# On the stability of the Jeffery-Hamel problem

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August 27, 2012

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# Outline of the talk

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1. Jeffery-Hamel solutions
2. Problem description
3. Dynamical system
4. Existence and uniqueness
5. Asymptotic behavior

# Radial scale invariant solutions

- Radial scale invariant solutions of the incompressible Navier-Stokes system in a wedge

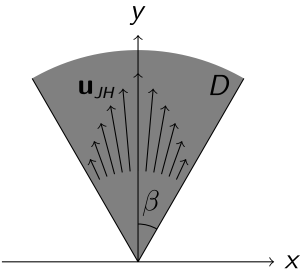
$$D = \left\{ (r\sin(\theta), r\cos(\theta)) \in \mathbb{R}^2 : \ r > 0 \text{ et } \theta \in (-\beta, \beta) \right\}.$$

- Jeffery-Hamel solution

$$\mathbf{u}_{JH}(r,\theta) = \frac{f(\theta)}{r}\mathbf{e}_r \,.$$

- Boundary condition

$$\mathbf{u}_{JH}\big|_{\partial D\setminus\{\mathbf{0}\}} = \mathbf{0}\,.$$



# Regularization of Jeffery-Hamel solutions

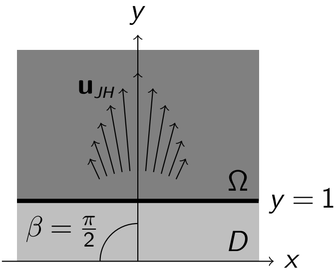
- From now on, consider $\beta = \pi/2$, *i.e.* the half-plane.
- Jeffery-Hamel solutions are singular at the origin.

- Choice of regularization : restrict the domain from $D$ to the half-space

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : \ y > 1\right\}.$$

- Singularity replaced by the inhomogeneous boundary condition

$$\mathbf{u}\big|_{\partial\Omega} = \mathbf{u}_{JH}\big|_{\partial\Omega}.$$



# Invariant quantities

Let $\mathbf{u}_{JH} = (u_{JH}, v_{JH})$.
Invariant quantity:

- Flux

$$\Phi_{JH} = \int_{\mathbb{R}} v_{JH}(x,1)\,\mathrm{d}x = \int_{\mathbb{R}} v_{JH}(x,y)\,\mathrm{d}x = \lim_{y\to\infty} \int_{\mathbb{R}} v_{JH}(x,y)\,\mathrm{d}x\,.$$

Supplementary invariant of Jeffery-Hamel solutions:

- Asymmetry

$$A_{JH} = \int_{\mathbb{R}} u_{JH}(x,1)\,\mathrm{d}x = \int_{\mathbb{R}} u_{JH}(x,y)\,\mathrm{d}x = \lim_{y\to\infty} \int_{\mathbb{R}} u_{JH}(x,y)\,\mathrm{d}x\,.$$

**Remark**

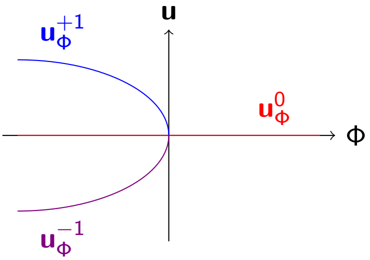
Motivation to analyze the problem as a dynamical system with $y$ playing the role of time.

# Jeffery-Hamel solutions with small flux

**Proposition**

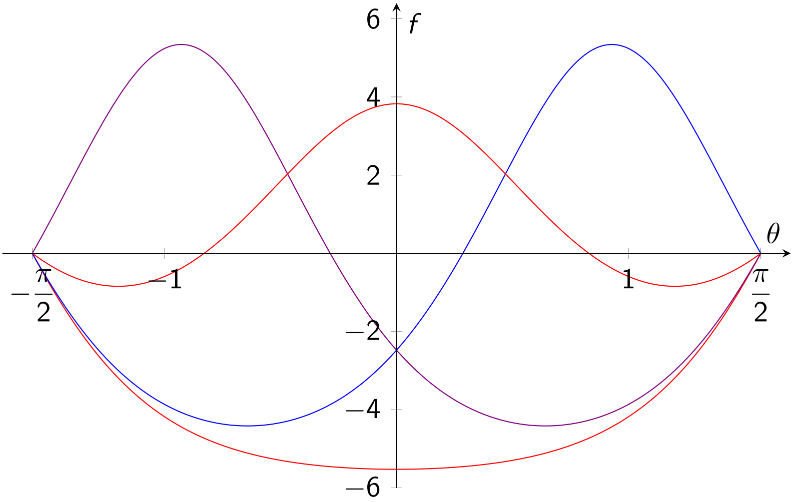
*For any small enough flux $\Phi$, there exists a symmetric Jeffery-Hamel solution $\mathbf{u}_\Phi^0$. Moreover, if $\Phi < 0$, there exist two quasi-antisymmetric solutions $\mathbf{u}_\Phi^{\pm 1}$.*

Tri-critical bifurcation at $\Phi = 0$:



# Graph of Jeffery-Hamel solutions

$f'' + f^2 + 4f = C$



# Outline of the talk

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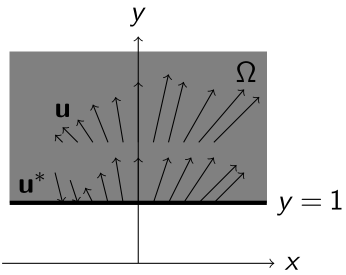
- Well-posedness of Navier-Stokes system in the half-space with fluid transport to infinity:




- Navier-Stokes and continuity equation:

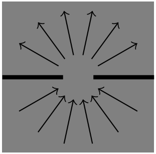
$$\Delta \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = 0\,, \qquad \nabla\cdot\mathbf{u} = 0\,.$$

- Boundary condition:

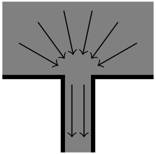
$$\mathbf{u}\big|_{\partial\Omega} = \mathbf{u}^*\,.$$

# History of results

- Symmetric aperture domain:
  G.P. Galdi, M. Padula & V.A. Solonnikov, *Existence, uniqueness and asymptotic behaviour of solutions of steady-state Navier-Stokes equations in a plane aperture domain*, Indiana University Mathematics Journal (1996)



- Half-plane channel junction:
  S. Nazarov, A. Sequeira & J. Videman, *Steady flows of Jeffrey-Hamel type from the half-plane into an infinite channel*, Journal de Mathématiques Pures et Appliquées (2001/2002)



- Open problem: asymmetric cases.

# Method of resolution

1. Write the Navier-Stokes equation as a Stokes system with an inhomogeneous term.
2. Rewrite the Stokes system as a dynamical system in $y$.
3. Fourier transform in $x$.
4. Integral equation and compatibility conditions for the Stokes system.
5. Fixed point argument to prove the existence of a solution for the Navier-Stokes equation.

# Outline of the talk

# Inhomogeneous Stokes system

- Stokes system

$$\Delta \mathbf{u} - \nabla p = \nabla \cdot \mathbf{Q}\,, \qquad \nabla \cdot \mathbf{u} = 0\,, \qquad \mathbf{u}|_{\partial\Omega} = \mathbf{u}^*\,.$$

- The Navier-Stokes equation corresponds to $\mathbf{Q} = \mathbf{u} \otimes \mathbf{u}$.
- Vorticity

$$\omega = \nabla \wedge \mathbf{u}\,.$$

- Stokes equation for the vorticity

$$\Delta\omega = \left(\partial_x^2 - \partial_y^2\right) Q_{12} + \partial_x \partial_y \left(Q_{22} - Q_{11}\right)\,.$$

# Dynamical system

- By defining $\gamma = \omega + Q_{12}$, the Stokes system becomes

$$\partial_y u = \partial_x v - \gamma + Q_{12}\,, \qquad \partial_y \gamma = \partial_x \eta + \partial_x \left(Q_{22} - Q_{11}\right)$$
$$\partial_y v = -\partial_x u\,, \qquad \partial_y \eta = -\partial_x \gamma + 2\partial_x Q_{12}\,.$$

- Fourier transform in $x$:

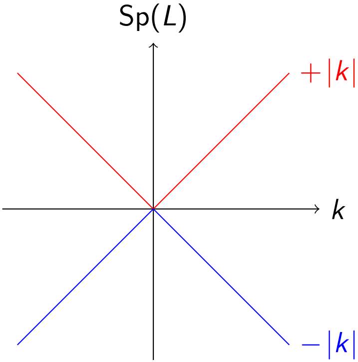
$$\partial_y \hat{\mathbf{r}} = L\hat{\mathbf{r}} + \hat{\mathbf{q}}\,, \qquad \hat{\mathbf{u}}(k,1) = \hat{\mathbf{u}}^*\,,$$

where

$$\hat{\mathbf{r}} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\gamma} \\ \hat{\eta} \end{pmatrix}, \quad L = \begin{pmatrix} 0 & ik & -1 & 0 \\ -ik & 0 & 0 & 0 \\ 0 & 0 & 0 & ik \\ 0 & 0 & -ik & 0 \end{pmatrix}, \quad \hat{\mathbf{q}} = \begin{pmatrix} \hat{Q}_{12} \\ 0 \\ 2ik\left(\hat{Q}_{22} - \hat{Q}_{11}\right) \\ 2ik\hat{Q}_{12} \end{pmatrix}.$$

# Spectrum of $L$

- Spectrum of $L$ consists of two stable branches and two unstable ones:



# Integral equations

- Bounded solutions: projection $P$ onto the stables branches.
- Integral equation

$$\hat{\mathbf{r}}(k,y) = \mathrm{e}^{L(y-1)}\hat{\mathbf{r}}_s(k) + \int_1^y P\mathrm{e}^{L(y-z)}\hat{\mathbf{q}}(k,z)\mathrm{d}z - \int_y^\infty (1-P)\,\mathrm{e}^{L(y-z)}\hat{\mathbf{q}}(k,z)\mathrm{d}z$$

$$\hat{\mathbf{r}}_s(k) = \int_1^\infty (1-P)\,\mathrm{e}^{L(1-z)}\hat{\mathbf{q}}(k,z)\mathrm{d}z + \hat{\mathbf{r}}^*(k)\,,$$

where $\hat{\mathbf{r}}^*$ satisfies the boundary conditions.

- Velocity field

$$\hat{\mathbf{u}} = \mathcal{S}(\hat{\mathbf{Q}},\hat{\mathbf{u}}^*)\,.$$

# Outline of the talk

# Function spaces

- Let $\mathcal{B}_{\alpha,p}$ be the closure of $C_0^\infty(\mathbf{R} \times [1,\infty))$ with respect to the weighted norm
$$\left\|\hat{f};\mathcal{B}_{\alpha,p}\right\| = \sup_{\mathbb{R}\times[1;\infty)} \frac{|\hat{f}|}{\mu_{\alpha,p}},$$
where the weight is given by
$$\mu_{\alpha,p}(k,y) = \frac{1}{y^p}\frac{1}{1+(|k|y)^\alpha}.$$
- Let $\mathcal{A}_{\alpha,p}$ the closure of $C_0^\infty(\mathbf{R})$ with respect to the weighted norm
$$\left\|\hat{f};\mathcal{A}_{\alpha,p}\right\| = \sup_{\mathbb{R}} \frac{|\hat{f}|}{\eta_{\alpha,p}} \qquad \text{with} \qquad \eta_{\alpha,p}(k) = \frac{|k|^p}{1+|k|^{\alpha+p}}.$$

# Stokes solutions

## Theorem

*For any $\alpha > 1$ and $p \geq 0$, the Stokes operator $\mathcal{S} : \mathcal{R}^4_{\alpha,p} \times \mathcal{T}^2_{\alpha,p} \to \mathcal{V}^2_{\alpha,p}$ , $\hat{\mathbf{u}} = \mathcal{S}(\hat{\mathbf{Q}}, \hat{\mathbf{u}}^*)$, is well-defined and continuous, provided the compatibility condition $\hat{\mathbf{u}}_s \in \mathcal{U}^2_{\alpha,p}$ holds.*

## Corollary

*For $p = 0$, there is no compatibility condition, for $0 < p \leq 1$, there is two compatibility conditions*

$$\hat{u}^*(0) + \int_1^\infty \hat{Q}_{12}(0, y)\,\mathrm{d}y = 0\,, \qquad \hat{v}^*(0) = 0\,,$$

*and for $1 < p \leq 2$, one has in addition a condition for $\partial_k \hat{\mathbf{u}}^*(0)$.*

# Existence of strong solutions

**Theorem (existence)**

*For all $\alpha > 3$, and all sufficiently small boundary condition $\hat{\mathbf{u}}^* \in \mathcal{T}^2_{\alpha,0}$, there exists a strong solution $u \in C^2(\Omega)$ of the Navier-Stokes equation which satisfies $y\mathbf{u} \in L^\infty(\Omega)$.*

**Proof.**

1. Navier-Stokes equation
$$\hat{\mathbf{u}} = \mathcal{S}(\hat{\mathbf{Q}}(\hat{\mathbf{u}},\hat{\mathbf{u}}),\hat{\mathbf{u}}^*)\,, \qquad \hat{\mathbf{Q}}(\hat{\mathbf{u}},\hat{\mathbf{u}}) = \hat{\mathbf{u}} \otimes \hat{\mathbf{u}}\,.$$
2. Continuity of the operators $\mathcal{S} : \mathcal{R}^4_{\alpha,0} \times \mathcal{T}^2_{\alpha,0} \to \mathcal{V}^2_{\alpha,0}$ and $\hat{\mathbf{Q}} : \mathcal{V}^2_{\alpha,0} \times \mathcal{V}^2_{\alpha,0} \to \mathcal{R}^4_{\alpha,0}$ with no compatibility conditions.
3. Fixed point argument.
4. Construction of the pressure.
5. Inverse Fourier transform.

$\square$

# Weak-strong uniqueness

**Theorem (existence of weak solutions)**

*For any boundary condition* $\mathbf{u}^*$ *which admits an extension* $\mathbf{a}$ *with* $\|\nabla \mathbf{a}\|_2 + \|y\mathbf{a}\|_\infty \leq \frac{1}{4}$, *there exists a weak solution* $\mathbf{u}$ *in* $\Omega$.

**Theorem (weak-strong uniqueness)**

*For any small enough boundary condition* $\mathbf{u}^*$, *any weak-solution coincides with the strong solution. In particular any weak solution with small enough boundary data satisfies* $y\mathbf{u} \in L^\infty(\Omega)$.

**Proof.**

Similar to: M. Hillairet & P. Wittwer, *Asymptotic description of solutions of the exterior Navier-Stokes problem in a half space*, Archive for Rational Mechanics and Analysis (2012) $\square$

# Outline of the talk

# Asymptotic behavior: difficulties and work-around

- The asymptotic behavior can not be extracted from the existence theorem.
- Search solutions of the form $\hat{\mathbf{u}} = \hat{\mathbf{u}}^0_\Phi + \tilde{\mathbf{u}}$.
- Since $\hat{\mathbf{u}}^0_\Phi \in \mathcal{V}_{\alpha,0}$, to detach the asymptotics from $\hat{\mathbf{u}}$ we have to take $\tilde{\mathbf{u}} \in \mathcal{V}_{\alpha,p}$ with $p > 0$.

## Problem

The Stokes operator $\mathcal{S}$ has one compatibility condition for $0 < p < 1$. We need an additional real parameter $\nu$, otherwise, we can only prove existence for the boundary conditions $\tilde{\mathbf{u}}^*$ in a subspace of co-dimension one.

# Invariant quantities

Two quantities influence the decay of solutions as a function of $y$:

- The flux

$$\Phi = \int_{\mathbb{R}} v(x,1)\mathrm{d}x = \int_{\mathbb{R}} v(x,y)\mathrm{d}x = \lim_{y\to\infty}\int_{\mathbb{R}} v(x,y)\mathrm{d}x\,.$$

- The asymmetry

$$A = \int_{\mathbb{R}} v(x,1)\mathrm{d}x \neq \int_{\mathbb{R}} v(x,y)\mathrm{d}x \neq \lim_{y\to\infty}\int_{\mathbb{R}} v(x,y)\mathrm{d}x\,.$$

**Remark**

To find the asymptotic behavior, we add a real parameter $\nu$ which allows to interpolate between the asymmetry at $y=1$ and the one at $y\to\infty$.

# Asymptotic behavior

## Theorem

*For any Jeffery-Hamel solution $\mathbf{u}_\Phi^0$ with flux $\Phi$ small enough, and all boundary conditions of the form*

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_\Phi^0\Big|_{\partial\Omega} + \lambda \mathbf{u}_1^\nu|_{\partial\Omega} + \mathbf{u}_2^*\,, \quad \int_{\partial\Omega} \mathbf{u}_1^\nu = (1,0)\,, \quad \int_{\partial\Omega} \mathbf{u}_2^* = \mathbf{0}\,,$$

*with*

$$|\lambda| \lesssim \Phi^2\,, \qquad \left\|\hat{\mathbf{u}}_2^*; \mathcal{U}_{\alpha,p}^2\right\| \lesssim |\lambda|/\Phi\,,$$

*there exists a solution $\mathbf{u}$ of the Navier-Stokes equation with*

$$\lim_{y\to\infty} y \left(\sup_{x\in\mathbb{R}} \left|\mathbf{u} - \mathbf{u}_\Phi^0\right|\right) = 0\,.$$

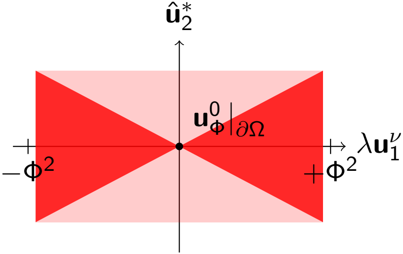
# Admissible boundary conditions

## Problem

The size of $\mathbf{u}_1^\nu$ is large compared to its asymmetry:

$$\|\lambda \hat{\mathbf{u}}_1^\nu\| \lesssim |\lambda| / |\Phi| \,, \qquad \int_{\partial\Omega} \lambda \mathbf{u}_1^\nu = (\lambda, 0) \,, \qquad |\lambda| \lesssim \Phi^2 \,.$$

Sketch of admissible boundary conditions:



# Conclusions

1. The existence theorem in $\mathcal{V}_{\alpha,0}$ reduces the possible asymptotic behavior to $y\mathbf{u} \in L^\infty(\Omega)$.
2. Spatial stability of the symmetric Jeffery-Hamel in a wedge.
3. Inversion of more than just the Stokes system. See M. Hillairet and P. Wittwer, *Asymptotic description of solutions of the exterior Navier-Stokes problem in a half space*, arXiv:1107.1028v1
4. Numerical results still inconclusive.

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