Optimal asymptotic behavior of the vorticity of a viscous flow past a two-dimensional body

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This note is about the asymptotic behavior of the vorticity for the steady incompressible Navier-Stokes equations in a two-dimensional exterior domain, in the case the velocity at infinity $\boldsymbol{u}_\infty$ is non-zero. It is well known that the asymptotic behavior of the velocity field is given by the fundamental solution of the Oseen system which is the linearization of the Navier-Stokes equation around $\boldsymbol{u}_\infty$. The vorticity has the property of decaying algebraically inside a parabolic region called the wake and exponentially outside. The previously proven asymptotic expansions of the vorticity are relevant only inside the wake because everywhere else the remainder is larger than the asymptotic term. Here we present an asymptotic expansion that removes this weakness. Surprisingly, the found asymptotic term is not given by the Oseen linearization and has a power of decay that depends on the data. This strange behavior is specific to the two dimensional problem and is not present in three dimensions.

1 Introduction

The stationary flow of an incompressible fluid past a body B is described by the Navier-Stokes equations,

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{\nabla} p &= \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{f}\,, & \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0\,, \\ \boldsymbol{u}|_{\partial B} &= \boldsymbol{u}^*\,, & \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u} &= \boldsymbol{u}_\infty\,, \end{aligned} \tag{1a}$$

in the domain $\Omega = \mathbb{R}^2 \setminus \bar{B}$, where $\boldsymbol{f}$ is the source force, $\boldsymbol{u}_\infty \neq \boldsymbol{0}$ the velocity at infinity and $\boldsymbol{u}^*$ is any boundary condition with no net flux,

$$\int_{\partial B} \boldsymbol{u}^* \cdot \boldsymbol{n} = 0\,. \tag{1b}$$

We assume that the body B is an open bounded domain with smooth boundary. In view of the symmetries of the equation, we assume without lost of generality that $\boldsymbol{u}_\infty = 2\boldsymbol{e}_1$ and $\boldsymbol{0} \in B$.

This system has been subject to many investigations, see Galdi (2011, Chapter XII) for a complete statement of the main results known for this problem. Leray (1933) has shown the existence of weak solutions, but with the procedure he used, he was unable to verify that $\boldsymbol{u}$ tends to $\boldsymbol{u}_\infty$ at large distances. Gilbarg & Weinberger (1974, 1978) have shown that any Leray solution $\boldsymbol{u}$ either converges at large distances to some constant vector $\boldsymbol{u}_0$, or the average of $\boldsymbol{u}$ of over circles in the L^2-norm diverges as the size of the circle grows. Later on, Amick (1988) proved that if $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{u}^* = \boldsymbol{0}$, then $\boldsymbol{u} \in L^\infty$ and therefore $\boldsymbol{u}$ converges to a constant $\boldsymbol{u}_0$ at infinity. However, the question if $\boldsymbol{u}_0 = \boldsymbol{u}_\infty$ is still open in general. In case $\boldsymbol{u}_\infty \neq \boldsymbol{0}$, Finn & Smith (1967); Galdi (1993, 2004) used the Oseen approximation and a fixed point technique to prove existence and uniqueness of solutions to (1) for small data. The asymptotic structure of the solutions was presented by Babenko (1970) who shows in particular that velocity behaves at infinity like the Oseen fundamental solution. The asymptotic expansion of the velocity was also given under more general assumptions by Galdi & Sohr (1995); Sazonov (1999). The asymptotic behavior of the vorticity was first given by Babenko (1970, Theorem 8.1) only in the wake, and then by Clark (1971, Theorem 3.5'). These two results are relevant only in the wake region, *i.e.* for $|\boldsymbol{x}| - x_1 \leq 1$, because otherwise, the remainder is bigger than the asymptotic term which is given by the Oseen linearization. In fact, we prove that the true asymptote which is also valid outside the wake region is not given by the Oseen linearization. Under smallness conditions, we show that the asymptote of the vorticity is given in polar coordinates (r, θ) by

$$\omega(\boldsymbol{x}) = r^{A(1-\cos\theta)+B\sin\theta}\left[\frac{\mu(\theta)}{r^{1/2}} + O\left(\frac{1}{r^{1/2+\varepsilon}}\right)\right]\mathrm{e}^{-r(1-\cos\theta)}\,,$$

for all $\varepsilon \in (0,1)$, where $A, B \in \mathbb{R}$ depend linearly on the net force $\boldsymbol{F}$ and μ is a 2π-periodic function depending on $\boldsymbol{f}$ and $\boldsymbol{u}^*$. Surprisingly the power of decay of the asymptote depends on the net force $\boldsymbol{F}$ and in particular this contradicts the statement of Theorem XII.8.4 in Galdi (2011) for any solution with $\boldsymbol{F} \neq \boldsymbol{0}$. For $\boldsymbol{x}, \boldsymbol{x}_0 \in \mathbb{R}^2$ we use the following notation

$$\begin{aligned}
r &= |\boldsymbol{x}|\,, & r_0 &= |\boldsymbol{x}_0|\,, & r_1 &= |\boldsymbol{x}-\boldsymbol{x}_0|\,,\\
\theta &= \angle\boldsymbol{x}\,, & \theta_0 &= \angle\boldsymbol{x}_0\,, & \theta_1 &= \angle\left(\boldsymbol{x}-\boldsymbol{x}_0\right).,
\end{aligned}$$

where $\angle\boldsymbol{x}$ denotes the angle $\theta \in (-\pi, \pi]$ such that $\boldsymbol{x} = |\boldsymbol{x}|(\cos\theta, \sin\theta)$. For a positive function $w : \Omega \to \mathbb{R}$, we write $A(\boldsymbol{x}, \boldsymbol{x}_0) = O(w(\boldsymbol{x}))$ if for $\boldsymbol{x}_0$ in a bounded domain, there exists $C > 0$ such that for all $\boldsymbol{x} \in \Omega$,

$$\left|A(\boldsymbol{x}, \boldsymbol{x}_0)\right| \leq C\,|w(\boldsymbol{x})|\ .$$

2 Asymptote for the linear problem

It is well-known that the problem (1) is related to the Oseen system which is the linearization of (1) around $\boldsymbol{u} = \boldsymbol{u}_\infty = 2\boldsymbol{e}_1$,

$$\begin{aligned}
\Delta\boldsymbol{u} - \boldsymbol{\nabla}p - 2\partial_1\boldsymbol{u} &= \boldsymbol{f}\,, & \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0\,,\\
\boldsymbol{u}|_{\partial B} &= \boldsymbol{u}^* - \boldsymbol{u}_\infty\,, & \lim_{|\boldsymbol{x}|\to\infty}\boldsymbol{u} &= \boldsymbol{0}\,,
\end{aligned} \tag{2}$$

The fundamental solution of the Oseen system is given by

$$\mathbf{E} = \begin{pmatrix} \partial_1\psi - G & \partial_2\psi \\ \partial_2\psi & -\partial_1\psi \end{pmatrix}, \qquad \boldsymbol{e} = -\boldsymbol{\nabla}H\,,$$

with

$$\psi = \frac{H+G}{2}\,, \qquad H = \frac{1}{2\pi}\log r\,, \qquad G = \frac{1}{2\pi}\mathrm{e}^{r\cos\theta}K_0(r)\,.$$

Denoting by $\mathbf{T}(\boldsymbol{u}, p)$ the stress tensor,

$$\mathbf{T}(\boldsymbol{u}, p) = \boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T - p\mathbf{1}\,,$$

the Green identity for the Oseen operator is

$$\int_\Omega\left(\boldsymbol{\nabla}\cdot\mathbf{T}(\boldsymbol{u},p) - 2\partial_1\boldsymbol{u}\right)\cdot\boldsymbol{v} - \int_\Omega\left(\boldsymbol{\nabla}\cdot\mathbf{T}(\boldsymbol{v},q) + 2\partial_1\boldsymbol{v}\right)\cdot\boldsymbol{u} = \int_{\partial\Omega}\left(\boldsymbol{v}\cdot\mathbf{T}(\boldsymbol{u},p) - \boldsymbol{u}\cdot\mathbf{T}(\boldsymbol{v},q) - 2\boldsymbol{u}\cdot\boldsymbol{v}\,\boldsymbol{e}_1\right)\cdot\boldsymbol{n}\,.$$

Therefore, the solution of the Oseen system is given by

$$\boldsymbol{u}(\boldsymbol{x}) = \int_\Omega \mathbf{E}(\boldsymbol{x}-\bullet)\boldsymbol{f} - \int_{\partial\Omega}\left[\mathbf{E}(\boldsymbol{x}-\bullet)\left(\mathbf{T}(\boldsymbol{u},p) - 2\boldsymbol{u}\otimes\boldsymbol{e}_1\right) + \boldsymbol{u}\cdot\mathbf{T}(\mathbf{E},\boldsymbol{w})(\boldsymbol{x}-\bullet)\right]\cdot\boldsymbol{n}\,,$$

with

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \mathbf{T}(\mathbf{E}, \boldsymbol{w})(\boldsymbol{x} - \bullet) \cdot \boldsymbol{n} = \int_{\partial\Omega} \left[\boldsymbol{n} \cdot \boldsymbol{\nabla}\mathbf{E}(\boldsymbol{x} - \bullet) \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\mathbf{E}(\boldsymbol{x} - \bullet) \cdot \boldsymbol{n} - \boldsymbol{w}(\boldsymbol{x} - \bullet)\, \boldsymbol{u} \cdot \boldsymbol{n}\right],$$

where $\bullet$ denotes a placeholder for the argument over which the Green function is integrated. In order to obtain the representation formula for the vorticity, we remark that for any $\boldsymbol{A} \in \mathbb{R}^2$,

$$\boldsymbol{\nabla} \wedge (\mathbf{E} \cdot \boldsymbol{A}) = \boldsymbol{\nabla} G \wedge \boldsymbol{A},$$

with G as defined above, so we obtain

$$\begin{aligned} \omega(\boldsymbol{x}) = & \int_{\Omega} \boldsymbol{\nabla} G(\boldsymbol{x} - \bullet) \wedge \boldsymbol{f} - \int_{\partial\Omega} \left[\boldsymbol{\nabla} G(\boldsymbol{x} - \bullet) \wedge \left(\mathbf{T}(\boldsymbol{u}, p) - 2\boldsymbol{u} \otimes \boldsymbol{e}_1\right)\right] \cdot \boldsymbol{n} \\ & - \int_{\partial\Omega} \left[\boldsymbol{\nabla}_{\boldsymbol{x}}\left(\boldsymbol{n} \cdot \boldsymbol{\nabla} G(\boldsymbol{x} - \bullet)\right) \wedge \boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{x}}\left(\boldsymbol{u} \cdot \boldsymbol{\nabla} G(\boldsymbol{x} - \bullet)\right) \wedge \boldsymbol{n}\right]. \end{aligned}$$

The asymptotic expansions of the fundamental solutions are given by

$$\begin{aligned} \mathbf{E}(\boldsymbol{x}) &= \frac{-1}{\sqrt{32\pi}} \left(\frac{1}{r^{1/2}} + O\left(\frac{1}{r^{3/2}}\right)\right) \mathrm{e}^{-r(1-\cos\theta)} \begin{pmatrix} 1 + \cos\theta & \sin\theta \\ \sin\theta & 1 - \cos\theta \end{pmatrix} + \frac{1}{4\pi r} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \\ \boldsymbol{\nabla} G(\boldsymbol{x}) &= \frac{1}{\sqrt{8\pi}} (1 - \cos\theta, -\sin\theta) \left(\frac{1}{r^{1/2}} + O\left(\frac{1}{r^{3/2}}\right)\right) \mathrm{e}^{-r(1-\cos\theta)}, \end{aligned}$$

and for $|\alpha| = 1$,

$$\begin{aligned} |D^\alpha \mathbf{E}| &\lesssim \left(\frac{|\theta|}{r^{1/2}} + \frac{1}{r^{3/2}}\right) \mathrm{e}^{-r(1-\cos\theta)} + \frac{1}{r^2}, \\ |D^\alpha \boldsymbol{\nabla} G| &\lesssim \left(\frac{|\theta|^2}{r^{1/2}} + \frac{1}{r^{3/2}}\right) \mathrm{e}^{-r(1-\cos\theta)}. \end{aligned}$$

If $\boldsymbol{f}$ has compact support, the solution of the Oseen equation (2) behaves like the fundamental solution,

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbf{E}(\boldsymbol{x})\boldsymbol{F} + O\left(\frac{|\theta|}{r^{1/2}} + \frac{1}{r^{3/2}}\right) \mathrm{e}^{-r(1-\cos\theta)} + O\left(\frac{1}{r^2}\right), \tag{3}$$

$$\omega(\boldsymbol{x}) = \boldsymbol{\nabla} G(\boldsymbol{x}) \cdot \boldsymbol{F}^{\perp} + O\left(\frac{|\theta|^2}{r^{1/2}} + \frac{1}{r^{3/2}}\right) \mathrm{e}^{-r(1-\cos\theta)}, \tag{4}$$

where $\boldsymbol{F}$ is the net force

$$\boldsymbol{F} = \int_{\Omega} \boldsymbol{f} + \int_{\partial B} \left(\mathbf{T}(\boldsymbol{u}, p) - 2\boldsymbol{u} \otimes \boldsymbol{e}_1\right) \cdot \boldsymbol{n}.$$

Explicitly, the asymptotic expansion of the velocity is given by

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_w(\boldsymbol{x}) + \boldsymbol{u}_h(\boldsymbol{x}) + O\left(\frac{|\theta|}{r^{1/2}} + \frac{1}{r^{3/2}}\right) \mathrm{e}^{-r(1-\cos\theta)} + O\left(\frac{1}{r^2}\right), \tag{5}$$

where $\boldsymbol{u}_w$ is the wake part and $\boldsymbol{u}_h$ a harmonic function,

$$\boldsymbol{u}_w(\boldsymbol{x}) = -\frac{F_1 \boldsymbol{e}_1}{\sqrt{8\pi}} \frac{1}{r^{1/2}} \mathrm{e}^{-r(1-\cos\theta)}, \tag{6}$$

$$\boldsymbol{u}_h(\boldsymbol{x}) = \frac{1}{4\pi r} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \boldsymbol{F} = \frac{F_1}{4\pi} \frac{\boldsymbol{e}_r}{r} - \frac{F_2}{4\pi} \frac{\boldsymbol{e}_\theta}{r}. \tag{7}$$

In contrast to the asymptotic expansion (3) or (5) of the velocity field, the asymptotic expansion (4) for the vorticity is only relevant inside the wake region, because outside the wake region, the remainder is larger than the asymptotic term. In order to obtain an asymptote that is relevant in all directions, we have to proceed differently, by using the following asymptotic expansion,

$$\boldsymbol{\nabla} G(\boldsymbol{x}-\boldsymbol{x}_0) = \boldsymbol{\nabla} G(\boldsymbol{x}) \mathrm{e}^{r_0\left(\cos(\theta-\theta_0)-\cos\theta_0\right)} + O\left(\frac{1}{r^{3/2}}\right) \mathrm{e}^{-r(1-\cos\theta)} .$$

By applying this result, we obtain

$$\omega(\boldsymbol{x}) = \boldsymbol{\nabla} G(\boldsymbol{x}) \cdot \boldsymbol{F}^{\perp}(\theta) + O\left(\frac{1}{r^{3/2}}\right) \mathrm{e}^{-r(1-\cos\theta)} , \tag{8}$$

where $\boldsymbol{F}(\theta)$ is now a function depending on the angle θ,

$$\boldsymbol{F}(\theta) = \int_{\Omega} \mathrm{e}^{r_0\left(\cos(\theta-\theta_0)-\cos\theta_0\right)} \boldsymbol{f} + \int_{\partial B} \mathrm{e}^{r_0\left(\cos(\theta-\theta_0)-\cos\theta_0\right)} \left(\mathbf{T}(\boldsymbol{u},p) - 2\boldsymbol{u}\otimes\boldsymbol{e}_1\right)\cdot\boldsymbol{n} .$$

To our knowledge, this asymptotic formula is nowhere mentioned in the literature. With this expression, the asymptote is now detached in all directions from the remainder.

3 Asymptote for the nonlinear problem

For the nonlinear problem (1), neither the asymptotic behavior (5) nor (8) is correct, as shown below. The best results concerning the asymptotic behavior of $\boldsymbol{u}$ and ω for the nonlinear problem (1) are due to Babenko (1970). In particular he shows the following result:

Theorem 1 (Babenko, 1970, Theorems 6.1 & 8.1)**.** *If $\boldsymbol{u}$ is a physically reasonable solution of* (1), i.e. *such that $\boldsymbol{u}-\boldsymbol{u}_\infty = O(r^{-1/4-\varepsilon})$ for some $\varepsilon > 0$ small enough, then the velocity satisfies*

$$\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_\infty = \mathbf{E}(\boldsymbol{x})\boldsymbol{F} + O\left(\frac{|\theta|\,(\log r, 1)}{r^{1/2}} + \frac{1}{r}\right) \mathrm{e}^{-r(1-\cos\theta)} + O\left(\frac{1}{r^{1+\varepsilon}}\right) ,$$

and the vorticity satisfies

$$\omega(\boldsymbol{x}) = \boldsymbol{\nabla} G(\boldsymbol{x}) \cdot \boldsymbol{F}^{\perp} + O\left(\frac{\log r}{r^{3/2}}\right) \mathrm{e}^{-\mu r(1-\cos\theta)} ,$$

for some $\mu \in (0,1)$, where $\boldsymbol{F} \in \mathbb{R}^2$ is the net force,

$$\boldsymbol{F} = \int_{\Omega} \boldsymbol{f} + \int_{\partial B} \left(\mathbf{T}(\boldsymbol{u},p) - \boldsymbol{u}\otimes\boldsymbol{u}\right)\cdot\boldsymbol{n} .$$

This result is optimal for the velocity in a sense that the remainder decays faster than the asymptotic terms. However, for the vorticity, this result is only relevant in the wake since the remainder is greater than the asymptotic term if $\theta \neq 0$, due to the fact that $\mu \in (0,1)$. In fact, we will show that the asymptotic behavior of the vorticity is not given by the Oseen tensor outside the wake. More precisely if we consider the vorticity which is given by

$$\Delta\omega - 2\partial_1\omega = \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega + \boldsymbol{\nabla}\wedge\boldsymbol{f} ,$$

then the Oseen approximation is given by $\boldsymbol{u} = \boldsymbol{0}$. We will show that the linearization that leads to the correct asymptotic behavior of the nonlinear system is given by linearizing around the harmonic function $\boldsymbol{u}_h$ of the Oseen tensor itself (7). We will show that this new linear system has an asymptotic behavior where the power of decay itself depends on $\boldsymbol{F}$:

Theorem 2. *If there exists $\varepsilon \in (0, 1/2)$ and $A, B \in \mathbb{R}$ such that the solution $\boldsymbol{u}$ of (1) satisfies*

$$\begin{aligned} \left|\left(\boldsymbol{u} - \boldsymbol{u}_\infty - \boldsymbol{u}_h\right) \cdot \boldsymbol{e}_r\right| &\leq \frac{\nu}{r^{1/2}} \mathrm{e}^{-r(1-\cos\theta)/2} + \frac{\nu}{r^{1+\varepsilon}} \,, \\ \left|\left(\boldsymbol{u} - \boldsymbol{u}_\infty - \boldsymbol{u}_h\right) \cdot \boldsymbol{e}_\theta\right| &\leq \frac{\nu}{r} \mathrm{e}^{-r(1-\cos\theta)/2} + \frac{\nu}{r^{1+\varepsilon}} \,, \end{aligned} \tag{9}$$

where

$$\boldsymbol{u}_h = 2\boldsymbol{\nabla}\left(A \log r + B\theta\right) = \frac{2A}{r}\boldsymbol{e}_r + \frac{2B}{r}\boldsymbol{e}_\theta \,,$$

for some $\nu > 0$ small enough, then the solution of (1) with $\boldsymbol{f}$ a source term of compact support satisfies for some $C > 0$,

$$|\omega| \leq C\, r^{A(1-\cos\theta)+B\sin\theta} r^{-1/2} \mathrm{e}^{-r(1-\cos\theta)} \,.$$

Moreover,

$$\omega(\boldsymbol{x}) = r^{A(1-\cos\theta)+B\sin\theta} \left[\frac{\mu(\theta)}{r^{1/2}} + O\left(\frac{1}{r^{1/2+\varepsilon}}\right)\right] \mathrm{e}^{-r(1-\cos\theta)} \,,$$

where μ is some 2π-periodic function.

Remark 3. For $\boldsymbol{u}_\infty$, $\boldsymbol{f}$, and $\boldsymbol{u}^*$ small enough, the hypothesis on $\boldsymbol{u}$ follows from theorem 1, with

$$A = \frac{F_1}{8\pi} \,, \qquad\qquad B = -\frac{F_2}{8\pi} \,.$$

Proof. Let $\boldsymbol{u}$ be a solution of (1) which by hypothesis can be written as $\boldsymbol{u} = \boldsymbol{u}_\infty + \boldsymbol{u}_h + \bar{\boldsymbol{u}}$, with $\bar{\boldsymbol{u}}$ satisfying

$$\left|\bar{\boldsymbol{u}} \cdot \boldsymbol{e}_r\right| \leq \frac{\nu}{r^{1/2}} \mathrm{e}^{-r(1-\cos\theta)/2} + \frac{\nu}{r^{1+\varepsilon}} \,, \qquad \left|\bar{\boldsymbol{u}} \cdot \boldsymbol{e}_\theta\right| \leq \frac{\nu}{r} \mathrm{e}^{-r(1-\cos\theta)/2} + \frac{\nu}{r^{1+\varepsilon}} \,.$$

This expression is to be understood as To prove the result, we consider the vorticity equation

$$\Delta\omega - 2\partial_1\omega - \boldsymbol{u}_h \cdot \boldsymbol{\nabla}\omega = \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\omega + \boldsymbol{\nabla} \wedge \boldsymbol{f} \,. \tag{10}$$

The change of variables,

$$\omega(r,\theta) = r^{A(1-\cos\theta)+B\sin\theta} \mathrm{e}^{-r(1-\cos\theta)} a(r,\theta) \,,$$

transforms the original equation (10) into

$$\Delta a - 2\partial_r a - \frac{a}{r} = \boldsymbol{v} \cdot \boldsymbol{\nabla} a + \varphi a + R \,, \tag{11}$$

where $\boldsymbol{v}$ and φ are linearly related to $\boldsymbol{u}_h$ and $\bar{\boldsymbol{u}}$ and satisfy

$$\left|\boldsymbol{v} \cdot \boldsymbol{e}_r\right| \lesssim \frac{\nu}{r^{1/2}} \,, \qquad \left|\boldsymbol{v} \cdot \boldsymbol{e}_\theta\right| \lesssim \frac{\nu \log r}{r} \,, \qquad |\varphi| \lesssim \frac{\nu}{r^{1+\varepsilon}} \,,$$

and where the source term R is given by

$$R(r,\theta) = r^{-A(1-\cos\theta)-B\sin\theta} \mathrm{e}^{r(1-\cos\theta)} \left(\boldsymbol{\nabla} \wedge \boldsymbol{f}\right) \,.$$

We will show that the solution a of (11) satisfies the bounds

$$|a| \lesssim \frac{1}{r^{1/2}} \,, \qquad \left|\boldsymbol{e}_r \cdot \boldsymbol{\nabla} a\right| \lesssim \frac{1}{r^{1+\varepsilon}} \,, \qquad \left|\boldsymbol{e}_\theta \cdot \boldsymbol{\nabla} a\right| \lesssim \frac{1}{r} \,, \tag{12}$$

which imply the bound claimed on ω. However in order to prove (12), a bootstrap argument in not sufficient. The idea is to consider the equation (11) as a linear equation in a with a and $\boldsymbol{\nabla} a$ given on $\partial\Omega$ for fixed $\boldsymbol{v}$, φ, and R and to construct a solution a by a fixed point argument. By uniqueness, the solution constructed by the fixed point argument is equal to the solution whose existence is assumed by hypothesis.

The fundamental solution of the linear operator defining the left hand-side of (11) is

$$W(\boldsymbol{x},\boldsymbol{x}_0)=\frac{1}{2\pi}\mathrm{e}^{r-r_0}K_0\left(|\boldsymbol{x}-\boldsymbol{x}_0|\right)\,, \tag{13}$$

where (r_0,θ_0) denotes the polar coordinates of $\boldsymbol{x}_0$. In view of the Green identity,

$$\int_\Omega\left(\Delta b+2\partial_r b+\frac{b}{r}\right)a=\int_\Omega\left(\Delta a-2\partial_r a-\frac{a}{r}\right)b+\int_{\partial\Omega}\left(a\boldsymbol{\nabla}b-b\boldsymbol{\nabla}a+ab\boldsymbol{e}_r\right)\cdot\boldsymbol{n}\,,$$

the solution can be written as

$$a(\boldsymbol{x})=\int_\Omega WR+\int_\Omega W\left(\varphi a+\boldsymbol{v}\boldsymbol{\nabla}a\right)+\int_{\partial\Omega}\left[a\boldsymbol{\nabla}_{\boldsymbol{x}_0}W-W\boldsymbol{\nabla}a+aW\boldsymbol{e}_r\right]\cdot\boldsymbol{n}\,, \tag{14}$$

where the integrations are performed over $\boldsymbol{x}_0$. For $i=1,2,3$, we denote by $a_i(\boldsymbol{x})$ the ith term in the expression (14).

The asymptotic expansion at large r, of the fundamental solution is given by

$$W(\boldsymbol{x},\boldsymbol{x}_0)=\frac{1}{\sqrt{8\pi}}\frac{1}{r^{1/2}}\mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))}+O\left(\frac{1}{r^{3/2}}\right)\,,$$

$$\boldsymbol{\nabla}_{\boldsymbol{x}_0}W(\boldsymbol{x},\boldsymbol{x}_0)=\frac{\left(\cos\theta-\cos\theta_0,\sin\theta-\sin\theta_0\right)}{\sqrt{8\pi}}\frac{1}{r^{1/2}}\mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))}+O\left(\frac{1}{r^{3/2}}\right)\,,$$

and since R has compact support and $\partial\Omega$ is a bounded, the first and the third term of (14) have the claimed asymptotic behavior,

$$a_1(\boldsymbol{x})=\frac{1}{\sqrt{8\pi}}\frac{1}{r^{1/2}}\int_{\mathbb{R}^2}\mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))}R(r_0,\theta_0)+O\left(\frac{1}{r^{3/2}}\right)=\frac{\mu_1(\theta)}{r^{1/2}}+O\left(\frac{1}{r^{3/2}}\right),$$

$$a_3(\boldsymbol{x})=\frac{\mu_3(\theta)}{r^{1/2}}+O\left(\frac{1}{r^{3/2}}\right)\,,$$

where μ_1 and μ_3 are 2π-periodic functions of the angle θ,

$$\mu_1(\theta)\ =\ \frac{1}{\sqrt{8\pi}}\int_\Omega r_0^{-A(1-\cos\theta_0)-B\sin\theta_0}\mathrm{e}^{r_0(\cos(\theta-\theta_0)-\cos\theta_0)}\left(\boldsymbol{\nabla}\wedge\boldsymbol{f}\right)\,,$$

$$\mu_3(\theta)\ =\ \frac{1}{\sqrt{8\pi}}\int_{\partial\Omega}\mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))}\left[a\left(\cos\theta-\cos\theta_0,\sin\theta-\sin\theta_0\right)-\boldsymbol{\nabla}a+a\boldsymbol{e}_r\right]\cdot\boldsymbol{n}\,.$$

In the same way,

$$\boldsymbol{\nabla}_{\boldsymbol{x}}W(\boldsymbol{x},\boldsymbol{x}_0)=O\left(\frac{1}{r^{3/2}}\right)\,,\qquad\qquad \boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{x}_0}W(\boldsymbol{x},\boldsymbol{x}_0)=O\left(\frac{1}{r^{3/2}}\right)\,,$$

and we obtain

$$\left|\boldsymbol{\nabla}a_1\right|\lesssim\frac{1}{r^{3/2}}\,,\qquad\qquad \left|\boldsymbol{\nabla}a_3\right|\lesssim\frac{1}{r^{3/2}}\,,$$

so a_1 and a_3 satisfy (12). To deal with the second term a_2, we first make a fixed point on a in the space defined by (12) so we have

$$|\varphi a+\boldsymbol{v}\boldsymbol{\nabla}a|\lesssim\frac{\nu}{r^{3/2+\varepsilon}}\,.$$

By using lemma 4 and lemma 5 we obtain that the second term a_2 of (14) satisfies (12). Since ν is small, a fixed point argument shows the bound (12) claimed on a and therefore the bound on ω.

In order to prove the asymptotic behavior of a_2, we show that

$$I = \int_\Omega \left| \sqrt{8\pi} W - \frac{1}{r^{1/2}} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \right| \frac{1}{r_0^{3/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 \,,$$

is bounded by $r^{-1/2-\varepsilon}$. By using the asymptotic expansion of the Bessel function K_0, we have $I \le I_1 + I_2$ where

$$I_1 = \int_{\mathbb{R}^2} \left| \frac{1}{r_1^{1/2}} \mathrm{e}^{r-r_0-r_1} - \frac{1}{r^{1/2}} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \right| \frac{1}{r_0^{3/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 \,,$$

$$I_2 = \int_{\mathbb{R}^2} \frac{1}{r_1^{3/2}} \mathrm{e}^{r-r_0-r_1} \frac{1}{r_0^{3/2+\varepsilon}} \mathrm{d}^2\boldsymbol{x}_0 \,,$$

with $r_1 = |\boldsymbol{x} - \boldsymbol{x}_0|$. In view of lemma 5, I_2 is bounded by $r^{-3/2}$. For $r_0 \ge r/2$, the term I_1 is bounded by $r^{-1/2+\varepsilon}$. For $r_0 \le r/2$, we have

$$\left| \frac{1}{r_1^{1/2}} - \frac{1}{r^{1/2}} \right| = \frac{1}{r_1^{1/2} r^{1/2}} \frac{r_0}{r^{1/2} + r_1^{1/2}} \frac{\left| r_0 - 2r\cos\left(\theta - \theta_0\right) \right|}{r + r_1} \lesssim \frac{r_0}{r_1^{1/2} r} \,,$$

$$\begin{aligned}
\left| \mathrm{e}^{r-r_0-r_1} - \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \right| &\lesssim \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{r_0^2 \sin^2\left(\theta - \theta_0\right)}{r_1 + r - r_0 \cos\left(\theta - \theta_0\right)} \\
&\lesssim \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{r_0^2 \left|\theta - \theta_0\right|^2}{r} \,.
\end{aligned}$$

Therefore, always for $r_0 \le r/2$, we have like in the proof of lemma 5,

$$\begin{aligned}
I_1 &\lesssim \int_{\mathbb{R}^2} \left| \frac{1}{r_1^{1/2}} - \frac{1}{r^{1/2}} \right| \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{1}{1 + r_0^{3/2+\varepsilon}} \mathrm{d}^2\boldsymbol{x}_0 \\
&\quad + \int_{\mathbb{R}^2} \frac{1}{r_1^{1/2}} \left| \mathrm{e}^{r-r_0-r_1} - \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \right| \frac{1}{1 + r_0^{3/2+\varepsilon}} \mathrm{d}^2\boldsymbol{x}_0 \\
&\lesssim \frac{1}{r} \int_{\mathbb{R}^2} \frac{1}{r_1^{1/2}} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{1 + r_0 \left|\theta - \theta_0\right|^2}{1 + r_0^{1/2+\varepsilon}} \mathrm{d}^2\boldsymbol{x}_0 \lesssim \frac{1}{r^{1/2+\varepsilon}} \,.
\end{aligned}$$

□

Lemma 4. *The Green function* (13) *satisfies*

$$|W| \lesssim \frac{1}{r_1^{1/2}} \mathrm{e}^{r-r_1-r_0} \,, \qquad \left| \boldsymbol{e}_r \cdot \boldsymbol{\nabla} W \right| \lesssim \frac{1}{r_1^{3/2}} \mathrm{e}^{(r-r_1-r_0)/2} \,, \qquad \left| \boldsymbol{e}_\theta \cdot \boldsymbol{\nabla} W \right| \lesssim \frac{r_0^{1/2}}{r_1^{3/2}} \mathrm{e}^{(r-r_1-r_0)/2} \,,$$

where $r_1 = |\boldsymbol{x} - \boldsymbol{x}_0|$.

Proof. First, the Bessel functions satisfy

$$K_0(r) \lesssim \frac{1}{r^{1/2}} \mathrm{e}^{-r} \,, \qquad K_1(r) \lesssim \frac{1}{r^{1/2}} \mathrm{e}^{-r} \,, \qquad K_1(r) - K_0(r) \lesssim \frac{1}{r^{3/2}} \mathrm{e}^{-r} \,,$$

so the first bound is proven. Since $r - r_0\cos(\theta-\theta_0) = r_1\cos(\theta_1-\theta)$, we have,

$$\begin{aligned}\partial_r W &= \frac{1}{2\pi r_1}\mathrm{e}^{r-r_0}\left[\left(r_0\cos(\theta-\theta_0)-r\right)K_1(r_1)+r_1K_0(r_1)\right] ,\\ &= \frac{1}{2\pi}\mathrm{e}^{r-r_0}\left[\frac{r_1-r+r_0\cos(\theta-\theta_0)}{r_1}K_0(r_1)-\cos(\theta-\theta_1)\left(K_1(r_1)-K_0(r_1)\right)\right]\end{aligned}$$

so

$$\begin{aligned}\left|\partial_r W\right| &\lesssim \frac{1}{r_1^{3/2}}\mathrm{e}^{r-r_1-r_0}+\frac{1}{r_1}\mathrm{e}^{r-r_0}\left(\left|r_0+r_1-r\right|+r_0\left|1-\cos(\theta-\theta_0)\right|\right)K_0(r_1)\\ &\lesssim \frac{1}{r_1^{3/2}}\mathrm{e}^{r-r_1-r_0}\left[1+\left|r_0+r_1-r\right|+r_0\left|1-\cos(\theta-\theta_0)\right|\right]\lesssim\frac{1}{r_1^{3/2}}\mathrm{e}^{(r-r_1-r_0)/2} .\end{aligned}$$

For the last bound, we have

$$\frac{1}{r}\partial_\theta W = \frac{1}{2\pi r_1}\mathrm{e}^{r-r_0}r_0\sin(\theta-\theta_0)K_1(r_1) ,$$

so

$$\left|\frac{1}{r}\partial_\theta W\right| \lesssim \frac{1}{r_1^{3/2}}\mathrm{e}^{r-r_1-r_0}r_0\left|\sin(\theta-\theta_0)\right| \lesssim \frac{r_0^{1/2}}{r_1^{3/2}}\mathrm{e}^{(r-r_1-r_0)/2} .$$

□

Lemma 5. *For $\alpha,\sigma\in(0,2)$ such that $\alpha+\sigma>3/2$, we have*

$$\int_{\mathbb{R}^2}\frac{1}{r_1^\alpha}\frac{1}{r_0^\sigma}\mathrm{e}^{r-r_1-r_0}\mathrm{d}\boldsymbol{x}_0 \lesssim \frac{1}{r^{\alpha+\sigma-3/2}}+\frac{|\log r|^{\delta_{\sigma,3/2}}}{r^\alpha}+\frac{|\log r|^{\delta_{\alpha,3/2}}}{r^\sigma} ,$$

where $r=|\boldsymbol{x}|$, $r_0=\left|\boldsymbol{x}_0\right|$, $r_1=\left|\boldsymbol{x}-\boldsymbol{x}_0\right|$ and $\delta_{\alpha,\sigma}$ denotes the Kronecker delta.

Proof. We have to estimate

$$I=\int_0^\infty\int_{-\pi}^{+\pi}\frac{1}{r_1^\alpha}\frac{1}{r_0^\sigma}\mathrm{e}^{r-r_1-r_0}r_0\mathrm{d}\theta_0\mathrm{d}r_0 .$$

First of all, since

$$r_1^2=r^2+r_0^2-2rr_0\cos(\theta-\theta_0)\geq r^2+r_0^2\cos^2(\theta-\theta_0)-2rr_0\cos(\theta-\theta_0)=\left(r-r_0\cos(\theta-\theta_0)\right)^2 ,$$

we have

$$r_1+r_0-r\geq r_0\left(1-\cos(\theta-\theta_0)\right) ,$$

and therefore

$$\int_{-\pi}^{+\pi}\mathrm{e}^{r-r_1-r_0}\mathrm{d}\theta_0\leq\int_{-\pi}^{+\pi}\mathrm{e}^{-r_0\left(1-\cos(\theta-\theta_0)\right)}\mathrm{d}\theta_0\leq\frac{1}{1+r_0^{1/2}} .$$

If $r_0\leq r/2$, then $r_1\geq r/2$ and therefore

$$\begin{aligned}I&\lesssim\frac{1}{r^\alpha}\int_0^{r/2}\int_{-\pi}^{+\pi}\frac{1}{r_0^\sigma}\mathrm{e}^{-r_0\left(1-\cos(\theta-\theta_0)\right)}r_0\mathrm{d}\theta_0\mathrm{d}r_0\\ &\lesssim\frac{1}{r^\alpha}\int_0^{r/2}\frac{1}{r_0^{\sigma-1}}\frac{1}{1+r_0^{1/2}}\mathrm{d}r_0\lesssim\frac{|\log r|^{\delta_{\sigma,3/2}}}{r^\alpha}+\frac{1}{r^{\alpha+\sigma-3/2}} .\end{aligned}$$

In the case where $r_1 \le r/2$, we have by symmetry the previous bound with α and σ exchanged,

$$I \lesssim \frac{|\log r|^{\delta_{\alpha,3/2}}}{r^{\sigma}} + \frac{1}{r^{\alpha+\sigma-3/2}} \,.$$

Therefore, it remains the case where $r_0 \ge r/2$ and $r_1 \ge r/2$, for which we have

$$I \lesssim \int_{r/2}^{\infty} \frac{1}{\left(r + |r - r_0|\right)^{\alpha}} \frac{1}{r_0^{\sigma-1/2}} \mathrm{d}r_0 \lesssim \frac{1}{r^{\alpha+\sigma-3/2}} \,.$$

$\square$

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