative techniques are used to prove the existence of a strong solution decaying at infinity like the fundamental solution of the Oseen equation. This solution is then again used as an input to a weak strong uniqueness argument in order to show the decay of weak solutions. These results can be found in [6].
- The case $\lambda = 0$ remains largely open. As we already pointed out, the problem is that the solution to the Stokes equation with boundary data $\mathbf{u}^* \neq 0$ diverges at infinity, unless one makes additional assumptions on the domain B and the data $\mathbf{u}^*$. Partial results for the Navier Stokes system with symmetric data can be found in [8, 18, 19, 17] .

From now on we limit the discussion to the case where B is a disk of radius one. We choose $\mathbf{x} = (x, y)$ Cartesian coordinates with the origin at the center of B, $(r, \theta) \in \Omega := (0, \infty) \times (-\pi, \pi)$ the associated polar coordinates, and $(\mathbf{e}_r, \mathbf{e}_\theta)$ the corresponding local orthonormal basis. For the function $\mathbf{u}$ we have in polar coordinates:

$$\mathbf{u}(r, \theta) = u_r(r, \theta)\mathbf{e}_r + u_\theta(r, \theta)\mathbf{e}_\theta, \quad \forall\ (r, \theta) \in \Omega\,. \tag{5}$$

The following theorem is our main result:

Theorem 2 *Let $\mu_0 > \mu_{crit} \equiv \sqrt{48}$ and $\mathbf{u}^* \in C^\infty(\partial B)$ satisfying (3) be sufficiently close to $\mathbf{u}^*_{\mu_0} := \mu_0 \mathbf{e}_\theta$. Then, the equations (1), (2), with boundary condition $\mathbf{u}^*$, have at least one solution $(\mathbf{u}, p) \in C^\infty(\mathbb{R}^2 \setminus \overline{B})^2 \times C^\infty(\mathbb{R}^2 \setminus \overline{B})$. Moreover, there exist μ close to μ_0 such that:*

$$\lim_{r \to \infty} r \left\| \mathbf{u}(r, \theta) - \frac{\mu \mathbf{e}_\theta}{r}; L^\infty(-\pi, \pi) \right\| = 0\ . \tag{6}$$

Remark 3 *If the pair $(u(x, y), v(x, y))$ is a solution for the boundary condition $(u^*(x, y), v^*(x, y))$, then the pair $(u(x, -y), -v(x, -y))$ is a solution for the boundary condition $(u^*(x, -y), -v^*(x, -y))$. Thus, our result extends to $\mu_0 < -\mu_{crit}$.*

Remark 4 *If $\mathbf{u}(r, \theta)$ is a solution for the boundary condition $\mathbf{u}^*$ on the complement of the unit disk, then for all $\lambda > 0$, $\lambda \mathbf{u}(\lambda r, \theta)$ is a solution for the boundary condition $\lambda \mathbf{u}^*$ on the complement of the disk of radius λ^{-1}.*

Remark 5 *The restriction to the case where B is a disk is for the sake of simplicity only. This permits to rewrite the system in polar coordinates, yielding explicit expressions for the solutions. We expect that with more work the results can be generalized to arbitrary smooth B.*

To prove Theorem 2 we proceed as follows: We fix $\mu > \mu_{crit}$ and consider the pair $(\mathbf{u}_\mu, p_\mu)$:

$$\mathbf{u}_\mu(r,\theta) = \frac{\mu \mathbf{e}_\theta}{r}\ , \qquad p_\mu(r,\theta) = -\frac{1}{2}\frac{\mu^2}{r^2}\ , \qquad \forall\,(r,\theta)\in\Omega\ , \tag{7}$$

which is an exact solution to (1), (2). Next we set, $(\mathbf{u}, p) = (\mathbf{u}_\mu + \mathbf{v}, p = p_\mu + q)$ and prove, that for all sufficiently small boundary conditions $\mathbf{v}^*$ satisfying

$$\int_{\partial B} \mathbf{v}^* \cdot \mathbf{n}\ \mathrm{d}\sigma = 0\ , \tag{8}$$

there existence of a solution $(\mathbf{v},\, q) \in C^\infty(\mathbb{R}^2 \setminus \overline{B})^2 \times C^\infty(\mathbb{R}^2 \setminus \overline{B})$ such that $\mathbf{v}|_{\partial B} = \mathbf{v}^* + \mu_*$, for some $\mu_* > \mu_{crit}$ depending on μ and $\mathbf{v}^*$. In a final step, we show that this function can be inverted, giving μ as a function of μ_* and $\mathbf{v}^*$, thus yielding Theorem 2.

The feasibility of our approach relies on the fact that the system obtained by linearizing (1), (2) around the explicit solution $(\mathbf{u}_\mu, p_\mu)$ can be analyzed explicitly. As mentioned above, when compared with the case $\mu = 0$, *i.e.*, the Stokes equation, the vorticity decays for $\mu > \mu_{crit}$ faster than $1/r^2$, instead of like $1/r$, such that $\mathbf{u}$ can be shown to decay faster than $1/r$ at infinity, making the nonlinearity subcritical. Introducing suitable function spaces, we are then able to solve the full non-linear system by a classical fixed-point argument.

2 Dynamical system formulation

Let $(\mathbf{u}, p) \in C^\infty(\mathbb{R}^2 \setminus B)$ be a solution to (1), (2), satisfying (3). We first make the construction of the stream-function ψ associated with $\mathbf{u}$ precise. Since $\mathbf{u} \in C^\infty(\mathbb{R}^2 \setminus B)$, we have in particular that $\mathbf{u}^* \in C^\infty(\partial B)$. Let $\mathbf{u}_{int} \in C^\infty(\overline{B})$ satisfy $\mathbf{u}_{int} = \mathbf{u}^*$ on ∂B. Such a function exists since $\mathbf{u}^*$ satisfies (3). For instance, $\mathbf{u}_{int}$ can be the solution to the Stokes equations on B, with boundary condition $\mathbf{u}^*$ on ∂B. Then, setting:

$$\bar{\mathbf{u}}(x,y) = \begin{cases} \mathbf{u} & \text{in } \mathbb{R}^2 \setminus \overline{B}\,, \\ \mathbf{u}_{int} & \text{in } \overline{B}\,, \end{cases}$$

we obtain a continuous divergence-free vector-field on the whole of $\mathbb{R}^2$. Furthermore, this function is smooth on both sides of ∂B so that there exists $\psi \in C^1(\mathbb{R}^2) \cap C^\infty(\overline{B}) \cap C^\infty(\mathbb{R}^2 \setminus B)$ satisfying $\mathbf{u} = \nabla^\perp \psi$.

Instead of (1), (2), we consider now the equation for the stream function ψ and the vorticity $\omega = \nabla \times \mathbf{u}$,

$$\begin{cases} \Delta\psi &= -\omega \\ \Delta\omega &= \mathbf{u}\cdot\nabla\omega \end{cases} \qquad \text{in } \mathbb{R}^2 \setminus \overline{B}\,.$$

For the function $\mathbf{u}$ we have in polar coordinates (5), and the vorticity becomes:

$$\omega = \frac{1}{r}\partial_r(r u_\theta) - \frac{1}{r}\partial_\theta u_r, \quad \forall\ (r,\theta)\in\Omega\,.$$

For the boundary data we have:

$$\mathbf{u}^*(\theta) = u_r^*(\theta)\mathbf{e}_r + u_\theta^*(\theta)\mathbf{e}_\theta\ , \quad \forall\ \theta \in (-\pi,\pi)\,.$$

In polar coordinates we get the following equations for the stream function ψ and the vorticity ω:

$$\begin{cases} \partial_{rr}\psi + \frac{1}{r}\partial_r\psi + \frac{1}{r^2}\partial_{\theta\theta}\psi &= -\omega\,, \\ \partial_{rr}\omega + \frac{1}{r}\partial_r\omega + \frac{1}{r^2}\partial_{\theta\theta}\omega &= u_r\partial_r\omega + \frac{u_\theta}{r}\partial_\theta\omega\,, \end{cases} \qquad \forall\ (r,\theta)\in\Omega\ , \tag{9}$$

and

$$\begin{cases} u_r &= \dfrac{\partial_\theta \psi}{r} \,, \\ u_\theta &= -\partial_r \psi \,, \end{cases} \quad \forall\ (r,\theta) \in \Omega \ , \tag{10}$$

together with the boundary conditions:

$$\begin{cases} u_r(1,\theta) &= u_r^*(\theta) \,, \quad \lim_{r\to\infty} u_r(r,\theta) &= 0 \,, \\ u_\theta(1,\theta) &= u_\theta^*(\theta) \,, \quad \lim_{r\to\infty} u_\theta(r,\theta) &= 0 \,, \end{cases} \quad \forall\ \theta \in (-\pi,\pi) \,. \tag{11}$$

For the exact solution $(\mathbf{u}_\mu, p_\mu)$ given by (7) we have in polar coordinates for the corresponding stream-function-vorticity pair (ψ_μ, ω_μ), for all $\mu \in \mathbb{R}$:

$$\begin{cases} \psi_\mu(r,\theta) &= -\mu \ln(r) \,, \\ \omega_\mu(r,\theta) &= 0 \,, \end{cases} \quad \forall\ (r,\theta) \in \Omega \,.$$

In order to prove Theorem 2 we construct, as explained above, a solution which is a perturbation of the explicit solutions $(\mathbf{u}_\mu, p_\mu)$. We therefore set $\psi = \psi_\mu + \gamma$ and $\omega = \omega_\mu + w$. Substituting this Ansatz into (9), (10), we obtain the following equivalent system for the unknowns (γ, w):

$$\begin{cases} \partial_{rr}\gamma + \frac{1}{r}\partial_r\gamma + \dfrac{1}{r^2}\partial_{\theta\theta}\gamma &= -w \,, \\ \partial_{rr}w + \frac{1}{r}\partial_r w + \dfrac{1}{r^2}\partial_{\theta\theta}w - \dfrac{\mu}{r^2}\partial_\theta w &= \dfrac{\partial_\theta\gamma}{r}\partial_r w - \dfrac{\partial_r\gamma}{r}\partial_\theta w \ , \end{cases} \quad \forall\ (r,\theta) \in \Omega \ , \tag{12}$$

with the boundary conditions:

$$\begin{cases} \partial_\theta\gamma(1,\theta) &= v_r^*(\theta) \ , \\ \partial_r\gamma(1,\theta) &= -v_\theta^*(\theta) \ , \\ \lim_{r\to\infty} \left(|\gamma(r,\theta)| + |\partial_r\gamma(r,\theta)|\right) &= 0 \ , \end{cases} \quad \forall\ \theta \in (-\pi,\pi) \ , \tag{13}$$

for certain $(v_r^*(\theta), v_\theta^*(\theta))$ to be defined later on, satisfying:

$$\int_{\partial B} v_r^* \ \mathrm{d}\sigma = 0 \ , \tag{14}$$

and which are small in a sense to be made precise.

Following the method developed in [14], we solve (12), (13), for data (v_r^*, v_θ^*), by interpreting the radial coordinate r as a time and by expanding in a Fourier series:

$$\gamma(r,\theta) = \sum_{n\in\mathbb{Z}} \gamma_n(r) e^{in\theta} \ , \qquad w(r,\theta) = \sum_{n\in\mathbb{Z}} w_n(r) e^{in\theta} \ .$$

Notation. To unburden the notation we write for the Fourier series of γ and w:

$$\hat{\gamma} = (\gamma_n)_{n\in\mathbb{Z}} \,, \quad \hat{w} = (w_n)_{n\in\mathbb{Z}} \,, \tag{15}$$

and analogously for all other functions.

From (12), (13) we obtain, for $n \in \mathbb{Z}$, the following system of ordinary differential equations:

$$\begin{cases} \partial_{rr}\gamma_n + \dfrac{1}{r}\partial_r\gamma_n - \dfrac{n^2}{r^2}\gamma_n &= -w_n \,, \\ \partial_{rr}w_n + \dfrac{1}{r}\partial_r w_n - \dfrac{i\mu n + n^2}{r^2} w_n &= F_n \,, \end{cases} \quad \text{on } (1,\infty) \,, \tag{16}$$

with the source term F_n given by:

$$F_n = -\frac{i}{r} \sum_{k+l=n} \left(k\, w_k\, \partial_r\gamma_l - l\,\gamma_l\,\partial_r w_k\right) \ , \tag{17}$$

and with the boundary conditions:

$$\left\{\begin{array}{lcl} in\gamma_n(1) & = & v^*_{r,n}\ , \\ -\partial_r\gamma_n(1) & = & v^*_{\theta,n}\ , \qquad \forall\, n\in\mathbb{Z}\setminus\{0\}\,. \\ \lim_{r\to\infty}\,\left(|\gamma_n(r)|+|\partial_r\gamma_n(r)|\right) & = & 0\ , \end{array}\right. \tag{18}$$

Note that $v^*_{r,0}=0$ by assumption (14) and that the value of $\gamma_0(1)$ is irrelevant, *i.e.*, the stream function is only unique up to an additive constant. As we show later in this section, the value $v^*_{\theta,0}$ cannot be chosen freely if one wants the solution γ_0 to satisfy the boundary condition at infinity.

For convenience, we first solve (16) with boundary conditions:

$$\left\{\begin{array}{lcl} \gamma_n(1) & = & \gamma^*_n\ , \\ w_n(1) & = & \omega^*_n\ , \qquad \forall\, n\in\mathbb{Z}\setminus\{0\}\,, \\ \lim_{r\to\infty}\,\left(|\gamma_n(r)|+|w_n(r)|\right) & = & 0\ , \end{array}\right. \tag{19}$$

instead of (19). Once the solution is constructed we then re-express the solution in terms of the original boundary conditions.

Assuming that the functions F_n are continuous and decay sufficiently rapidly at infinity, there exits exactly one solution to (16) satisfying (19). Since the Green's function of equations (16) are $r\mapsto r^{\pm|n|}$ and $r\mapsto r^{\pm\zeta_n}$, respectively, where $\zeta_n=\sqrt{n^2+i\mu n}$, with $\mathcal{R}e(\sqrt{z})>0$ for $z\in\mathbb{C}\setminus(-\infty,0]$, the solutions are given by the following explicit expressions:

$$\left\{\begin{array}{lcl} \gamma_n(r) & = & \dfrac{\overline{\gamma}_n}{r^{|n|}}+\displaystyle\int_r^\infty \frac{sw_n(s)}{2|n|}\left(\frac{r}{s}\right)^{|n|}\mathrm{d}s+\int_1^r \frac{sw_n(s)}{2|n|}\left(\frac{s}{r}\right)^{|n|}\mathrm{d}s\,, \\ w_n(r) & = & \dfrac{\overline{w}_n}{r^{\zeta_n}}-\displaystyle\int_r^\infty \frac{sF_n(s)}{2\zeta_n}\left(\frac{r}{s}\right)^{\zeta_n}\mathrm{d}s-\int_1^r \frac{sF_n(s)}{2\zeta_n}\left(\frac{s}{r}\right)^{\zeta_n}\mathrm{d}s\,, \end{array}\right. \quad \text{for } n\in\mathbb{Z}\setminus\{0\}\,. \tag{20}$$

with:

$$\left\{\begin{array}{lcl} \overline{\gamma}_n & = & \gamma^*_n-\displaystyle\int_1^\infty \frac{sw_n(s)}{2|n|}\left(\frac{1}{s}\right)^{|n|}\mathrm{d}s\,, \\ \overline{w}_n & = & w^*_n+\displaystyle\int_1^\infty \frac{sF_n(s)}{2\zeta_n}\left(\frac{1}{s}\right)^{\zeta_n}\mathrm{d}s\,, \end{array}\right. \quad \text{for } n\in\mathbb{Z}\setminus\{0\}\,. \tag{21}$$

For $n=0$, there still exist solutions to (16) decaying at infinity, but these solutions exist only for exactly one boundary condition. The reason is that for $n=0$ the Green's functions for the equations in (16) are $r\mapsto 1$ and $r\mapsto\ln r$, which do not decay at infinity. The solutions decaying at infinity are:

$$\left\{\begin{array}{lcl} \gamma_0(r) & = & \displaystyle\int_r^\infty \frac{1}{s}\int_s^\infty t\,w_0(t)\,\mathrm{d}t\,\mathrm{d}s\,, \\ w_0(r) & = & -\displaystyle\int_r^\infty \frac{1}{s}\int_s^\infty t\,F_0(t)\,\mathrm{d}t\,\mathrm{d}s\,. \end{array}\right. \tag{22}$$

We recall that the value of $\gamma_0(1)$ is irrelevant. The value of $w_0(1)$ fixes the value of $\partial_r\gamma_0(1)=-v^*_{\theta,0}$ as a function of μ. Once the solution is constructed, we will show that $\mu\mapsto\mu-\partial_r\gamma_0(1)=:\mu^*$ can be inverted, which then shows the existence of a solution for an open set of boundary conditions.

3 Functional framework and main result

We now introduce the function spaces which we use to solve the system (18)–(22). We use the notation introduced in (15):

Definition 6 *Given $\kappa > 0$, $\alpha > 0$ and $m \in \mathbb{N}$, such that $m < \kappa$, we set:*

$$\begin{aligned} \mathcal{B}_\kappa &:= \{\hat\varphi^* \in \mathbb{C}^{\mathbb{Z}} \text{ such that } \sup_{n\in\mathbb{Z}}(1+|n|)^\kappa |\varphi_n^*| < \infty\} \,, \\ \mathcal{B}_\kappa^0 &:= \{\hat\varphi^* \in \mathcal{B}_\kappa \text{ such that } \varphi_0^* = 0\} \,, \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_{\alpha,\kappa} &:= \{\hat\varphi \in (C([1,\infty);\mathbb{C}))^{\mathbb{Z}}, \text{ such that } \sup_{n\in\mathbb{Z}} \sup_{r\in[1,\infty)} r^\alpha (1+|n|)^\kappa |\varphi_n(r)| < \infty\} \,, \\ \mathcal{U}_{\alpha,\kappa}^m &:= \{\hat\varphi \in (C^m([1,\infty);\mathbb{C}))^{\mathbb{Z}}, \text{ such that } (\partial_r^l \varphi_n)_{n\in\mathbb{Z}} \in \mathcal{B}_{\alpha+l,\kappa-l}, \text{ for all } 0 \le l \le m\} \,. \end{aligned}$$

These function spaces are reminiscent of weighted Sobolev spaces, and permit to obtain sharp estimates on the decay of solutions to (20)–(22). The spaces with one lower index (mainly $\mathcal{B}_\kappa^0$) are used for the boundary data, whereas the spaces with two lower indices (mainly $\mathcal{U}_{\alpha,\kappa}^m$) will be used for solving (20)–(22).

The spaces introduced in Definition 6 satisfy the following straightforward properties. Given $\alpha > 0$, $\kappa > 0$, and $m \in \mathbb{N}$, such that $m < \kappa$, we have:

1. The spaces $\mathcal{B}_\kappa$, $\mathcal{B}_{\alpha,\kappa}$, $\mathcal{U}_{\alpha,\kappa}^m$ are Banach spaces when equipped with their respective norms:

$$\|\hat w \; ; \; \mathcal{B}_\kappa\| = \sup_{n\in\mathbb{N}} (1+|n|)^\kappa |w_n| , \qquad \|\hat\varphi \; ; \; \mathcal{B}_{\alpha,\kappa}\| = \sup_{n\in\mathbb{Z}} \sup_{r\in[1,\infty)} r^\alpha (1+|n|)^\kappa |\varphi_n(r)| \,,$$

$$\|\hat\varphi \; ; \; \mathcal{U}_{\alpha,\kappa}^m\| = \sum_{l=0}^m \|(\partial_r^l \varphi_n)_{n\in\mathbb{Z}} \; ; \; \mathcal{B}_{\alpha+l,\kappa-l}\| \,.$$

2. Given $\alpha \ge \alpha'$ and $\kappa \ge \kappa'$ we have the embedding $\mathcal{B}_{\alpha,\kappa} \subset \mathcal{B}_{\alpha',\kappa'}$ together with the bound:

$$\|\hat\varphi \, ; \mathcal{B}_{\alpha',\kappa'}\| \le \|\hat\varphi \, ; \mathcal{B}_{\alpha,\kappa}\| \,, \qquad \forall \, \hat\varphi \in \mathcal{B}_{\alpha,\kappa} \,. \tag{23}$$

3. The space $\mathcal{B}_\kappa^0$ is a closed subspace of $\mathcal{B}_\kappa$, and thus also a Banach space.

We now formulate the problem of finding a solution to (18)–(22) in such a way that we can apply the inverse map theorem on our function spaces:

Lemma 7 *Given $\mu > \mu_{crit}$, let $\alpha > 0$ be sufficiently small and $\kappa > 0$. Then, the map $\mathcal{S}_\mu \colon \mathcal{B}_{4+2\alpha,\kappa} \times \mathcal{B}_{\kappa+4}^0 \times \mathcal{B}_{\kappa+2}^0 \to \mathcal{U}_{\alpha,\kappa+4}^2 \times \mathcal{U}_{\alpha+2,\kappa+2}^2$, which associates to the triple $(\hat F, \hat\gamma^*, \hat w^*)$ the pair $(\hat\gamma, \hat w)$ by virtue of equations (20), (22), together with (19), (21), is linear and continuous.*

The notion of α small enough will be made precise in the last section.

Lemma 8 *Let $\alpha > 0$ and $\kappa > 0$. Then, the map $\mathcal{NL} \colon \left(\mathcal{U}_{\alpha,\kappa+4}^2 \times \mathcal{U}_{\alpha+2,\kappa+2}^2\right)^2 \to \mathcal{B}_{4+2\alpha,\kappa+1}$, defined by*

$$\mathcal{NL}[(\hat\gamma_a, \hat w_a), (\hat\gamma_b, \hat w_b)] = \left(r \mapsto -\frac{i}{r} \sum_{k+l=n} \left(k\, w_k^a(r)\, \partial_r \gamma_l^b(r) - l\, \gamma_l^a(r)\, \partial_r w_k^b(r) \right) \right)_{n\in\mathbb{Z}} \,,$$

is bilinear and continuous.

The proofs of these lemmas are postponed to Section 5. We also introduce the trace operator Γ_1:

$$\begin{array}{rrcl} \Gamma_1 \colon & \mathcal{U}_{\alpha_1,\kappa_1}^2 \times \mathcal{U}_{\alpha_2,\kappa_2}^2 & \to & \mathcal{B}_{\kappa_1-1}^0 \times \mathcal{B}_{\kappa_1-1}^0 \,, \\ & (\hat\gamma, \hat w) & \longmapsto & ((in\gamma_n(1))_{n\in\mathbb{Z}}, ((\delta_{n,0}-1)\partial_r \gamma_n(1))_{n\in\mathbb{Z}}) \,, \end{array}$$

where $\delta_{n,m}$ is the Kronecker symbol. This map is linear and continuous for arbitrary $(\alpha_i, \kappa_i) \in (0,\infty)^2$, $(i = 1, 2)$.

To compute solutions to (18)–(22), we introduce a map Φ_μ, which allows to solve the differential equations and constrain the trace on $r = 1$ in one step. Namely, given $\mu > \mu_{crit}$, α sufficiently small and $\kappa > 0$, we set:

$$\begin{aligned}\Phi_\mu: \quad (\mathcal{U}^2_{\alpha,\kappa+4} \times \mathcal{U}^2_{\alpha+2,\kappa+2}) \times (\mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2}) \quad &\longrightarrow \quad (\mathcal{U}^2_{\alpha,\kappa+4} \times \mathcal{U}^2_{\alpha+2,\kappa+2}) \times (\mathcal{B}^0_{\kappa+3} \times \mathcal{B}^0_{\kappa+3}) \\ \begin{pmatrix} \hat{x} = (\hat{\gamma}, \hat{w}) \\ \hat{x}^* = (\hat{\gamma}^*, \hat{w}^*) \end{pmatrix} \quad &\longmapsto \quad \begin{pmatrix} \mathcal{S}_\mu(\mathcal{NL}(\hat{x}, \hat{x}), \hat{x}^*) - \hat{x} \\ \Gamma_1[\mathcal{S}_\mu(\mathcal{NL}(\hat{x}, \hat{x}), \hat{x}^*)] \end{pmatrix}\end{aligned} \tag{24}$$

By definition, if $(\hat{x} = (\hat{\gamma}, \hat{w}), \hat{x}^* = (\hat{\gamma}^*, \hat{w}^*))$ is a solution to $\Phi_\mu(\hat{x}, \hat{x}^*) = (0, (\hat{v}^*_r, \hat{v}^*_\theta))$, then $(\hat{\gamma}, \hat{w})$ satisfies (16)-(18). This motivates the following notion of κ–solutions:

Definition 9 *Given an exponent $\kappa > 0$, an angular velocity $\mu > \mu_{crit}$, and a boundary condition $\hat{\mathbf{v}}^* := (\hat{v}^*_r, \hat{v}^*_\theta) \in \mathcal{B}^0_{\kappa+3} \times \mathcal{B}^0_{\kappa+3}$, we call* κ–solution for the boundary condition $\hat{\mathbf{v}}^*$ and the asymptotic angular velocity μ *a pair $\hat{x} = (\hat{\gamma}, \hat{w})$, such that, for sufficiently small $\alpha > 0$ and some $\hat{x}^* = (\hat{\gamma}^*, \hat{w}^*) \in \mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2}$:*

- $\hat{x} \in \mathcal{U}^2_{\alpha+4,\kappa} \times \mathcal{U}^2_{\alpha+2,\kappa+2}$,
- $\Phi_\mu(\hat{x}, \hat{x}^*) = (0, \hat{\mathbf{v}}^*)$.

The remaining sections are devoted to the proof of the following result:

Theorem 10 *Given $\kappa > 0$ and $\mu_0 > \mu_{crit}$ there exists $\varepsilon_{\kappa,\mu_0} > 0$ and an open interval $I_{\kappa,\mu_0} \ni \mu_0$ such that, given $\hat{\mathbf{v}}^* \in B(\mathcal{B}^0_{\kappa+3} \times \mathcal{B}^0_{\kappa+3}\ ;\ \varepsilon_{\kappa,\mu_0})$ and $\mu^* \in I_{\kappa,\mu_0}$, there exists $\mu > \mu_{crit}$ and a κ–solution $(\hat{\gamma}, \hat{w})$ for the boundary condition $\hat{\mathbf{v}}^*$ and the asymptotic angular velocity μ, satisfying the condition $\mu - \partial_r \gamma_0(1) = \mu^*$.*

As mentioned above, the notion of α small enough will be made precise in the last section. Before entering into the details of the proof of Theorem 10, we explain why it implies Theorem 2.

Proof of Theorem 2. Let $\mu_0 > \mu_{crit}$ and $\kappa > 1$. Applying Theorem 10 yields a ball of initial conditions with positive radius $\varepsilon_{\kappa,\mu_0} > 0$ and an open neighborhood I_{κ,μ_0} of μ_0.

For $\mathbf{u}^* \in C^\infty(\partial B)$, we define :

$$\mu^* = \frac{1}{2\pi} \int_0^{2\pi} u^*_\theta(s)\, \mathrm{d}s\,,$$

and the sequences $\hat{u}^*$ and $\hat{v}^*$ by $u_0 = v_0 = 0$ and:

$$u^*_n = \frac{1}{2\pi} \int_0^{2\pi} u^*_r(\theta) e^{-in\theta}\, \mathrm{d}\theta\,, \qquad v^*_n = \frac{1}{2\pi} \int_0^{2\pi} u_\theta(\theta) e^{-in\theta}\, \mathrm{d}\theta\,, \qquad \forall\, n \in \mathbb{Z} \setminus \{0\}\,.$$

The regularity of $\mathbf{u}^*$ yields that $(\hat{u}^*, \hat{v}^*) \in \mathcal{B}^0_{\kappa+3} \times \mathcal{B}^0_{\kappa+3}$ and

$$u^*_r(\theta) = \sum_{n\in\mathbb{Z}} u^*_n e^{in\theta}\,, \qquad u^*_\theta(\theta) = \mu^* + \sum_{n\in\mathbb{Z}} v^*_n e^{in\theta}\,.$$

We now assume that:

$$\|(\hat{u}^*, \hat{v}^*)\ ;\ \mathcal{B}^0_{\kappa+3} \times \mathcal{B}^0_{\kappa+3}\| < \varepsilon_{\kappa,\mu_0}\,, \quad \mu^* \in I_{\kappa,\mu_0}\,, \tag{25}$$

which makes the meaning of $\mathbf{u}^*$ sufficiently close to $\mu_0 \mathbf{e}_\theta$ in the statement of Theorem 2 precise.

Consequently, the assumptions of Theorem 10 are satisfied, which yields that there exists $\mu > \mu_{crit}$ and a κ–solution $(\hat{\gamma}, \hat{w})$ for the boundary data $(\hat{u}^*, \hat{v}^*)$ satisfying $\mu - \partial_r \gamma_0(1) = \mu^*$. Let

$$w(r,\theta) = \sum_{n\in\mathbb{Z}} w_n(r) e^{in\theta}\,, \qquad \gamma(r,\theta) = \sum_{n\in\mathbb{Z}} \gamma_n(r) e^{in\theta}\,, \quad \forall\,(r,\theta) \in \Omega\,.$$

Because $\kappa > 1$, classical results from the theory of Fourier series yield that:

- $w \in C^2(\mathbb{R}^2 \setminus \overline{B})$ and $\gamma \in C^4(\mathbb{R}^2 \setminus \overline{B})$ so that $\mathbf{v} = \nabla^\perp \gamma \in C^3(\mathbb{R}^2 \setminus \overline{B})$,

- $\mathbf{v}\cdot\nabla w \in C^1(\mathbb{R}^2\setminus\overline{B})$ with:

$$\mathbf{v}\cdot\nabla w(r,\theta) = \sum_{n\in\mathbb{Z}}\left[-\frac{i}{r}\sum_{k+l=n}\left(l\,w_l(r)\,\partial_r\gamma_k(r) - k\,\gamma_k(r)\,\partial_r w_l(r)\right)\right]e^{in\theta}\,,\quad \forall\,(r,\theta)\in\Omega\,.$$

Let

$$\Delta w - \frac{\mu}{r}\,\mathbf{e}_\theta\cdot\nabla w - \mathbf{v}\cdot\nabla w =: \varphi \in C([1,\infty);C([-\pi,\pi]))\,.$$

Because $(\hat{\gamma},\hat{w})$ satisfies (16), all the Fourier coefficients of φ vanish identically on $[1,\infty)$. Hence, (γ,w) is a solution of

$$\left\{\begin{array}{rcl}\Delta\gamma &=& -w\,,\\ \Delta w - \dfrac{\mu}{r}\,\mathbf{e}_\theta\cdot\nabla w &=& \mathbf{v}\cdot\nabla w\,,\end{array}\right.\quad \text{in } \mathbb{R}^2\setminus\overline{B}\,. \tag{26}$$

Straightforward manipulations of the Fourier series of γ yield that:

$$|\mathbf{v}(r,\theta)| \le \frac{\|\gamma\,;\,\mathcal{U}^2_{\alpha,\kappa+4}\|}{r^{\alpha+1}}\,,\quad \forall\,(r,\theta)\in\Omega\,. \tag{27}$$

Since $\mathbf{u}^*$ has zero flux, *i.e.*, since v_r^* has zero average, we have that $\mathbf{v}(1,\theta) = v_r^*(\theta)e_r + v_\theta^*(\theta)\mathbf{e}_\theta$, for all $\theta\in(-\pi,\pi)$, where:

$$\begin{aligned} v_r^*(\theta) &= \sum_{n\in\mathbb{Z}} in\gamma_n(1)e^{in\theta} = \sum_{n\in\mathbb{Z}} u_n^* e^{in\theta} = u_r^*(\theta)\,,\\ v_\theta^*(1,\theta) &= -\sum_{n\in\mathbb{Z}}\partial_r\gamma_n(1)e^{in\theta} = u_\theta^*(\theta) - \partial_r\gamma_0(1) - \mu^*\,.\end{aligned}$$

Therefore, if we set $\psi := \psi_\mu + \gamma$, $\omega := w$, $\mathbf{u} := \nabla^\perp\psi = \mathbf{u}_\mu + \mathbf{v}$, then the pair (ψ,ω) is a solution to

$$\left\{\begin{array}{rcl}\Delta\psi &=& -\omega\,,\\ \Delta\omega &=& \mathbf{u}\cdot\nabla\omega\,,\end{array}\right.\quad \text{in } \mathbb{R}^2\setminus\overline{B}\,, \tag{28}$$

and the following boundary conditions are satisfied (recall that $\mu - \partial_r\gamma_0(1) = \mu^*$ by construction of $\hat{\gamma}$):

$$\mathbf{u} = \mathbf{u}^*\,,\quad \text{on } \partial B\,,\qquad \lim_{r\to\infty}|\mathbf{u}(r,\theta)| = 0\,.$$

The inequality (27) implies that the boundary condition at infinity is satisfied in the following more precise sense:

$$\lim_{r\to\infty}\left\|r\left(\mathbf{u}(r,\theta) - \frac{\mu\mathbf{e}_\theta}{r}\right);L^\infty((-\pi,\pi))\right\| = \lim_{r\to\infty} r\|v(r,\theta);L^\infty((-\pi,\pi))\| = 0\,.$$

To complete the proof, we need to show how to obtain the Navier Stokes equations (1) from the relations between $\mathbf{u}$, ψ and ω, together with (28). First, multiplying (28) by $\varphi\in C_c^\infty(\mathbb{R}^2\setminus\overline{B})$ yields:

$$-\int_{\mathbb{R}^2\setminus\overline{B}}\nabla w\colon\nabla\varphi = \int_{\mathbb{R}^2\setminus\overline{B}}[\mathbf{u}\cdot\nabla w]\,\varphi\,.$$

We have the basic identities:

$$w = \nabla\times\mathbf{u}\,,\qquad \mathbf{u}\cdot\nabla w = \nabla\times[\mathbf{u}\cdot\nabla\mathbf{u}]\,, \tag{29}$$

from which we obtain, after integration by parts, that for any given $\varphi\in C_c^\infty(\mathbb{R}^2\setminus\overline{B})$:

$$-\int_{\mathbb{R}^2\setminus\overline{B}}\nabla\mathbf{u}\colon\nabla\nabla^\perp\varphi = \int_{\mathbb{R}^2\setminus\overline{B}}[\mathbf{u}\cdot\nabla\mathbf{u}]\cdot\nabla^\perp\varphi\,. \tag{30}$$

This identity yields the pressure p *via* De Rham's theory, modulo the difficulty, that not all the divergence-free velocity-fields $\mathbf{w}\in C_c^\infty(\mathbb{R}^2\setminus\overline{B})$ of compact support can be written in the form $\nabla^\perp\varphi$ with $\varphi\in$

$C_c^\infty(\mathbb{R}^2 \setminus \overline{B})$. More precisely, if a smooth velocity-field $\mathbf{v}$ satisfies $\nabla \times \mathbf{v} = 0$ in $\mathbb{R}^2 \setminus \overline{B}$, then $\mathbf{v}$ is the gradient of a function up to a contribution of the form $C\mathbf{x}^\perp/|\mathbf{x}|^2$. We now show that this contribution vanishes in our case.

Let $\Phi_0 \in C^\infty(\mathbb{R})$ be such that $\mathrm{supp}(\Phi_0') \subset\subset (1,2)$ and $\Phi_0(0) = 0$, $\Phi_0(2) = 1$, and let $\mathbf{w}_0 = \nabla^\perp \Phi_0$. In polar coordinates we have $\mathbf{w}_0(r,\theta) = -\Phi_0'(r)\mathbf{e}_\theta$, for all $(r,\theta) \in \Omega$. Given a divergence-free $\mathbf{w} \in C_c^\infty(\mathbb{R}^2 \setminus \overline{B})$ we define $\widetilde{\mathbf{w}}$ and φ by:

$$\widetilde{\mathbf{w}} = \mathbf{w} - M_{\mathbf{w}}\mathbf{w}_0 \quad \text{with} \quad \left[\int_1^\infty w_\theta(r,0)\,\mathrm{d}r\right] =: M_{\mathbf{w}}, \qquad \varphi(r,\theta) = \int_0^r w_\theta(s,\theta)\,\mathrm{d}s - M_{\mathbf{w}}\Phi_0(r)\,.$$

By definition of $M_{\mathbf{w}}$, we have that $\varphi \in C_c^\infty(\mathbb{R}^2 \setminus \overline{B})$ and $\nabla^\perp \varphi = \widetilde{\mathbf{w}}$, so that (30) implies:

$$-\int_{\mathbb{R}^2\setminus\overline{B}} \nabla\mathbf{u} \colon \nabla\widetilde{\mathbf{w}} = \int_{\mathbb{R}^2\setminus\overline{B}} [\mathbf{u}\cdot\nabla\mathbf{u}]\cdot\widetilde{\mathbf{w}}\,.$$

Replacing $\widetilde{\mathbf{w}}$ by its definition yields that, for any divergence-free $\mathbf{w} \in C_c^\infty(\mathbb{R}^2 \setminus \overline{B})$, we have:

$$-\int_{\mathbb{R}^2\setminus\overline{B}} \nabla\mathbf{u} \colon \nabla\mathbf{w} = \int_{\mathbb{R}^2\setminus\overline{B}} [\mathbf{u}\cdot\nabla\mathbf{u}]\cdot\mathbf{w} - M_{\mathbf{w}} \int_{\mathbb{R}^2\setminus\overline{B}} \left([\mathbf{u}\cdot\nabla\mathbf{u}]\cdot\mathbf{w}_0 + \nabla\mathbf{u}\colon\nabla\mathbf{w}_0\right)\,.$$

Let I_0 be the last integral in the previous equality. We then have:

$$I_0 = \lim_{N\to\infty} \int_{B(\mathbb{R}^2,N)\setminus\overline{B}} \left([\mathbf{u}\cdot\nabla\mathbf{u}]\cdot\mathbf{w}_0 + \nabla\mathbf{u}\colon\nabla\mathbf{w}_0\right)\,.$$

Integrating by parts, we obtain, for all $N > 1$:

$$\begin{aligned}
\int_{B(\mathbb{R}^2,N)\setminus\overline{B}} \left([\mathbf{u}\cdot\nabla\mathbf{u}]\cdot\mathbf{w}_0 + \nabla\mathbf{u}\colon\nabla\mathbf{w}_0\right) &= \int_{\partial B(\mathbb{R}^2,N)} \left([\Phi_0\mathbf{u}\cdot\nabla\mathbf{u}]\cdot\mathbf{n}^\perp + \Phi_0'\partial_r\mathbf{u}\cdot\mathbf{n}^\perp\right)\,\mathrm{d}\sigma \\
&\quad - \int_{B(\mathbb{R}^2,N)} \left(\Phi_0\mathbf{u}\cdot\nabla\omega + \nabla\omega\cdot\nabla\Phi_0\right) \\
&= \int_{\partial B(\mathbb{R}^2,N)} \left(\Phi_0\left[(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{n}^\perp - \partial_{\mathbf{n}}\omega\right] + \Phi_0'\partial_r\mathbf{u}\cdot\mathbf{n}^\perp\right)\,\mathrm{d}\sigma\,,
\end{aligned}$$

where, in order to get the last identity, we have again used that $\mathbf{u}\cdot\nabla\omega = \Delta\omega$, in $\mathbb{R}^2 \setminus \overline{B}$. Since $\mathbf{u}$ decays like $1/r$, $\nabla\mathbf{u}$ decays like $1/r^2$, and $\nabla\omega$ like $1/r^3$. This yields that $I_0 = 0$ in the limit $N \to \infty$. Finally, we have:

$$-\int_{\mathbb{R}^2\setminus\overline{B}} \nabla\mathbf{u} \colon \nabla\mathbf{w} = \int_{\mathbb{R}^2\setminus\overline{B}} [\mathbf{u}\cdot\nabla\mathbf{u}]\cdot\mathbf{w}\,,$$

for any divergence-free vector-field $\mathbf{w} \in C_c^\infty(\mathbb{R}^2 \setminus \overline{B})$, and De Rham's theory (see [21, Remark 1.5]) implies the existence of a pressure p such that (1) is satisfied. ■

4 Proof of Theorem 10

In this section $\kappa > 0$ and $\mu_0 > \mu_{crit}$ are fixed. First, we set $\mu_- = (\mu_0 + \mu_{crit})/2$ and $\mu_+ = (2\mu_0 + \mu_{crit})/2$ so that $I := [\mu_-, \mu_+]$ satisfies

$$\mu_0 \in [\mu_-, \mu_+] \subset (\mu_{crit}, \infty)\,.$$

We also set:

$$\alpha := \frac{1}{4}\min\left(\frac{1}{\sqrt{2}}\left[\sqrt{1+|\mu_-|^2}+1\right]^{1/2} - 2, 1\right)\,. \tag{31}$$

We emphasize that, because $\mu_0 > \mu_{crit}$, we have $\alpha \in (0, 1/4]$. Let:

$$\begin{aligned}
\Phi_\mu\colon \quad (\mathcal{U}^2_{\alpha,\kappa+4} \times \mathcal{U}^2_{\alpha+2,\kappa+2}) \times (\mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2}) \quad &\longrightarrow \quad (\mathcal{U}^2_{\alpha,\kappa+4} \times \mathcal{U}^2_{\alpha+2,\kappa+2}) \times (\mathcal{B}^0_{\kappa+3} \times \mathcal{B}^0_{\kappa+3}) \\
\begin{pmatrix} \hat{x} = (\hat{\gamma}, \hat{w}) \\ \hat{x}^* = (\hat{\gamma}^*, \hat{w}^*) \end{pmatrix} \quad &\longmapsto \quad \begin{pmatrix} \mathcal{S}_\mu(\mathcal{NL}(\hat{x},\hat{x}), \hat{x}^*) - \hat{x} \\ \Gamma_1[\mathcal{S}_\mu(\mathcal{NL}(\hat{x},\hat{x}), \hat{x}^*)] \end{pmatrix}
\end{aligned} \tag{32}$$

We will show in the next section that for $\mu \in I$ the map Φ_μ is well defined. We split the proof of Theorem 10 into two steps. First, we show that we can construct a κ–solution for any sufficiently small boundary condition $(\hat{u}^*, \hat{v}^*) \in \mathcal{B}^0_{\kappa+3} \times \mathcal{B}^0_{\kappa+3}$ and an interval of asymptotic angular velocities μ. In a second step, we analyze the dependence of this solution as a function of μ.

We have the following abstract result:

Proposition 11 *Let I be a compact interval and X, Y two Banach spaces. Assume that $\Phi \colon \mu \in I \mapsto \Phi_\mu$ satisfies:*

- $\Phi \in C(I; C^1(X;Y))$,
- $\Phi_\mu(x) = 0$ *for all* $\mu \in I$,
- $D\Phi_\mu(0)$ *is one-to-one and onto, and has a continuous inverse for all* $\mu \in I$.

Then, there exist positive constants η_x and η_y, such that Φ_μ is a C^1-diffeomorphism from $B_X(0,\eta_x)$ onto $\Phi_\mu(B_X(0,\eta_x)) \supset B_Y(0,\eta_y)$. Furthermore, the family of inverse maps $\Phi^{-1} \colon \mu \in I \mapsto (\Phi_\mu)^{-1}$ satisfies

$$\Phi^{-1} \in C(I; C^1(B_Y(0,\eta_y); B_X(0,\eta_x)))\,.$$

Proof. The proof is standard, but for the sake of completeness we recall the main ingredients. Given the hypothesis of Proposition 11, the map Φ_μ satisfies the assumptions of the inverse function theorem for arbitrary $\mu \in I$, so that there exits $\eta_{x,\mu} > 0$ and $\eta_{y,\mu} > 0$ such that Φ_μ is a C^1-diffeomorphism from $B_X(0,\eta_{x,\mu})$ onto $\Phi_\mu(B_X(0,\eta_{x,\mu})) \supset B_Y(0,\eta_{\mu,y})$. Since Φ is continuous, it is clear that these constants can be chosen independently of μ, locally in μ. By a compactness argument, we can therefore find constants $\eta_x > 0$ and $\eta_y > 0$ such that $\Phi_\mu \colon B_X(0,\eta_x) \to \Phi_\mu(B_X(0,\eta_x)) \supset B_Y(0,\eta_y)$ is a C^1 diffeomorphism for arbitrary $\mu \in I$. We now show that $\Phi^{-1} \in C(I \times B_Y(0,\eta_y))$. The proof that $\Phi^{-1} \in C(I; C^1(B_Y(0,\eta_y); B_X(0,\eta_x)))$ is then obtained by differentiating (with respect to x) the identity $\Phi_\mu^{-1} \circ \Phi_\mu(x) = x$ which holds true on $B_X(0,\eta_x)$.

Given $(\mu,\tilde{\mu}) \in I^2$ and $(y,\tilde{y}) \in [B_Y(0,\eta_y)]^2$, we denote:

$$x = \Phi_\mu^{-1}(y)\,, \qquad \tilde{x} = \Phi_{\tilde{\mu}}^{-1}(\tilde{y})\,.$$

By construction we have:

$$\begin{aligned}
y - \tilde{y} &= \Phi_\mu(x) - \Phi_{\tilde{\mu}}(\tilde{x}), \\
&= D\Phi_\mu(0)[x-\tilde{x}] + \Phi_\mu(\tilde{x}) - \Phi_{\tilde{\mu}}(\tilde{x}) + o(|x-\tilde{x}|) \\
&= D\Phi_\mu(0)[x-\tilde{x}] + o(|\mu-\tilde{\mu}|) + o(\|x-\tilde{x};X\|)\,.
\end{aligned}$$

Consequently, reducing the size of η_x and η_y if necessary, we get that:

$$\|x-\tilde{x}\,;X\| \leq 2\|[D\Phi_\mu(0)]^{-1}; \mathcal{L}_c(Y;X)\|\,[\|y-\tilde{y};Y\| + o(|\mu-\tilde{\mu}|)]\,,$$

where $\mathcal{L}_c(Y;X)$ denotes the set of continuous linear map $Y \to X$. This completes the proof. ■

We now show that we can apply Proposition 11 to the map Φ as defined in (32) To this end, we remark that Φ_μ depends on μ only through $\mu \mapsto \mathcal{S}_\mu$, and that for all $\mu \in I$ the differential $D\Phi_\mu(0)$ is:

$$D\Phi_\mu(0)[\hat{x},\hat{x}^*] = (\Gamma_1\mathcal{S}_\mu(0,\hat{x}^*), -\hat{x})\,, \quad \forall\,(\hat{x},\hat{x}^*) \in (\mathcal{U}^2_{\alpha,\kappa+4} \times \mathcal{U}^2_{\alpha+2,\kappa+2}) \times (\mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2})\,.$$

Let $\mathcal{T}_\mu(\hat{x}^*) := \Gamma_1\mathcal{S}_\mu(0,\hat{x}^*)$. Since Φ is a combination of linear and bilinear maps, it suffices to apply Lemma 8 and the following lemma in order to check that the assumptions of Proposition 11 are satisfied:

Lemma 12 *Let α be given by* (31)*, then the restriction of the map $\mathcal{S} \colon \mu \mapsto \mathcal{S}_\mu$ to $I := [\mu_-, \mu_+]$ satisfies:*

i) $\mathcal{S} \in C(I; \mathcal{L}_c(\mathcal{B}_{\kappa,2\alpha+4} \times (\mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2})\ ;\ \mathcal{U}^2_{\alpha,\kappa+4} \times \mathcal{U}^2_{\alpha+2,\kappa+2}))$,

ii) *$\mathcal{T}_\mu$ is a one to one and onto map $\mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2} \to \mathcal{B}^0_{\kappa+3} \times \mathcal{B}^0_{\kappa+3}$ with continuous inverse.*

We postpone the proof of this technical lemma to the next section. We now apply Proposition 11 to Φ, but restrict the image to the component $\hat{x}$. This yields the following result:

Theorem 13 *There exists a map $\Psi\colon \mu \mapsto \Psi_\mu$ satisfying*

$$\Psi \in C(I; C^1(B_{\mathcal{B}^0_{\kappa+3}\times\mathcal{B}^0_{\kappa+3}}(0,\varepsilon_\kappa)\ ;\ \mathcal{U}^2_{\alpha,\kappa+4}\times\mathcal{U}^2_{\alpha+2,\kappa+2}))\ ,$$

such that, for all $\mathbf{v}^* \in B_{\mathcal{B}^0_{\kappa+3}\times\mathcal{B}^0_{\kappa+3}}(0,\varepsilon_\kappa)$ and $\mu \in I$, $\Psi_\mu(\mathbf{v}^*)$ is a κ–solution for the boundary condition $\mathbf{v}^*$ with respect to the asymptotic angular velocity μ.

In a final step we show how to prescribe the zero mode of the solution $\Psi_\mu(\mathbf{v}^*)$ by using the dependence of the solution on μ. Let $\eta = (\mu_0 - \mu_{crit})/4$, and consider a boundary condition

$$\mathbf{v}^* \in B_{\mathcal{B}^0_{\kappa+3}\times\mathcal{B}^0_{\kappa+3}}(0,\varepsilon_\kappa)\,.$$

Let $(\hat{\psi}_\mu,\hat{w}_\mu) := \Psi_\mu(\mathbf{v}^*)$, for all $\mu \in I$. Using that $\Psi_\mu(0) = 0$, and restricting the size of ε_κ if necessary, we can assume that

$$|\partial_r\psi_{\mu,0}(1)| \le \|\hat{\psi}_\mu\,;\mathcal{U}^2_{\alpha,\kappa}\| \le \eta\,, \qquad \forall\,\mu\in I\,.$$

Consequently, the map $\mu \mapsto \mu - \partial_r\psi_{\mu,0}(0)$, which is continuous from I to $\mathbb{R}$, because $\mu \mapsto \Psi_\mu(0,\mathbf{v}^*)$ is continuous, satisfies:

$$\mu_- - \partial_r\psi_{\mu_-,0}(1) \le \mu_- + \eta < \mu_0\,, \qquad \mu_+ - \partial_r\psi_{\mu_+,0}(0) \ge \mu_+ - \eta > \mu_0\,.$$

Hence the image of this map contains an open interval I_{κ,μ_0} containing μ_0. This completes the proof of Theorem 10.

5 Proof of main lemmas

This section contains the proof of the technical lemmas which have been used without proof in the previous sections. First we prove Lemma 8, which is standard. We then give proofs of Lemmas 7 and 12 which are more delicate.

Proof of Lemma 8. Let $\hat{F} = \mathcal{NL}((\hat{\gamma}^a,\hat{w}^a),(\hat{\gamma}^b,\hat{w}^b))$, for $(\hat{\gamma}^i,\hat{w}^i) \in \mathcal{U}^2_{\alpha,\kappa+4}\times\mathcal{U}^2_{\alpha+2,\kappa+2}$, $i=\{a,b\}$. First, we note that for $\kappa > 0$ and $n \in \mathbb{N}$, the following series converge:

$$\sum_{l\in\mathbb{Z}} \frac{1}{(1+|l|)^{\kappa+1}(1+|n-l|)^{\kappa+3}}\,, \qquad \sum_{k\in\mathbb{Z}} \frac{1}{(1+|k|)^{\kappa+3}(1+|n-k|)^{\kappa+1}}\,. \tag{33}$$

Consequently, we have for all $(n,k,l) \in \mathbb{Z}^3$:

$$|l w^a_l(r)\partial_r\gamma^b_{n-l}(r)| \le \frac{1}{r^{3+2\alpha}}\frac{\|(w^a_n)_{n\in\mathbb{Z}}\ ;\ \mathcal{B}_{\alpha+2,\kappa+2}\|\,\|(\partial_r\gamma^b_n)_{n\in\mathbb{Z}}\ ;\ \mathcal{B}_{\alpha+1,\kappa+3}\|}{(1+|l|)^{\kappa+1}(1+|n-l|)^{\kappa+3}}\,,$$

and

$$|k\gamma^a_k(r)\partial_r w^b_{n-k}(r)| \le \frac{1}{r^{3+2\alpha}}\frac{\|(\gamma^a_n)_{n\in\mathbb{Z}}\ ;\ \mathcal{B}_{\alpha,\kappa+4}\|\,\|(\partial_r w^b_n)_{n\in\mathbb{Z}}\ ;\ \mathcal{B}_{\alpha+3,\kappa+1}\|}{(1+|k|)^{\kappa+3}(1+|n-k|)^{\kappa+1}}\,,$$

and therefore the series defining F_n is converging. We now bound the series (33). By symmetry, it is sufficient to consider only the first series and $n \ge 0$. We split the sum into two parts:

$$\begin{aligned} S^-_{n/2} &= \sum_{l=-\infty}^{[n/2]} \frac{1}{(1+|l|)^{\kappa+1}(1+|n-l|)^{\kappa+3}} \\ &\le C_\kappa\left(\frac{1}{1+|n|}\right)^{\kappa+3}\sum_{l\in\mathbb{Z}}\frac{1}{(1+|l|)^{\kappa+1}} \le C_\kappa\left(\frac{1}{1+|n|}\right)^{\kappa+3}\,, \end{aligned}$$

and

$$
\begin{aligned}
S^+_{n/2} &= \sum_{[n/2]+1}^{\infty} \frac{1}{(1+|l|)^{\kappa+1}(1+|n-l|)^{\kappa+3}} \\
&\le C_\kappa \left(\frac{1}{1+|n|}\right)^{\kappa+1} \sum_{l\in\mathbb{Z}} \frac{1}{(1+|l|)^{\kappa+3}} \le C_\kappa \left(\frac{1}{1+|n|}\right)^{\kappa+1} .
\end{aligned}
$$

This shows that $\hat{F} \in \mathcal{B}_{4+2\alpha,\kappa+1}$. ■

5.1 Proof of Lemma 7

From now on, we assume $\kappa > 0$. We recall that:

$$\zeta_n := \left[n^2 + i\mu n\right]^{\frac{1}{2}} , \quad \forall\, n \in \mathbb{Z} .$$

In what follows, we use without mention the following properties of ζ_n:

$$|\zeta_n| = |n| \left(1 + \left(\frac{\mu}{n}\right)^2\right)^{\frac{1}{4}} , \quad \forall\, n \in \mathbb{Z} \setminus \{0\} ,$$

and

$$
\left.
\begin{aligned}
\xi_n := \mathcal{R}e(\zeta_n) &= \frac{|n|}{\sqrt{2}} \left[\left(1 + \left(\frac{\mu}{n}\right)^2\right)^{\frac{1}{2}} + 1\right]^{\frac{1}{2}} , \\
\mathcal{I}m(\zeta_n) &= \frac{n}{\sqrt{2}} \left[\left(1 + \left(\frac{\mu}{n}\right)^2\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}} ,
\end{aligned}
\right. \quad \forall\, n \in \mathbb{Z} \setminus \{0\} . \tag{34}
$$

We note that ξ_n is an increasing function of $|n|$ so that its minimal value (over $n \in \mathbb{Z} \setminus \{0\}$) is reached for $n = \pm 1$ and is equal to:

$$\rho_\mu := \frac{1}{\sqrt{2}} \left[\left(1 + \mu^2\right)^{\frac{1}{2}} + 1\right]^{\frac{1}{2}} .$$

Let

$$\alpha_\mu := \frac{1}{2} \min(\rho_\mu - 2, 1) .$$

For $\mu > \mu_{crit}$ we have $\rho_\mu > 2$, so that $\alpha_\mu > 0$. We choose $\alpha \in (0, \alpha_\mu)$ from now on. This is the smallness condition that is mentioned in Lemma 7.

With the above conventions, we first analyze the equations which determine $\hat{w}$:

Proposition 14 *Given $\hat{F} \in \mathcal{B}_{2\alpha+4,\kappa}$ and $\hat{w}^* \in \mathcal{B}^0_{\kappa+2}$, the equations:*

$$w_n(r) = \frac{\overline{w}_n}{r^{\zeta_n}} + \int_r^\infty \frac{sF_n(s)}{2\zeta_n} \left(\frac{r}{s}\right)^{\zeta_n} \mathrm{d}s + \int_1^r \frac{sF_n(s)}{2\zeta_n} \left(\frac{s}{r}\right)^{\zeta_n} \mathrm{d}s , \tag{35}$$

$$w_0(r) = -\int_r^\infty \frac{1}{s} \int_s^\infty t\, F_0(t)\, \mathrm{d}t\, \mathrm{d}s , \tag{36}$$

with:

$$\overline{w}_n = w^*_n - \int_1^\infty \frac{sF_n(s)}{2\zeta_n} \left(\frac{1}{s}\right)^{\zeta_n} \mathrm{d}s ,$$

define C^2 functions. Moreover, we have $\hat{w} \in \mathcal{U}^2_{\alpha+2,\kappa+2}$, and there exists a constant $C_{\alpha,\mu} < \infty$, depending only on α and μ, such that

$$\|(w_n)_{n\in\mathbb{Z}} \ ; \ \mathcal{U}^2_{\alpha+2,\kappa+2}\| \le C_{\alpha,\mu} \left[\|(F_n)_{n\in\mathbb{Z}} \ ; \ \mathcal{B}_{4+2\alpha,\kappa+2}\| + \|(w^*_n)_{n\in\mathbb{Z}} \ ; \ \mathcal{B}^0_{\kappa+2}\|\right] . \tag{37}$$

Proof. We only prove (37); existence and continuity follow in a straightforward way. We first treat the case $n \neq 0$ and then the case $n = 0$. Throughout the proof, we use the shorthand M_F for $\|(F_n)_{n\in\mathbb{Z}}\ ;\ \mathcal{B}_{4+2\alpha,\kappa}\|$.

Case $n \neq 0$. We split the expression defining w_n according to (35):

$$w_n(r) = \frac{\overline{w}_n}{r^{\zeta_n}} + I_1^n(r) + I_\infty^n(r)\,.$$

By definition of the norm on the space $\mathcal{B}_{4+2\alpha,\kappa}$, we have:

$$|F_n(s)| \le \frac{M_F}{(1+|n|)^\kappa\, s^{4+2\alpha}}\,, \qquad \forall\, s \ge 1\,.$$

Using that $\xi_n \ge \rho_\mu > \alpha+2$, and that $c|n| \le |\zeta_n| \le C(1+\sqrt{|\mu|})|n|$ and $|n| \le \xi_n \le (1+\sqrt{|\mu|})|n|$ for large values of $|n|$, we get that:

$$\begin{aligned}
|I_1^n(r)| &= \left|\int_1^r \frac{sF_n(s)}{2\zeta_n}\left(\frac{s}{r}\right)^{\zeta_n}\,\mathrm{d}s\right| \\
&\le \frac{M_F}{(1+|n|)^\kappa|\zeta_n|}\,\frac{1}{r^{\xi_n}}\int_1^r s^{1+\xi_n-(4+2\alpha)}\,\mathrm{d}s \\
&\le C_{\alpha,\mu}\,\frac{M_F}{(1+|n|)^{\kappa+2}}\,\frac{\left(1+r^{\xi_n-(2\alpha+2)}\right)}{r^{\xi_n}}\,, \qquad (38)
\end{aligned}$$

for all $r \ge 1$. Here we used that, by our smallness condition on α, we have $\xi_n - (2\alpha+2) > -1$. We also have:

$$\begin{aligned}
|I_\infty^n(r)| &= \left|\int_r^\infty \frac{sF_n(s)}{2\zeta_n}\left(\frac{r}{s}\right)^{\zeta_n}\,\mathrm{d}s\right| \\
&\le \frac{M_F}{(1+|n|)^\kappa|\zeta_n|}\,r^{\xi_n}\int_r^\infty s^{1-\xi_n-(2\alpha+4)}\,\mathrm{d}s \\
&\le C_{\alpha,\mu}\,\frac{M_F}{(1+|n|)^{\kappa+2}}\,\frac{1}{r^{2\alpha+2}}\,, \qquad (39)
\end{aligned}$$

for all $r \ge 1$. Using these bounds for $r = 1$, we obtain:

$$|\overline{w}_n| \le |w_n^*| + C_{\alpha,\mu}\,\frac{M_F}{(1+|n|)^{\kappa+2}}\,. \qquad (40)$$

Plugging (38), (39) and (40) into (35) and recalling that $\xi_n > \alpha+2$ yields:

$$|w_n(r)| \le C_{\alpha,\mu}\,\frac{M_F + \|\hat{w}^*\,;\,\mathcal{B}^0_{\kappa+2}\|}{(1+|n|)^{\kappa+2}}\,\frac{1}{r^{\alpha+2}}\,, \qquad \forall\, r \ge 1\,.$$

Differentiating (35) with respect to r, we obtain:

$$\partial_r w_n(r) = -\frac{\zeta_n\overline{w}_n}{r^{\zeta_n+1}} + \frac{\zeta_n}{r}I_1^n(r) - \frac{\zeta_n}{r}I_\infty^n(r)\,, \quad \forall\, r \ge 1\,.$$

To summarize, when we differentiate w_n with respect to r, the decay in r increases by one power, and the decay in n decreases by one power. This observation allows us to bound $\partial_r w_n$ in the indicated function spaces. Finally, since the expression defining $\hat{w}$ define a solution of (16), we plug the bounds on w_n and $\partial_r w_n$ into this equation and get a bound for $\partial_{rr} w_n(r)$. We obtain that there exists a constant $C_{\alpha,\mu}$, depending only on α and μ, such that

$$\frac{r^2|\partial_{rr}w_n(r)|}{(1+|n|)^2} + \frac{r|\partial_r w_n(r)|}{(1+|n|)} + |w_n(r)| \le C_{\alpha,\mu}\,\frac{M_F + \|\hat{w}^*\,;\,\mathcal{B}^0_{\kappa+2}\|}{(1+|n|)^{\kappa+2}}\,\frac{1}{r^{\alpha+2}}\,, \qquad \forall\, r \ge 1\,.$$

We emphasize here that the constant $C_{\alpha,\mu}$ depends on α and μ. Nevertheless, it is clear from the computations above that, when α is fixed and μ varies in a a compact interval $I \subset \mathbb{R}$, this constant remains uniformly bounded.

Case $n = 0$**.** Proceeding as in the case $n \neq 0$, we get the bound:

$$|w_0(r)| \le M_F \int_r^\infty \frac{1}{s} \int_s^\infty \frac{1}{t^{4+2\alpha}}\, \mathrm{d}t\, \mathrm{d}s \le C_\alpha \frac{M_F}{r^{2+\alpha}}\,, \quad \forall\, r \ge 1\,.$$

Similarly, one shows

$$|\partial_r w_0(r)| \le \frac{M_F}{r} \int_r^\infty \frac{1}{s^{4+2\alpha}}\, \mathrm{d}s \, \le C_\alpha \frac{M_F}{r^{3+\alpha}}\,, \quad \forall\, r \ge 1\,,$$

and we again conclude, by recalling the differential equation satisfied by w_0 (see (16) for $n = 0$), that:

$$r^2|\partial_{rr} w_0(r)| + r|\partial_r w_0(r)| + |w_0(r)| \le C_\alpha \frac{M_F + \|\hat{w}^* \,;\, \mathcal{B}^0_{\kappa+2}\|}{r^{2+\alpha}}\,, \qquad \forall\, r \ge 1\,.$$

This completes the proof. ■

We next consider the equation satisfied by $\hat{\gamma}$:

Proposition 15 *Given $\hat{\phi} \in \mathcal{B}_{\alpha+2,\kappa+2}$ and $\hat{\gamma}^* \in \mathcal{B}^0_{\kappa+4}$, the equations:*

$$\gamma_n(r) = \frac{\overline{\gamma}_n}{r^{|n|}} - \int_r^\infty \frac{s\phi_n(s)}{2|n|} \left(\frac{r}{s}\right)^{|n|} \mathrm{d}s - \int_1^r \frac{s\phi_n(s)}{2|n|} \left(\frac{s}{r}\right)^{|n|} \mathrm{d}s\,, \tag{41}$$

$$\gamma_0(r) = \int_r^\infty \frac{1}{s} \int_s^\infty t\, \phi_0(t)\, \mathrm{d}t\, \mathrm{d}s\,, \tag{42}$$

with

$$\overline{\gamma}_n = \gamma^*_n + \int_1^\infty \frac{s\phi_n(s)}{2|n|} \left(\frac{1}{s}\right)^{|n|} \mathrm{d}s\,,$$

define C^2 functions. Moreover, $\hat{\gamma} \in \mathcal{U}^2_{\alpha,\kappa+4}$ and there exists a constant $C_\alpha < \infty$, depending only on α, such that

$$\|(\gamma_n)_{n\in\mathbb{Z}} \;;\; \mathcal{U}^2_{\alpha,\kappa+4}\| \le C_\alpha \left[\|(\phi_n)_{n\in\mathbb{Z}} \;;\; \mathcal{B}_{\alpha+2,\kappa+2}\| + \|(\gamma^*_n)_{n\in\mathbb{Z}} \;;\; \mathcal{B}^0_{\kappa+4}\|\right]\,. \tag{43}$$

The proof is identical to the proof of Proposition 14 and is left to the reader. Lemma 7 is a straightforward consequence of Proposition 15 and Proposition 14.

5.2 Proof of Lemma 12, first item

Let $I = [\mu_-, \mu_+] \subset (\mu_{crit}, \infty)$ and α be given by (31). In particular, we have $\alpha < \min\{\alpha_\mu, \mu \in I\}$ so that, applying the results of the preceding section, it follows that $\mathcal{S}_\mu \colon \mathcal{B}_{4+2\alpha,\kappa} \times (\mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2}) \longrightarrow \mathcal{U}^2_{\alpha,\kappa+4} \times \mathcal{U}^2_{\alpha+2,\kappa+2}$ is a well-defined continuous linear map, for all values of $\mu \in I$. We now show that the map $\mathcal{S} \colon \mu \mapsto \mathcal{S}_\mu$ is also continuous. This amounts to show that, for arbitrary $\mu_0 \in I$ and $\mu \in I$, there exists a constant C_μ which converges to zero as μ converges to μ_0, such that, for arbitrary $(\hat{F}, \hat{x}^*) \in \mathcal{B}_{4+2\alpha,\kappa} \times (\mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2})$:

$$\|\mathcal{S}_{\mu_0}(\hat{F}, \hat{x}^*) - \mathcal{S}_\mu(\hat{F}, \hat{x}^*)\,;\, \mathcal{U}^2_{\alpha,\kappa+4} \times \mathcal{U}^2_{\alpha+2,\kappa+2}\| \le C_\mu \|(\hat{F}, \hat{x}^*)\,;\, \mathcal{B}_{4+2\alpha,\kappa} \times (\mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2})\|\,.$$

Given $\mu \in I$, we let $(\hat{\gamma}[\mu], \hat{w}[\mu]) := \mathcal{S}_\mu(\hat{F}, \hat{x}^*)$. Since $\hat{\gamma}[\mu]$ is obtained from $\hat{w}[\mu]$ *via* an equation which does not depend on μ, we can apply directly Proposition 14, yielding that, for arbitrary $(\mu, \tilde{\mu}) \in I^2$:

$$\|\hat{\gamma}[\mu] - \hat{\gamma}[\tilde{\mu}]\,;\, \mathcal{U}^2_{\alpha,\kappa+4}\| \le C_\alpha \|\hat{w}[\mu] - \hat{w}[\tilde{\mu}]\,;\, \mathcal{U}^2_{\alpha+2,\kappa+2}\|\,.$$

Hence, it suffices to prove that $\hat{w}[\mu]$ is continuous with respect to μ, in order to obtain the continuity of $\mathcal{S}$.

To show the continuity of $\hat{w}[\mu]$, we first remark that $w_0[\mu]$ does not depend μ, so that we only detail the case $n \neq 0$. Let $n \neq 0$, we split $w_n[\mu]$ into three terms:

$$w_n[\mu](r) = W_b^n[\mu](r) + I_1^n[\mu](r) + I_\infty^n[\mu](r)\,,$$

where:

$$W_b^n[\mu](r) = \frac{\bar{w}_n[\mu]}{r^{\zeta_n[\mu]}}\,,\quad I_1^n[\mu](r) = \int_1^r \frac{sF_n(s)}{2\zeta_n[\mu]} \left(\frac{s}{r}\right)^{\zeta_n[\mu]} \mathrm{d}s\,,\quad I_\infty^n[\mu](r) = \int_r^\infty \frac{sF_n(s)}{2\zeta_n[\mu]} \left(\frac{r}{s}\right)^{\zeta_n[\mu]} \mathrm{d}s\,.$$

We recall that

$$\partial_r w_n[\mu](r) = -\frac{\zeta_n[\mu]}{r} W_b^n[\mu](r) - \frac{\zeta_n[\mu]}{r} I_1^n[\mu](r) + \frac{\zeta_n[\mu]}{r} I_\infty^n[\mu](r)\,, \tag{44}$$

$$\partial_{rr} w_n[\mu](r) = F_n(r) - \frac{\partial_r w_n[\mu](r)}{r} + \frac{(n^2 + i\mu n)}{r^2} w_n[\mu](r)\,. \tag{45}$$

Note also that $\zeta_n[\mu] = (n^2 + i\mu n)^{1/2}$ is a continuous function of μ uniformly in n. Indeed, since the square root is analytic in a neighborhood of 1, we have for sufficiently large n (uniformly in $\mu \in I$):

$$|\zeta_n[\mu] - \zeta_n[\tilde{\mu}]| = |n| \left| \left(1 + \frac{i\mu}{n}\right)^{\frac{1}{2}} - \left(1 + \frac{i\tilde{\mu}}{n}\right)^{\frac{1}{2}} \right| \leq C|\mu - \tilde{\mu}|\,.$$

We also have the bound $|\zeta_n[\mu]| \leq c|n|$, with c independent of $\mu \in I$. Introducing these bounds into (44)-(45) shows that the continuity of $\hat{w}[\mu]$ follows from the continuity of $(\hat{W}_b[\mu], \hat{I}_1[\mu], \hat{I}_\infty[\mu])$ in $\mathcal{B}_{\alpha+2,\kappa+2}$. For consistency, the three sequences, which are only defined for $n \neq 0$, are completed by 0 for $n = 0$.

To begin with, we consider the continuity of $\mu \mapsto \hat{I}_1[\mu]$. Let $(\mu, \tilde{\mu}) \in I^2$, and assume that $|\zeta_n[\mu] - \zeta_n[\tilde{\mu}]| < \alpha/2$, uniformly in n. We have:

$$I_1^n[\mu] - I_1^n[\tilde{\mu}] = J_1(r) + J_2(r)\,,$$

where:

$$J_1(r) = \int_1^r \frac{sF_n(s)}{2\zeta_n[\mu]} \left[\frac{\zeta_n[\tilde{\mu}] - \zeta_n[\mu]}{\zeta_n[\tilde{\mu}]}\right] \left(\frac{s}{r}\right)^{\zeta_n[\mu]} \mathrm{d}s\,,$$

$$J_2(r) = \int_1^r \frac{sF_n(s)}{2\zeta_n[\tilde{\mu}]} \left[1 - \left(\frac{s}{r}\right)^{\zeta_n[\mu] - \zeta_n[\tilde{\mu}]}\right] \left(\frac{s}{r}\right)^{\zeta_n[\tilde{\mu}]} \mathrm{d}s\,.$$

We have, uniformly in n:

$$\left|\frac{\zeta_n[\tilde{\mu}] - \zeta_n[\mu]}{\zeta_n[\tilde{\mu}]}\right| \leq c|\zeta_n[\mu] - \zeta_n[\tilde{\mu}]| = o(1)\,,$$

$$\left|1 - \left(\frac{s}{r}\right)^{\zeta_n[\mu] - \zeta_n[\tilde{\mu}]}\right| \leq c\,|\zeta_n[\mu] - \zeta_n[\tilde{\mu}]| \ln(r) r^{|\zeta_n[\mu] - \zeta_n[\tilde{\mu}]|} = o(1) \ln(r) r^{\alpha/2}\,.$$

We introduce these uniform bounds in J_1 and J_2 and redo the computations in the proof of Proposition 14 (see (38)). We get:

$$|J_1(r)| \leq \frac{\|\hat{F}; \mathcal{B}_{4+2\alpha}\|}{(1 + |n|^{\kappa+2})} \frac{o(1)}{r^{2+2\alpha}}\,,\quad |J_2(r)| \leq \frac{\|\hat{F}; \mathcal{B}_{4+2\alpha}\|}{(1 + |n|^{\kappa+2})} \frac{o(1)\ln(r)}{r^{2+3\alpha/2}}\,,$$

where the term $o(1)$ denotes a constant converging to 0 when $\mu - \tilde{\mu} \to 0$, uniformly in n. Finally, we have that, for all $n \in \mathbb{Z} \setminus \{0\}$:

$$|I_1^n[\mu] - I_1^n[\tilde{\mu}]| \leq \frac{\|\hat{F}; \mathcal{B}_{4+2\alpha}\|}{(1 + |n|^{\kappa+2})} \frac{o(1)}{r^{2+\alpha}}\,, \tag{46}$$

We now prove the continuity of $\mu \mapsto \hat{I}_\infty[\mu]$. For any $(\mu, \tilde{\mu}) \in I$ and $n \neq 0$, we perform a similar splitting:

$$I_\infty^n[\mu] - I_\infty^n[\tilde{\mu}] = J_1(r) + J_2(r)\,,$$

where:

$$J_1(r) = \int_r^\infty \frac{sF_n(s)}{2\zeta_n[\mu]} \left[\frac{\zeta_n[\tilde{\mu}] - \zeta_n[\mu]}{\zeta_n[\tilde{\mu}]}\right] \left(\frac{r}{s}\right)^{\zeta_n[\mu]} \mathrm{d}s\,,$$

$$J_2(r) = \int_r^\infty \frac{sF_n(s)}{2\zeta_n[\tilde{\mu}]} \left[1 - \left(\frac{r}{s}\right)^{\zeta_n[\mu]-\zeta_n[\tilde{\mu}]}\right] \left(\frac{r}{s}\right)^{\zeta_n[\tilde{\mu}]} \mathrm{d}s\,.$$

As in the preceding bound we have, uniformly in n:

$$\left|\frac{\zeta_n[\tilde{\mu}] - \zeta_n[\mu]}{\zeta_n[\tilde{\mu}]}\right| \leq c|\zeta_n[\mu] - \zeta_n[\tilde{\mu}]| = o(1)\,,$$

$$\left|1 - \left(\frac{s}{r}\right)^{\zeta_n[\mu]-\zeta_n[\tilde{\mu}]}\right| \leq o(1)\ln\left(\frac{s}{r}\right)\left(\frac{s}{r}\right)^{\alpha/2}\,,$$

where we have used that $|\zeta_n[\mu] - \zeta_n[\tilde{\mu}]| \leq \alpha/2$. We can therefore redo the computations in the proof of Proposition 14 (see (39)). This yields, for all $n \in \mathbb{Z} \setminus \{0\}$:

$$|J_1(r)| + |J_2(r)| \leq \frac{\|\hat{F}; \mathcal{B}_{4+2\alpha,\kappa}\|}{(1+|n|^{\kappa+2})} \frac{o(1)}{r^{2+\alpha}}\,.$$

As in the preceding estimate we conclude that:

$$\|\hat{I}_\infty[\mu] - \hat{I}_\infty[\tilde{\mu}]; \mathcal{B}_{\alpha_-+2,\kappa+2}\| = o(1)\|\hat{F}; \mathcal{B}_{4+2\alpha,\kappa}\|\,.$$

Finally, we prove the continuity of $\mu \mapsto \hat{W}_b[\mu]$:

$$\begin{aligned}
|W_b^n[\mu](r) - W_b^n[\tilde{\mu}](r)| &= \left|\frac{\bar{w}_n[\mu] - \bar{w}_n[\tilde{\mu}]}{r^{\zeta_n[\mu]}} + \frac{\bar{w}_n[\tilde{\mu}]}{r^{\zeta_n[\mu_-]}}\left(\frac{1}{r^{\zeta_n[\mu]-\zeta_n[\mu_-]}} - \frac{1}{r^{\zeta_n[\tilde{\mu}]-\zeta_n[\mu_-]}}\right)\right|\,,\\
&\leq \left|\frac{\bar{w}_n[\mu] - \bar{w}_n[\tilde{\mu}]}{r^{\zeta_n[\mu]}}\right| + \left|\frac{\bar{w}_n[\tilde{\mu}]}{r^{\zeta_n[\mu_-]}}\left(\frac{1}{r^{\zeta_n[\mu]-\zeta_n[\mu_-]}} - \frac{1}{r^{\zeta_n[\tilde{\mu}]-\zeta_n[\mu_-]}}\right)\right|\,,\\
&\leq \frac{|\bar{w}_n[\mu] - \bar{w}_n[\tilde{\mu}]|}{r^{2+2\alpha}} + \frac{|\bar{w}_n[\tilde{\mu}]|}{r^{2+\alpha}}\frac{1}{r^\alpha}\left|\left(\frac{1}{r^{\zeta_n[\mu]-\zeta_n[\mu_-]}} - \frac{1}{r^{\zeta_n[\tilde{\mu}]-\zeta_n[\mu_-]}}\right)\right|\,,
\end{aligned}$$

where we have used that $\mathcal{R}e(\zeta_n[\mu]) \geq \mathcal{R}e(\zeta_n[\mu_-]) > 2 + 2\alpha$. At this point we note that the bound which we obtained above for I_∞^n in $r = 1$ yields:

$$|\bar{w}_n[\mu] - \bar{w}_n[\tilde{\mu}]| = \frac{o(1)}{(1+|n|)^{\kappa+2}}\|\hat{F}; \mathcal{B}_{4+2\alpha,\kappa}\|\,.$$

As $\mu \mapsto \zeta_n[\mu]$ is continuous in μ (uniformly in n) and satisfies $\mathcal{R}e(\zeta_n[\mu]) \geq \mathcal{R}e(\zeta_n[\mu_-])$, for all $\mu \in I$, we also have:

$$\left\|\frac{1}{r^\alpha}\left(\frac{1}{r^{\zeta_n[\mu]-\zeta_n[\mu_-]}} - \frac{1}{r^{\zeta_n[\tilde{\mu}]-\zeta_n[\mu_-]}}\right)\,;\, L^\infty(1,\infty)\right\| = o(1)\,,$$

where $o(1)$ is uniform in n. By combination, this yields, for all $n \in \mathbb{Z} \setminus \{0\}$:

$$|W_b^n[\mu](r) - W_b^n[\tilde{\mu}](r)| \leq \frac{o(1)}{(1+|n|^{\kappa+2})r^{2+\alpha}}\left[\|\hat{F}; \mathcal{B}_{4+2\alpha,\kappa}\| + \|\hat{w}^*; \mathcal{B}^0_{\kappa+2}\|\right]\,.$$

This completes the proof of the first item in Lemma 12.

5.3 Proof of Lemma 12, second item

In this paragraph, we prove that the map $\mathcal{T}_1$ is one-to-one and onto with a continuous inverse. Given $\hat{x}^* = (\hat{\gamma}^*, \hat{w}^*) \in \mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2}$, we set $(\hat{\gamma}, \hat{w}) = \mathcal{S}_\mu(0, \hat{x}^*)$. A straightforward computation shows:

$$w_n(r) = \frac{w_n^*}{r^{\zeta_n}} \qquad \gamma_n(r) = \frac{\bar{\gamma}_n}{r^{|n|}} + \int_r^\infty \frac{s w_n(s)}{2|n|} \left(\frac{r}{s}\right)^{|n|} \mathrm{d}s + \int_1^r \frac{s w_n(s)}{2|n|} \left(\frac{s}{r}\right)^{|n|} \mathrm{d}s\,, \quad \forall\, r \geq 1\,,$$

where:

$$\bar{\gamma}_n = \gamma_n^* - \int_1^\infty \frac{s w_n(s)}{2|n|} \left(\frac{1}{s}\right)^{|n|} \mathrm{d}s\,.$$

Therefore, we have, for all $n \in \mathbb{Z} \setminus \{0\}$:

$$\gamma_n(1) = \gamma_n^*\,,$$

together with:

$$\partial_r \gamma_n(1) = -|n|\gamma_n^* + \int_1^\infty s w_n(s) \left(\frac{1}{s}\right)^{|n|} \mathrm{d}s = -|n|\gamma_n^* - \frac{w_n^*}{2 - |n| - \zeta_n}\,,$$

so that $(\hat{v}_r^*, \hat{v}_\theta^*) = \mathcal{T}_1(\hat{x}^*)$ satisfies:

- $v_{r,0}^* = v_{\theta,0}^* = 0$,
- $v_{r,n}^* = in\gamma_n^*$, and $v_{\theta,n}^* = -\partial_r \gamma_n(1) = |n|\gamma_n^* + \dfrac{w_n^*}{2 - |n| - \zeta_n}$, for all $n \in \mathbb{Z} \setminus \{0\}$.

This shows that the map $\mathcal{T}_1$ is one-to-one and onto. Indeed, the inverse map is given by:

$$\mathcal{T}_1^{-1}[\hat{v}_r', \hat{v}_\theta')] = (\hat{\gamma}', \hat{w}')\,,$$

where:

- $\gamma_0' = w_0' = 0$,
- $\gamma_n' = \dfrac{v_{r,n}'}{in}$, and $w_n' = (2 - |n| - \zeta_n)\left(v_{\theta,n}' - \dfrac{|n| v_{r,n}'}{in}\right)$ for all $n \in \mathbb{Z} \setminus \{0\}$.

It is therefore clear that $\mathcal{T}_1^{-1} \in \mathcal{L}_c(\mathcal{B}^0_{\kappa+3} \times \mathcal{B}^0_{\kappa+3}\ ;\ \mathcal{B}^0_{\kappa+4} \times \mathcal{B}^0_{\kappa+2})$. This completes the proof.

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