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Optimal asymptotic behavior of the vorticity of a viscous flow past a two-dimensional body

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A B S T R A C T

The asymptotic behavior of the vorticity for the steady incompressible Navier–Stokes equations in a two-dimensional exterior domain is described in the case where the velocity at infinity $\boldsymbol{u}_\infty$ is nonzero. It is well known that the asymptotic behavior of the velocity field is given by the fundamental solution of the Oseen system which is the linearization of the Navier–Stokes equation around $\boldsymbol{u}_\infty$. Concerning the vorticity, the previous asymptotic expansions were relevant only inside a parabolic region called the wake region. Here we present an asymptotic expansion for the vorticity relevant everywhere. Surprisingly, the found asymptotic behavior is not given by the Oseen linearization and has a power of decay that depends on the data. This strange behavior is specific to the two-dimensional problem and is not present in three dimensions.

R É S U M É

Le comportement asymptotique de la vorticité pour les équations de Navier–Stokes incompressibles dans un domaine extérieur bidimensionnel est décrit dans le cas où la vitesse à l'infini $\boldsymbol{u}_\infty$ est non nulle. Il est bien connu que le développement asymptotique du champ de vitesses est donné par la solution fondamentale de l'équation d'Oseen, qui correspond à la linéarisation des équations de Navier–Stokes autour de $\boldsymbol{u}_\infty$. Concernant la vorticité, les développements asymptotiques antérieurs étaient pertinents seulement dans la région parabolique du sillage. Dans cet article, on présente un développement asymptotique pour la vorticité pertinent partout. De manière surprenante, le comportement asymptotique n'est pas donné par la linéarisation d'Oseen et possède une décroissance dépendant des données. Cette particularité est spécifique au problème bidimensionnel et n'est pas présente en trois dimensions.

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1. Introduction

The stationary flow of an incompressible fluid past a body is described by the Navier–Stokes equations,

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{\nabla} p &= \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{f} \,, & \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \,, \\ \boldsymbol{u}|_{\partial\Omega} &= \boldsymbol{u}^* \,, & \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u} &= \boldsymbol{u}_\infty \,, \end{aligned} \tag{1a}$$

in the exterior domain $\Omega = \mathbb{R}^2 \setminus \bar{B}$, where B is a bounded simply connected Lipschitz domain, $\boldsymbol{f}$ a source force of compact support in Ω, $\boldsymbol{u}_\infty \neq \boldsymbol{0}$ the velocity at infinity and $\boldsymbol{u}^*$ any boundary condition with no net flux,

$$\int_{\partial\Omega} \boldsymbol{u}^* \cdot \boldsymbol{n} = 0 \,. \tag{1b}$$

In view of the symmetries of the equation, we assume without loss of generality that $\boldsymbol{u}_\infty = 2\boldsymbol{e}_1$.

This system has been subject to many investigations under different smoothness hypotheses on the domain, on $\boldsymbol{f}$ and on $\boldsymbol{u}^*$, see Galdi [1, Chapter XII] for a complete statement of the main results known for this problem. Leray [2] has shown the existence of a weak solution $\boldsymbol{u} \in \dot{H}^1(\Omega)$, but with the procedure he used, he was unable to verify that $\boldsymbol{u}$ tends to $\boldsymbol{u}_\infty$ at large distances. Gilbarg and Weinberger [3,4] have shown that any Leray solution $\boldsymbol{u}$ either converges at large distances in a weak sense to some constant vector $\boldsymbol{u}_0$ or diverges in the same weak sense. Later on, Amick [5] proved that if $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{u}^* = \boldsymbol{0}$, then $\boldsymbol{u} \in L^\infty(\Omega)$ and therefore $\boldsymbol{u}$ converges to a constant $\boldsymbol{u}_0$ at infinity. However, the question if $\boldsymbol{u}_0 = \boldsymbol{u}_\infty$ is still open in general.

The convergence of weak solutions to $\boldsymbol{u}_\infty$ at large distances being unknown, we consider the subspace of physically reasonable solutions. We recall that a solution $\boldsymbol{u} \in \dot{H}^1(\Omega)$ is physically reasonable in the sense of Smith [6, §4] if $\boldsymbol{u} - \boldsymbol{u}_\infty = O(r^{-1/4-\varepsilon})$ for some $\varepsilon > 0$. Finn and Smith [7], Galdi [8,9,1] used the Oseen approximation and a fixed point technique to prove existence and uniqueness of physically reasonable solutions to (1) under smallness and regularity assumptions on B, $\boldsymbol{f}$, and $\boldsymbol{u}^*$. We note that without smallness assumptions such an existence result is still open.

The asymptotic structure of the physically reasonable solutions was presented by Babenko [10] who shows in particular that the velocity behaves at infinity like the Oseen fundamental solution. The asymptotic expansion of the velocity was also given under more general assumptions by Galdi and Sohr [11], Sazonov [12]. The asymptotic behavior of the vorticity was first given by Babenko [10, Theorem 8.1] but only in the wake region, and then by Clark [13, Theorem 3.5']. These two results are relevant only in the wake region, *i.e.* for $|\boldsymbol{x}| - x_1 \leq 1$, because otherwise, the remainder decays slower than the asymptotic term which is given by the Oseen linearization. Here we prove that the asymptote which is also valid outside the wake region is in fact not given by the Oseen linearization. Our main result states that the asymptote of the vorticity $\omega = \boldsymbol{\nabla} \wedge \boldsymbol{u}$ of a physically reasonable solution is given in polar coordinates (r, θ) by

$$\omega(\boldsymbol{x}) = r^{A_1(1-\cos\theta) - A_2 \sin\theta} \left[\frac{\mu(\theta)}{r^{1/2}} + O\!\left(\frac{1}{r^{1/2+\varepsilon}} \right) \right] \mathrm{e}^{-r(1-\cos\theta)} \,,$$

as $r \to \infty$ for all $\varepsilon \in \left(0, \frac{1}{4}\right)$, where $\boldsymbol{A} = (A_1, A_2) \in \mathbb{R}^2$ is a vector depending linearly on the net force

$$\boldsymbol{F} = \int_{\Omega} \boldsymbol{f} - \int_{\partial\Omega} \left(\mathbf{T}(\boldsymbol{u}, p) - \boldsymbol{u} \otimes \boldsymbol{u} \right) \cdot \boldsymbol{n} \,, \qquad \mathbf{T}(\boldsymbol{u}, p) = \boldsymbol{\nabla} \boldsymbol{u} + (\boldsymbol{\nabla} \boldsymbol{u})^{\mathrm{T}} - p\mathbf{1} \,, \tag{2}$$

and μ is a 2π-periodic continuous function depending on $\boldsymbol{f}$ and $\boldsymbol{u}^*$ (see Theorem 2 for the precise formulation). This result is quite astonishing, because the decay rate of the vorticity depends on the net force $\boldsymbol{F}$

and is valid under no smallness assumptions. More precisely, the decay properties of the vorticity is given by the linearization of the vorticity equation around $\boldsymbol{u}_\infty + \boldsymbol{u}_h$ where $\boldsymbol{u}_h$ is the critically decaying part of $\boldsymbol{u}$, *i.e.* the part that decays like $|\boldsymbol{x}|^{-1}$. We will see how the critical part $\boldsymbol{u}_h$ influences the behavior of the vorticity at infinity and to our knowledge this is the first time that an asymptotic behavior with a power of decay depending continuously on the data is shown in the context of the Navier–Stokes system. The main idea behind the proof is to eliminate the linear terms induced by $\boldsymbol{u}_h$ with an explicit change of variables. The importance of critically decaying functions was emphasized in similar contexts by Seregin et al. [14] and Guillod [15].

We remark that the asymptotic behavior presented here contradicts the statement of Theorem XII.8.4 in Galdi [1] for any solution with $\boldsymbol{F} \neq \boldsymbol{0}$. We suppose that this result, whose proof is not given, was inspired by a misunderstanding of the work of Babenko [10] and Clark [13]. The result of Babenko [10] explicitly included a parameter μ which makes the exponential decay of the vorticity outside the wake not optimal. The result of Clark [13, Theorem 3.5'] is much more difficult to understand due to many printing mistakes and to the fact that the proofs are given only for the three-dimensional case. We remark that in three dimensions, the asymptotic behavior of the vorticity is actually given by the Oseen linearization, *i.e.* around $\boldsymbol{u}_\infty$ only.

Notation For $\boldsymbol{x}, \boldsymbol{x}_0 \in \mathbb{R}^2$ we use the following notation

$$
\begin{aligned}
r &= |\boldsymbol{x}|\,, & r_0 &= |\boldsymbol{x}_0|\,, & r_1 &= |\boldsymbol{x}-\boldsymbol{x}_0|\,,\\
\theta &= \angle\boldsymbol{x}\,, & \theta_0 &= \angle\boldsymbol{x}_0\,, & \theta_1 &= \angle(\boldsymbol{x}-\boldsymbol{x}_0)\,,
\end{aligned}
$$

where $\angle\boldsymbol{x}$ denotes the angle $\theta \in (-\pi, \pi]$ such that $\boldsymbol{x} = r(\cos\theta, \sin\theta)$. For $i = 1, 2$, we denote by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ the orthonormal basis associated with the Cartesian coordinates $\boldsymbol{x} = (x_1, x_2)$ and by $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ the orthonormal basis related to the polar coordinates (r, θ). The closed ball of radius ρ centered at the origin is denoted by $\bar{B}_\rho$ and we define $\Omega_\rho = \mathbb{R}^2 \setminus \bar{B}_\rho$. The orthogonal of a vector field $\boldsymbol{v} = (v_1, v_2)$ is defined by $\boldsymbol{v}^\perp = (-v_2, v_1)$. The curl of a scalar field φ is defined by $\boldsymbol{\nabla} \wedge \varphi = \boldsymbol{\nabla}^\perp\varphi = (-\partial_2\varphi, \partial_1\varphi)$ and the curl of a vector field $\boldsymbol{u}$ by $\boldsymbol{\nabla} \wedge \boldsymbol{u} = \partial_1 u_2 - \partial_2 u_1$.

In integrations, we denote by $\boldsymbol{n}$ the outgoing unit vector normal to the boundary of the integration domain and we use $\bullet$ as a placeholder for the argument over which the integration is done. If a and b are two positive functions of the same variables, we write $a \lesssim b$ if there exists $C > 0$ such that $a \leq Cb$ for all values of the variables. For a positive function $w : \Omega \to \mathbb{R}$ and two scalar fields a and a_0, we write $a(\boldsymbol{x}) = a_0(\boldsymbol{x}) + O(w(\boldsymbol{x}))$ if there exists $C > 0$ and $\rho > 0$ such that $|a(\boldsymbol{x}) - a_0(\boldsymbol{x})| \leq Cw(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$ with $|\boldsymbol{x}| \geq \rho$. The term $a_0(\boldsymbol{x})$ is called the asymptotic term and $O(w(\boldsymbol{x}))$ the remainder. The asymptotic expansion $a(\boldsymbol{x}) = a_0(\boldsymbol{x}) + O(w(\boldsymbol{x}))$ is optimal if for any $C > 0$, the set $\{\boldsymbol{x} \in \Omega : |a_0(\boldsymbol{x})| \leq Cw(\boldsymbol{x})\}$ is bounded.

The space of smooth functions having compact support in Ω or in $\bar{\Omega}$ are denoted by $C_0^\infty(\Omega)$ and $C_0^\infty(\bar{\Omega})$ respectively. The completion of $C_0^\infty(\Omega)$ in the natural norm associated with the Sobolev space $H^1(\Omega)$ and its homogeneous counterpart $\dot{H}^1(\Omega)$ are denoted by $H_0^1(\Omega)$ and $\dot{H}_0^1(\Omega)$ respectively. We note that functions in $C_0^\infty(\Omega)$, $H_0^1(\Omega)$, and $\dot{H}_0^1(\Omega)$ are vanishing at the boundary, whereas functions in $C_0^\infty(\bar{\Omega})$ have bounded support but might be nonzero on the boundary.

2. Asymptote for the linear problem

The aim of this section is to introduce the definition of the fundamental solutions of the Oseen system and to initiate the concept of optimal asymptotic decay. Since the representation formula for the Oseen system will not be needed in the proof of our main result, they are deduced at a formal level in order to keep this section short, but it is possible to make them rigorous, see for example Galdi [1, VII.6].

It is well-known that the problem (1) is related to the Oseen system which is the linearization of (1) around $\boldsymbol{u} = \boldsymbol{u}_\infty = 2\boldsymbol{e}_1$,

$$\begin{aligned} \Delta\boldsymbol{u} - 2\partial_1\boldsymbol{u} - \boldsymbol{\nabla} p &= \boldsymbol{f}\,, & \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0\,, \\ \boldsymbol{u}|_{\partial B} &= \boldsymbol{u}^* - \boldsymbol{u}_\infty\,, & \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u} &= \boldsymbol{0}\,. \end{aligned} \tag{3}$$

The fundamental solution of the Oseen system $(\mathbf{E}, \boldsymbol{w})$ is a tensor of rank two $\mathbf{E}$ and a vector field $\boldsymbol{w}$, such that

$$\Delta\mathbf{E} - 2\partial_1\mathbf{E} - \boldsymbol{\nabla}\boldsymbol{w} = \delta_0\mathbf{1}\,,$$

where δ_0 is the delta function centered at the origin. Explicitly, the fundamental solution is given by

$$\mathbf{E} = \begin{pmatrix} \partial_1\psi - G & \partial_2\psi \\ \partial_2\psi & -\partial_1\psi \end{pmatrix}, \qquad \boldsymbol{w} = -\boldsymbol{\nabla} H\,, \qquad \psi = \frac{H+G}{2}\,,$$

with G and H satisfying $\Delta H = \delta_0$ and $\Delta G - 2\partial_1 G = -\delta_0$ respectively. Explicitly, these two fundamental solutions are given by

$$H = \frac{1}{2\pi}\log r\,, \qquad G = \frac{1}{2\pi}\mathrm{e}^{r\cos\theta}K_0(r)\,,$$

where K_0 denotes the modified Bessel function of the second kind.

Recalling the definition (2) of the stress tensor $\mathbf{T}(\boldsymbol{u}, p)$, the Green identity for the Oseen operator is

$$\int_\Omega (\boldsymbol{\nabla}\cdot\mathbf{T}(\boldsymbol{u},p) - 2\partial_1\boldsymbol{u})\cdot\boldsymbol{v} - \int_\Omega (\boldsymbol{\nabla}\cdot\mathbf{T}(\boldsymbol{v},q) + 2\partial_1\boldsymbol{v})\cdot\boldsymbol{u} = \int_{\partial\Omega} (\boldsymbol{v}\cdot\mathbf{T}(\boldsymbol{u},p) - \boldsymbol{u}\cdot\mathbf{T}(\boldsymbol{v},q) - 2\boldsymbol{u}\cdot\boldsymbol{v}\,\boldsymbol{e}_1)\cdot\boldsymbol{n}$$

for any smooth divergence-free vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ with enough decay at infinity.

By taking $\boldsymbol{v}(\boldsymbol{y}) = \boldsymbol{E}_i(\boldsymbol{x}-\boldsymbol{y})$ and $q(\boldsymbol{y}) = -w_i(\boldsymbol{x}-\boldsymbol{y})$ for $i = 1, 2$, we obtain the representation formula for the Oseen system

$$\boldsymbol{u}(\boldsymbol{x}) = \int_\Omega \mathbf{E}(\boldsymbol{x}-\bullet)\boldsymbol{f} - \int_{\partial\Omega} \left[\mathbf{E}(\boldsymbol{x}-\bullet)\left(\mathbf{T}(\boldsymbol{u},p) - 2\boldsymbol{u}\otimes\boldsymbol{e}_1\right) + \boldsymbol{u}\cdot\mathbf{T}(\mathbf{E},\boldsymbol{w})(\boldsymbol{x}-\bullet)\right]\cdot\boldsymbol{n}\,,$$

where $\mathbf{T}(\mathbf{E}, \boldsymbol{w})$ is a tensor of rank three with

$$\int_{\partial\Omega} \boldsymbol{u}\cdot\mathbf{T}(\mathbf{E},\boldsymbol{w})(\boldsymbol{x}-\bullet)\cdot\boldsymbol{n} = \int_{\partial\Omega} \left[\boldsymbol{n}\cdot\boldsymbol{\nabla}\mathbf{E}(\boldsymbol{x}-\bullet)\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\mathbf{E}(\boldsymbol{x}-\bullet)\cdot\boldsymbol{n} - \boldsymbol{w}(\boldsymbol{x}-\bullet)\,\boldsymbol{u}\cdot\boldsymbol{n}\right], \tag{4}$$

and where $\bullet$ denotes a placeholder for the argument over which the fundamental solution is integrated. In order to obtain the representation formula for the vorticity $\omega = \boldsymbol{\nabla}\wedge\boldsymbol{u}$, we remark that for any $\boldsymbol{A}\in\mathbb{R}^2$,

$$\boldsymbol{\nabla}\wedge(\mathbf{E}\cdot\boldsymbol{A}) = \boldsymbol{\nabla} G\cdot\boldsymbol{A}^\perp\,,$$

with G as defined above, so we obtain

$$\begin{aligned} \omega(\boldsymbol{x}) = &\int_\Omega \boldsymbol{\nabla} G(\boldsymbol{x}-\bullet)\cdot\boldsymbol{f}^\perp - \int_{\partial\Omega} \boldsymbol{\nabla} G(\boldsymbol{x}-\bullet)\cdot\left[(\mathbf{T}(\boldsymbol{u},p) - 2\boldsymbol{u}\otimes\boldsymbol{e}_1)\cdot\boldsymbol{n}\right]^\perp \\ &- \int_{\partial\Omega} \left[\boldsymbol{\nabla}_{\boldsymbol{x}}\big(\boldsymbol{n}\cdot\boldsymbol{\nabla} G(\boldsymbol{x}-\bullet)\big)\cdot\boldsymbol{u}^\perp + \boldsymbol{\nabla}_{\boldsymbol{x}}\left(\boldsymbol{u}\cdot\boldsymbol{\nabla} G(\boldsymbol{x}-\bullet)\right)\cdot\boldsymbol{n}^\perp\right]. \end{aligned}$$

By using the asymptotic expansion of the Bessel functions

$$K_n(r) = \sqrt{\frac{\pi}{2}} \left(\frac{1}{r^{1/2}} + O\left(\frac{1}{r^{3/2}}\right)\right) \mathrm{e}^{-r} , \tag{5}$$

the asymptotic expansions of the fundamental solutions are given by

$$\begin{aligned} \mathbf{E}(\boldsymbol{x}) &= \frac{-1}{\sqrt{32\pi}} \left(\frac{1}{r^{1/2}} + O\Big(\frac{1}{r^{3/2}}\Big)\right) \mathrm{e}^{-r(1-\cos\theta)} \begin{pmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix} + \frac{1}{4\pi r} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \\ \boldsymbol{\nabla} G(\boldsymbol{x}) &= \frac{1}{\sqrt{8\pi}} (1-\cos\theta, -\sin\theta) \left(\frac{1}{r^{1/2}} + O\Big(\frac{1}{r^{3/2}}\Big)\right) \mathrm{e}^{-r(1-\cos\theta)} , \end{aligned} \tag{6}$$

and for $|\alpha| = 1$,

$$\begin{aligned} |D^\alpha \mathbf{E}| &\lesssim \left(\frac{|\theta|}{r^{1/2}} + \frac{1}{r^{3/2}} \right) \mathrm{e}^{-r(1-\cos\theta)} + \frac{1}{r^2} , \\ |D^\alpha \boldsymbol{\nabla} G| &\lesssim \left(\frac{|\theta|^2}{r^{1/2}} + \frac{1}{r^{3/2}} \right) \mathrm{e}^{-r(1-\cos\theta)} . \end{aligned}$$

By assuming that the support of $\boldsymbol{f}$ is compact, the previous calculations naturally prove that the solution of the Oseen equation (3) behaves like the fundamental solution at large distances,

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbf{E}(\boldsymbol{x})\boldsymbol{L} + O\bigg(\left(\frac{|\theta|}{r^{1/2}} + \frac{1}{r^{3/2}} \right) \mathrm{e}^{-r(1-\cos\theta)} + \frac{1}{r^2} \bigg) , \tag{7}$$

$$\omega(\boldsymbol{x}) = \boldsymbol{\nabla} G(\boldsymbol{x}) \cdot \boldsymbol{L}^\perp + O\bigg(\left(\frac{|\theta|^2}{r^{1/2}} + \frac{1}{r^{3/2}} \right) \mathrm{e}^{-r(1-\cos\theta)} \bigg) , \tag{8}$$

where $\boldsymbol{L}$ is the counterpart for (3) of the net force (2),

$$\boldsymbol{L} = \int_\Omega \boldsymbol{f} - \int_{\partial\Omega} (\mathbf{T}(\boldsymbol{u}, p) - 2\boldsymbol{u} \otimes \boldsymbol{e}_1) \cdot \boldsymbol{n} . \tag{9}$$

We remark that the term $\boldsymbol{w}(\boldsymbol{x} - \bullet)\, \boldsymbol{u} \cdot \boldsymbol{n}$ in (4) does not contribute to the leading order due to the boundary condition $\boldsymbol{u}|_{\partial B} = \boldsymbol{u}^* - \boldsymbol{u}_\infty$ in (3) and the zero flux condition (1b).

We now make some remarks on the asymptotic behaviors (7) and (8). The precise definition of the big O was defined in the notations at the end of the introduction, but for example (7) is equivalent to saying that $\boldsymbol{u}(\boldsymbol{x}) - \mathbf{E}(\boldsymbol{x})\boldsymbol{L}$ is bounded by the function in the argument of the big O up to a multiplicative constant for all $\boldsymbol{x}$ outside a compact set. In a more intuitive way, this means the asymptotic behavior is valid at large values of r uniformly in the angle θ. In (7), we see the two basic scales of the problem: an exponential scale characterized by $\mathrm{e}^{-r(1-\cos\theta)}$ and a polynomial scale having an algebraic decay in r. The exponential scale has a so-called wake behavior since $\mathrm{e}^{-r(1-\cos\theta)}$ is bounded by a constant in the parabolic wake region $r\,|\theta|^2 \le 1$, whereas it decays exponentially fast in any direction characterized by a constant $\theta \ne 0$.

The asymptotic expansion (7) of the velocity field is optimal (see the notations at the end of the introduction), in particular the asymptotic term decays strictly slower than the remainder in any direction. Indeed, in any direction characterized by a constant $\theta \ne 0$, the asymptote decays like r^{-1} and the remainder like r^{-2}. Inside the wake region $r\,|\theta|^2 \le 1$, the first component of the velocity has an asymptote decaying like $r^{-1/2}$ and the remainder like $|\theta|\, r^{-1/2} + r^{-3/2} \lesssim r^{-1}$. In fact, we remark the following fact about the presence of factors of $|\theta|$ in front of the exponential scale: for any $\alpha \ge 0$, there exists a $C > 0$ such that

$$|\theta|^{\alpha}\, \mathrm{e}^{-r(1-\cos\theta)} \leq 3\,(1-\cos\theta)^{\alpha/2}\, \mathrm{e}^{-r(1-\cos\theta)} \leq \frac{C}{r^{\alpha/2}} \mathrm{e}^{-r(1-\cos\theta)/2} \,. \tag{10}$$

The second bound follows from the fact that $z^{\alpha/2}\mathrm{e}^{-z} \lesssim \mathrm{e}^{-z/2}$ for $z \geq 0$.

However, the asymptotic expansion (8) for the vorticity is not optimal since outside the wake region, the asymptotic term can be absorbed into the remainder. In order to obtain an asymptote that is relevant in all directions, we have to proceed differently. At large $\boldsymbol{x}$ for fixed values of $\boldsymbol{x}_0$, the asymptotic expansion of $r_1 = |\boldsymbol{x} - \boldsymbol{x}_0|$ is

$$r_1 = \sqrt{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)} = r - r_0\cos(\theta-\theta_0) + O\big(r^{-1}\big)\,,$$

so

$$\mathrm{e}^{-r_1} = \mathrm{e}^{-r}\mathrm{e}^{r_0\cos(\theta-\theta_0)}\left(1 + O\big(r^{-1}\big)\right). \tag{11}$$

Therefore, we obtain the following asymptotic expansion of the fundamental solution for fixed values of $\boldsymbol{x}_0$,

$$D^{\alpha}\boldsymbol{\nabla} G(\boldsymbol{x}-\boldsymbol{x}_0) = D^{\alpha}\boldsymbol{\nabla} G(\boldsymbol{x})\, \mathrm{e}^{r_0(\cos(\theta-\theta_0)-\cos\theta_0)} + O\!\left(\frac{1}{r^{3/2}}\mathrm{e}^{-r(1-\cos\theta)}\right),$$

for $|\alpha| \leq 1$ which implies that

$$\omega(\boldsymbol{x}) = \boldsymbol{\nabla} G(\boldsymbol{x}) \boldsymbol{\cdot} \boldsymbol{Z}^{\perp}(\theta) - \boldsymbol{\nabla}^2 G(\boldsymbol{x}) : \mathbf{J}(\theta) + O\!\left(\frac{1}{r^{3/2}}\mathrm{e}^{-r(1-\cos\theta)}\right), \tag{12}$$

where $\boldsymbol{Z}(\theta)$ is a vector and $\mathbf{J}(\theta)$ a tensor of rank two depending on the angle θ,

$$\boldsymbol{Z}(\theta) = \int_{\Omega} \mathrm{e}^{r_0(\cos(\theta-\theta_0)-\cos\theta_0)} \boldsymbol{f} - \int_{\partial\Omega} \mathrm{e}^{r_0(\cos(\theta-\theta_0)-\cos\theta_0)} \big(\mathbf{T}(\boldsymbol{u},p) - 2\boldsymbol{u}\otimes\boldsymbol{e}_1\big)\boldsymbol{\cdot}\boldsymbol{n}\,,$$

$$\mathbf{J}(\theta) = \int_{\partial\Omega} \mathrm{e}^{r_0(\cos(\theta-\theta_0)-\cos\theta_0)} \big(\boldsymbol{u}^{\perp}\otimes\boldsymbol{n} + \boldsymbol{u}\otimes\boldsymbol{n}^{\perp}\big)\,,$$

where the integrations are performed over $\boldsymbol{x}_0$. Since the domains of integration are compact, $\boldsymbol{Z}(\theta)$ and $\mathbf{J}(\theta)$ are smooth. In particular, we have $\boldsymbol{Z}(0) = \boldsymbol{L}$, where $\boldsymbol{L}$ is net force (9) and we can recover (8). By using the asymptotic expansions of $\boldsymbol{\nabla} G$ and its derivatives, (12) can be written as

$$\omega(\boldsymbol{x}) = \left(\frac{\mu(\theta)}{r^{1/2}} + O\!\left(\frac{1}{r^{3/2}}\right)\right)\mathrm{e}^{-r(1-\cos\theta)}\,, \tag{13}$$

where $\mu(\theta)$ is some 2π-periodic smooth function.

The asymptotic expansion (12) or (13) is now optimal and to our knowledge is nowhere mentioned in the literature. The aim of the next section is to obtain the analogous optimal result but for the nonlinear problem.

3. Asymptote for the nonlinear problem

For the nonlinear problem (1), neither the asymptotic behavior (7), (8) nor (13) is correct, as shown below. The best results concerning the asymptotic behavior of $\boldsymbol{u}$ and ω for the nonlinear problem (1) are due to Babenko [10]. In particular he shows the following result:

Theorem 1 *(Babenko [10, Theorems 6.1 & 8.1]). If $\boldsymbol{u}$ is a physically reasonable solution of (1),* i.e. *such that $\boldsymbol{u} - \boldsymbol{u}_\infty = O(r^{-1/4-\varepsilon})$ for some $\varepsilon \in \left(0, \frac{1}{4}\right)$, then there exists $\boldsymbol{F} \in \mathbb{R}^2$ such that the velocity satisfies*

$$\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_\infty = \mathbf{E}(\boldsymbol{x})\boldsymbol{F} + O\left(\left[\left(\frac{|\theta| \log r}{r^{1/2}} + \frac{1}{r}\right) \mathrm{e}^{-r(1-\cos\theta)} + \frac{1}{r^{1+\varepsilon}}\right] \boldsymbol{e}_1 + \frac{1}{r^{5/4+\varepsilon}} \boldsymbol{e}_2\right), \tag{14}$$

and the vorticity $\omega = \boldsymbol{\nabla} \wedge \boldsymbol{u}$ satisfies

$$\omega(\boldsymbol{x}) = \boldsymbol{\nabla} G(\boldsymbol{x}) \cdot \boldsymbol{F}^\perp + O\left(\frac{\log r}{r^{3/2}} \mathrm{e}^{-\mu r(1-\cos\theta)}\right),$$

for some $\mu \in (0, 1)$. In particular if Ω, $\boldsymbol{f}$, and $\boldsymbol{u}^*$ are smooth, $\boldsymbol{F}$ is equal to the net force (2).

The asymptotic expansion of the velocity field is optimal, however, the one for the vorticity is not and is only relevant in the wake since the remainder decays slower at infinity than the asymptotic term for large r and fixed $\theta \neq 0$, due to the fact that $\mu \in (0, 1)$. In fact, we show that the asymptotic behavior of the vorticity is not given by the Oseen tensor outside the wake and has a power of decay depending on $\boldsymbol{F}$:

Theorem 2. *If $\boldsymbol{u}$ is a physically reasonable solution of (1),* i.e. *such that $\boldsymbol{u} - \boldsymbol{u}_\infty = O(r^{-1/4-\varepsilon})$ for some $\varepsilon \in \left(0, \frac{1}{4}\right)$, then there exists a vector $\boldsymbol{A} = (A_1, A_2) \in \mathbb{R}^2$ and a 2π-periodic and 2ε-Hölder continuous function μ such that*

$$\omega(\boldsymbol{x}) = r^{A_1(1-\cos\theta) - A_2 \sin\theta} \left[\frac{\mu(\theta)}{r^{1/2}} + O\left(\frac{1}{r^{1/2+\varepsilon}}\right)\right] \mathrm{e}^{-r(1-\cos\theta)}. \tag{15}$$

In particular there exist two constants $C > 0$ and $\rho > 0$ such that

$$|\omega| \leq C\, r^{A_1(1-\cos\theta) - A_2 \sin\theta} r^{-1/2} \mathrm{e}^{-r(1-\cos\theta)}, \tag{16}$$

for all $\boldsymbol{x} \in \mathbb{R}^2$ such that $|\boldsymbol{x}| \geq \rho$. Moreover, if Ω, $\boldsymbol{f}$, and $\boldsymbol{u}^*$ are smooth, then $\boldsymbol{A}$ depends linearly on the net force $\boldsymbol{F}$,

$$\boldsymbol{A} = \frac{\boldsymbol{F}}{8\pi}, \qquad \boldsymbol{F} = \int_\Omega \boldsymbol{f} - \int_{\partial\Omega} \left(\mathbf{T}(\boldsymbol{u}, p) - \boldsymbol{u} \otimes \boldsymbol{u}\right) \cdot \boldsymbol{n}.$$

Remark 3. We cannot ensure that the asymptotic behavior (15) is optimal since μ might have zeros. However, for θ not being a zero of μ, the asymptotic behavior is optimal.

We now sketch the main steps of the proof of Theorem 2. By taking the curl of (1a), we obtain the equation for the vorticity $\omega = \boldsymbol{\nabla} \wedge \boldsymbol{u}$,

$$\Delta\omega = \boldsymbol{u} \cdot \boldsymbol{\nabla}\omega + \boldsymbol{\nabla} \wedge \boldsymbol{f}, \tag{17}$$

and we will show that the correct asymptotic behavior of the vorticity for the nonlinear system is not given by the Oseen approximation, *i.e.* the linearization of (17) around $\boldsymbol{u} = \boldsymbol{u}_\infty$, but by linearizing (17) around $\boldsymbol{u} = \boldsymbol{u}_\infty + \boldsymbol{u}_h$, where $\boldsymbol{u}_h$ is the following harmonic function

$$\boldsymbol{u}_h = 2\boldsymbol{\nabla}\left(A_1 \log r - A_2\theta\right) = \frac{2A_1}{r} \boldsymbol{e}_r - \frac{2A_2}{r} \boldsymbol{e}_\theta.$$

In the asymptotic expansion (14), the term $\mathbf{E}(\boldsymbol{x})\boldsymbol{F}$ contains $\boldsymbol{u}_h$ in view of (6), and therefore $\bar{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_\infty - \boldsymbol{u}_h$ will have some better decay. In particular outside the wake $\bar{\boldsymbol{u}}$ will decay like $r^{-1-\varepsilon}$ at infinity. Therefore, (17) can be written as

$$\Delta\omega - 2\partial_1\omega - \boldsymbol{u}_h \cdot \boldsymbol{\nabla}\omega = \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\omega + \boldsymbol{\nabla} \wedge \boldsymbol{f}\,, \tag{18}$$

and the idea is to look at this equation as a linear equation for ω with $\boldsymbol{u}$ and $\boldsymbol{u}_h$ fixed and a priori not related to ω. By some fixed point argument in a large enough domain, we will prove the existence of a solution ω to this equation satisfying (15) and (16). A uniqueness argument (see section §4) ensures that the constructed field ω is the vorticity, *i.e.* $\omega = \boldsymbol{\nabla}\wedge\boldsymbol{u}$. The idea behind the analysis of the linear system (18) is to invert the linear operator $\Delta - 2\partial_1 - \boldsymbol{u}_h \cdot \boldsymbol{\nabla}$ defined by the left-hand side of (18) on $\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\omega + \boldsymbol{\nabla} \wedge \boldsymbol{f}$ and to use a fixed point argument. The fixed point map will be a contraction at large distances even without any smallness assumptions due to some kind of compactness of $\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\omega$. We remark that the fixed point argument is crucial here since a bootstrapping argument cannot give the exponential factor $\mathrm{e}^{-r(1-\cos\theta)}$ in the vorticity. However, in order to close the fixed point argument, some very detailed bounds on the inverse of the linear operator $\Delta - 2\partial_1 - \boldsymbol{u}_h \cdot \boldsymbol{\nabla}$ are required. The main idea behind these linear estimates is to perform an explicit change of variable (see (35)) so that $\Delta - 2\partial_1 - \boldsymbol{u}_h \cdot \boldsymbol{\nabla}$ is transformed into $\Delta - 1$ modulo some lower order terms. The precise analysis of the auxiliary system $\Delta b - b = R$ and the asymptote of its solution will be studied in section §5.

4. Uniqueness for the vorticity equation

We prove the uniqueness of the solution to the vorticity equation in the space $\dot{H}^1_0(\Omega) \cap L^4(\Omega)$, which is large enough to contain the vorticity of any weak solution of the Navier–Stokes equation but also the solution constructed latter on by a fixed point argument.

Proposition 4 *(uniqueness). For a weakly divergence-free vector field $\boldsymbol{u} \in L^\infty(\Omega)$, let $\omega \in \dot{H}^1_0(\Omega)$ be a weak solution of*

$$\Delta\omega - \boldsymbol{u} \cdot \boldsymbol{\nabla}\omega = 0\,.$$

If $\omega \in L^4(\Omega)$, then $\omega = 0$.

Proof. Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth cutoff function such that $\eta(r) = 1$ if $r \le 1/2$ and $\eta(r) = 0$ if $r \ge 1$. For $k > 0$ large enough, let $\eta_k(\boldsymbol{x}) = \eta(|\boldsymbol{x}|/k)$, so that $\eta_k(\boldsymbol{x}) = 1$ if $|\boldsymbol{x}| \le k/2$ and $\eta_k(\boldsymbol{x}) = 0$ if $|\boldsymbol{x}| \ge k$. An explicit calculation shows that

$$|\boldsymbol{\nabla}\eta_k(\boldsymbol{x})| \le \frac{\|\eta'\|_\infty}{k}\,, \qquad |\Delta\eta_k(\boldsymbol{x})| \le \frac{2\,\|\eta'\|_\infty + \|\eta''\|_\infty}{k^2}\,.$$

Since ω is a weak solution, we have

$$\big(\boldsymbol{\nabla}\omega, \boldsymbol{\nabla}\varphi\big) + \big(\boldsymbol{u} \cdot \boldsymbol{\nabla}\omega, \varphi\big) = 0\,,$$

for all $\varphi \in C_0^\infty(\Omega)$, where $(\cdot,\cdot)$ denotes the inner product on $L^2(\Omega)$. We can extend the set of test functions to allow $\varphi = \eta_k\omega^3$, which gives

$$3\big(\boldsymbol{\nabla}\omega, \eta_k\omega^2\boldsymbol{\nabla}\omega\big) + \big(\boldsymbol{\nabla}\omega, \omega^3\boldsymbol{\nabla}\eta_k\big) + \big(\boldsymbol{u} \cdot \boldsymbol{\nabla}\omega, \eta_k\omega^3\big) = 0\,. \tag{19}$$

Integrating by parts, we have,

$$4\big|\big(\boldsymbol{\nabla}\omega, \omega^3\boldsymbol{\nabla}\eta_k\big)\big| = \big|\big(\omega^4, \Delta\eta_k\big)\big| \le \|\omega\|_4^4\,\|\Delta\eta_k\|_\infty\,,$$

and using that $\boldsymbol{u}$ is divergence-free,

$$4\left|\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega,\eta_k\omega^3\right)\right| = \left|\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\eta_k,\omega^4\right)\right| \leq \|\omega\|_4^4\,\|\boldsymbol{u}\|_\infty\,\|\boldsymbol{\nabla}\eta_k\|_\infty\,,$$

so both terms vanish in the limit $k\to\infty$. Consequently, by using (19), we obtain that $\left\|\eta_k^{1/2}\omega\boldsymbol{\nabla}\omega\right\|_2$ is zero in the limit $k\to\infty$, so $\|\omega\boldsymbol{\nabla}\omega\|_2 = 0$, *i.e.* $\omega = 0$ since $\omega\in L^4(\Omega)$ by assumption. □

5. Analysis of the auxiliary system

In this section we analyze the auxiliary system $\Delta - 1$ in some standard spaces and also in specific spaces in order to characterize its exponential asymptotic behavior. All the results of this section will be used later only in the domain $\Omega_\rho = \mathbb{R}^2\setminus\bar{B}_\rho$ for ρ large, where $\bar{B}_\rho$ denotes the closed ball of radius ρ centered at the origin. Therefore, we may assume in this section that Ω is smooth and that $\Omega\cap\bar{B}_1=\emptyset$.

First of all, we have the following very standard result:

Lemma 5 *(weak solution). For $R\in L^2(\Omega)$, there exists a unique weak solution $b\in H_0^1(\Omega)\cap H^2(\Omega)$ of*

$$\Delta b - b = R\,, \tag{20}$$

and this solution satisfies $\|b\|_{H^1(\Omega)}\leq\|R\|_{L^2(\Omega)}$.

Proof. By using the Lax–Milgram theorem, we obtain existence and uniqueness of the weak solution $b\in H_0^1(\Omega)$. Moreover, we have $\|b\|_{H^1(\Omega)}\leq\|R\|_{L^2(\Omega)}$ and by elliptic regularity $b\in H^2(\Omega)$. □

We prove the following representation formula of the solution b:

Proposition 6 *(representation formula). For $R\in L^2(\Omega)$, the solution b of Lemma 5 can be represented as*

$$b = \mathcal{L}R - \mathcal{M}b\,, \tag{21}$$

with the operators $\mathcal{L}$ and $\mathcal{M}$ satisfying for $|\alpha|\leq 1$,

$$D^\alpha\mathcal{L}R(\boldsymbol{x}) = \int_\Omega D^\alpha K(\boldsymbol{x}-\boldsymbol{x}_0)R(\boldsymbol{x}_0)\,,\qquad D^\alpha\mathcal{M}b(\boldsymbol{x}) = \int_{\partial\Omega} D^\alpha K(\boldsymbol{x}-\boldsymbol{x}_0)\boldsymbol{\nabla}b(\boldsymbol{x}_0)\cdot\boldsymbol{n}\,, \tag{22}$$

where the integrations are perform on $\boldsymbol{x}_0$ and

$$K(\boldsymbol{x}) = \frac{1}{2\pi}K_0(|\boldsymbol{x}|)\,.$$

Proof. Let $\eta:\mathbb{R}\to[0,1]$ be a smooth cutoff function such that $\eta(r)=1$ if $r\leq\frac{1}{2}$ and $\eta(r)=0$ if $r\geq 1$. For $k\geq 1$, let $G_k(\boldsymbol{x}) = \eta(|\boldsymbol{x}|/k)K(\boldsymbol{x})$. For $\boldsymbol{x}\in\mathbb{R}^2\setminus\{\boldsymbol{0}\}$, we define

$$H_k = \Delta G_k - G_k\,,$$

and $H_k(\boldsymbol{0})=0$. Since K is the fundamental solution of $\Delta-1$, we have that H_k has support in $\frac{k}{2}\leq|\boldsymbol{x}|\leq k$ and a simple calculation shows that $\|H_k\|_{H^1(\mathbb{R}^2)}\lesssim k^{-1}$. We have the following Green's identity,

$$\int_\Omega (\Delta g - g)\,b = \int_\Omega (\Delta b - b)\,g + \int_{\partial\Omega}(b\boldsymbol{\nabla}g - g\boldsymbol{\nabla}b)\cdot\boldsymbol{n}\,,$$

and at fixed values of $\boldsymbol{x}\in\Omega$, by setting $g(\boldsymbol{x}_0) = G_k(\boldsymbol{x}-\boldsymbol{x}_0)$, we obtain for $0\leq|\alpha|\leq 1$,

$$D^\alpha b(\boldsymbol{x}) = \int_\Omega D^\alpha G_k(\boldsymbol{x}-\boldsymbol{x}_0)R(\boldsymbol{x}_0) - \int_{\partial\Omega} D^\alpha G_k(\boldsymbol{x}-\boldsymbol{x}_0)\boldsymbol{\nabla} b(\boldsymbol{x}_0)\cdot\boldsymbol{n} - \int_\Omega D^\alpha H_k(\boldsymbol{x}-\boldsymbol{x}_0)b(\boldsymbol{x}_0)\,, \tag{23}$$

where the integrations are performed over $\boldsymbol{x}_0$. Since $D^\alpha K \in L^2(\Omega)$ and $R \in L^2(\Omega)$, the first term in (23) converges to $D^\alpha \mathcal{L}R$ defined by (22). Since the boundary is compact, the second term in (23) is equal to $D^\alpha \mathcal{M}b$ for k large enough. Finally, since $\left\|H_k\right\|_{H^1(\mathbb{R}^2)} \lesssim k^{-1}$ and $b \in L^2(\Omega)$, the last term in (23) vanishes in the limit $k \to \infty$, so (21) is proven. □

We have the following standard lemma on the operator $\mathcal{L}$:

Lemma 7. *The operator $\mathcal{L}$ maps $L^2(\Omega)$ to $H^1(\Omega)$ and $\|\mathcal{L}R\|_{H^1(\Omega)} \le 2\|R\|_{L^2(\Omega)}$.*

Proof. This follows from the classical Young inequality, since $\|K\|_{L^1(\mathbb{R}^2)} = 1$ and $\|\boldsymbol{\nabla} K\|_{L^1(\mathbb{R}^2)} = \frac{\pi}{2}$. □

We now introduce weighted Sobolev spaces to analyze more carefully the operators $\mathcal{L}$ and $\mathcal{M}$:

Definition 8 *(Banach spaces).* Let $\mathcal{A}(\Omega)$ be the following weighted Sobolev space

$$\mathcal{A}(\Omega) = \left\{a \in L^\infty(\Omega) \,:\, \|a\|_{\mathcal{A}(\Omega)} < \infty\right\}, \tag{24}$$

where the norm is given by

$$\|a\|_{\mathcal{A}(\Omega)} = \sup_{\boldsymbol{x}\in\Omega}\left(r^{1/2}\,|a|\right) + \sup_{\boldsymbol{x}\in\Omega}\left(r^{3/2}\,|\boldsymbol{e}_r\cdot\boldsymbol{\nabla} a|\right) + \sup_{\boldsymbol{x}\in\Omega}\left(r\,|\boldsymbol{e}_\theta\cdot\boldsymbol{\nabla} a|\right).$$

Then, we define the following Banach space for the solution b,

$$\mathcal{B}(\Omega) = \left\{b \in H^1(\Omega) \,:\, \mathrm{e}^r b \in \mathcal{A}(\Omega)\right\}, \qquad \|b\|_{\mathcal{B}(\Omega)} = \|\mathrm{e}^r b\|_{\mathcal{A}(\Omega)}\,. \tag{25}$$

Finally, we define for $\varepsilon > 0$ the following weighted space for the right-hand side of (20),

$$\mathcal{R}_\varepsilon(\Omega) = \left\{R \in L^\infty(\Omega) \,:\, \|R\|_{\mathcal{R}_\varepsilon(\Omega)} < \infty\right\}, \qquad \|R\|_{\mathcal{R}_\varepsilon(\Omega)} = \sup_{\boldsymbol{x}\in\Omega}\left(\mathrm{e}^r r^{3/2+\varepsilon}\,|R|\right). \tag{26}$$

We remark that we have $\mathcal{B}(\Omega) \subset H^1(\Omega)$ and $\mathcal{R}_\varepsilon(\Omega) \subset L^2(\Omega)$ with continuous injections.

We first treat the boundary term $\mathcal{M}b$:

Proposition 9 *(properties and asymptote of $\mathcal{M}$). If b is smooth in a neighborhood of $\partial\Omega$, then $\mathcal{M}b \in \mathcal{B}(\Omega)$ and moreover*

$$\mathcal{M}b = \frac{\mathrm{e}^{-r}}{r^{1/2}}\left(\mu_{\mathcal{M}}(\theta) + O(r^{-1})\right),$$

where $\mu_{\mathcal{M}}$ is a continuously differentiable 2π-periodic function depending on b.

Proof. By using (5) and (11), the asymptotic expansion of the fundamental solution for fixed $\boldsymbol{x}_0$ is given by

$$K(\boldsymbol{x}-\boldsymbol{x}_0) = \frac{1}{\sqrt{8\pi}}\frac{\mathrm{e}^{-r}}{r^{1/2}}\left(\mathrm{e}^{r_0\cos(\theta-\theta_0)} + O(r^{-1})\right),$$

$$\boldsymbol{\nabla} K(\boldsymbol{x}-\boldsymbol{x}_0) = \frac{1}{\sqrt{8\pi}}\frac{\mathrm{e}^{-r}}{r^{1/2}}\left(-\mathrm{e}^{r_0\cos(\theta-\theta_0)}\boldsymbol{e}_r + O(r^{-1})\right),$$

and since $\partial\Omega$ is compact,

$$\mathcal{M}b(\boldsymbol{x}) = \frac{\mathrm{e}^{-r}}{r^{1/2}}\left(\mu_{\mathcal{M}}(\theta) + O(r^{-1})\right), \qquad \boldsymbol{\nabla}\mathcal{M}b(\boldsymbol{x}) = \frac{\mathrm{e}^{-r}}{r^{1/2}}\left(-\mu_{\mathcal{M}}(\theta)\boldsymbol{e}_r + O(r^{-1})\right),$$

where

$$\mu_{\mathcal{M}}(\theta) = \frac{1}{\sqrt{8\pi}}\int_{\partial\Omega} \mathrm{e}^{r_0\cos(\theta-\theta_0)}\boldsymbol{\nabla}b(\boldsymbol{x}_0)\cdot\boldsymbol{n}\,.$$

Since $\partial\Omega$ is compact, the function μ_2 is continuously differentiable. By defining $a = \mathrm{e}^r\mathcal{M}b$, we obtain $|a| \lesssim r^{-1/2}$ and $|\boldsymbol{\nabla}a| \lesssim r^{-3/2}$ *i.e.* $a \in \mathcal{A}(\Omega)$, and this ends the proof. □

The analysis of the operator $\mathcal{L}$ is more delicate. The aim is to show that $\mathcal{L}$ maps $\mathcal{R}_\varepsilon(\Omega)$ to $\mathcal{B}(\Omega)$ and since both spaces have an exponential weight, we first introduce the following function

$$W(\boldsymbol{x},\boldsymbol{x}_0) = \mathrm{e}^{r-r_0}K(\boldsymbol{x}-\boldsymbol{x}_0)\,, \tag{27}$$

in order to explicitly cancel the exponential factors and deal with polynomial decays. We first show the following preparatory lemmas:

Lemma 10. *The function (27) satisfies*

$$|W| \lesssim \frac{1}{r_1^{1/2}}\mathrm{e}^{r-r_1-r_0}\,, \qquad |\boldsymbol{e}_r\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}W| \lesssim \frac{1}{r_1^{3/2}}\mathrm{e}^{(r-r_1-r_0)/2}\,, \qquad |\boldsymbol{e}_\theta\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}W| \lesssim \frac{r_0^{1/2}}{r_1^{3/2}}\mathrm{e}^{(r-r_1-r_0)/2}\,,$$

where $r_1 = |\boldsymbol{x}-\boldsymbol{x}_0|$.

Proof. First of all, since

$$r_1^2 = r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0) \geq r^2 + r_0^2\cos^2(\theta-\theta_0) - 2rr_0\cos(\theta-\theta_0) = \left(r - r_0\cos(\theta-\theta_0)\right)^2\,,$$

we have

$$r_1 + r_0 - r \geq r_0\left(1-\cos(\theta-\theta_0)\right)\,. \tag{28}$$

The Bessel functions satisfy

$$K_0(r) \lesssim \frac{1}{r^{1/2}}\mathrm{e}^{-r}\,, \qquad K_1(r) \lesssim \frac{1}{r^{1/2}}\mathrm{e}^{-r}\,, \qquad K_1(r) - K_0(r) \lesssim \frac{1}{r^{3/2}}\mathrm{e}^{-r}\,,$$

so the first bound is proven. Since $r - r_0\cos(\theta-\theta_0) = r_1\cos(\theta_1-\theta)$, we have,

$$\begin{aligned}\partial_r W &= \frac{1}{2\pi r_1}\mathrm{e}^{r-r_0}\left[\left(r_0\cos(\theta-\theta_0) - r\right)K_1(r_1) + r_1K_0(r_1)\right]\\ &= \frac{1}{2\pi}\mathrm{e}^{r-r_0}\left[\frac{r_1 - r + r_0\cos(\theta-\theta_0)}{r_1}K_0(r_1) - \cos(\theta-\theta_1)\left(K_1(r_1)-K_0(r_1)\right)\right],\end{aligned}$$

so by using (28),

$$|\partial_r W| \lesssim \frac{1}{r_1^{3/2}} \mathrm{e}^{r-r_1-r_0} + \frac{1}{r_1} \mathrm{e}^{r-r_0} \left(|r_0+r_1-r| + r_0 \left|1-\cos(\theta-\theta_0)\right| \right) K_0(r_1)$$
$$\lesssim \frac{1}{r_1^{3/2}} \mathrm{e}^{r-r_1-r_0} \left[1 + |r_0+r_1-r| + r_0 \left|1-\cos(\theta-\theta_0)\right| \right] \lesssim \frac{1}{r_1^{3/2}} \mathrm{e}^{(r-r_1-r_0)/2} .$$

In the same way, we have

$$\frac{1}{r}\partial_\theta W = \frac{1}{2\pi r_1} \mathrm{e}^{r-r_0} r_0 \sin(\theta-\theta_0) K_1(r_1) ,$$

so by using (28),

$$\left| \frac{1}{r}\partial_\theta W \right| \lesssim \frac{1}{r_1^{3/2}} \mathrm{e}^{r-r_1-r_0} r_0 \left|\sin(\theta-\theta_0)\right| \lesssim \frac{r_0^{1/2}}{r_1^{3/2}} \mathrm{e}^{(r-r_1-r_0)/2} . \quad \square$$

Lemma 11. *For $\alpha, \sigma \in (0,2)$ such that $\alpha+\sigma > 3/2$, we have*

$$\int_{\mathbb{R}^2} \frac{1}{r_1^\alpha} \frac{1}{r_0^\sigma} \mathrm{e}^{r-r_1-r_0} \mathrm{d}\boldsymbol{x}_0 \lesssim \frac{1}{r^{\alpha+\sigma-3/2}} + \frac{|\log r|^{\delta_{\sigma,3/2}}}{r^\alpha} + \frac{|\log r|^{\delta_{\alpha,3/2}}}{r^\sigma} ,$$

where $r = |\boldsymbol{x}|$, $r_0 = |\boldsymbol{x}_0|$, $r_1 = |\boldsymbol{x}-\boldsymbol{x}_0|$ and $\delta_{\alpha,\sigma}$ denotes the Kronecker delta.

Proof. We have to estimate

$$I = \int_0^\infty \int_{-\pi}^{+\pi} \frac{1}{r_1^\alpha} \frac{1}{r_0^\sigma} \mathrm{e}^{(r-r_1-r_0)} r_0 \mathrm{d}\theta_0 \mathrm{d}r_0 .$$

First of all, by using (28), we have

$$\int_{-\pi}^{+\pi} \mathrm{e}^{(r-r_1-r_0)} \mathrm{d}\theta_0 \leq \int_{-\pi}^{+\pi} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \mathrm{d}\theta_0 \leq \frac{1}{1+r_0^{1/2}} . \tag{29}$$

If $r_0 \leq r/2$, then $r_1 \geq r/2$ and therefore

$$I \lesssim \frac{1}{r^\alpha} \int_0^{r/2} \int_{-\pi}^{+\pi} \frac{1}{r_0^\sigma} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} r_0 \mathrm{d}\theta_0 \mathrm{d}r_0 \lesssim \frac{1}{r^\alpha} \int_0^{r/2} \frac{1}{r_0^{\sigma-1}} \frac{1}{1+r_0^{1/2}} \mathrm{d}r_0 \lesssim \frac{|\log r|^{\delta_{\sigma,3/2}}}{r^\alpha} + \frac{1}{r^{\alpha+\sigma-3/2}} .$$

In the case where $r_1 \leq r/2$, we have, by symmetry, the previous bound with α and σ exchanged,

$$I \lesssim \frac{|\log r|^{\delta_{\alpha,3/2}}}{r^\sigma} + \frac{1}{r^{\alpha+\sigma-3/2}} .$$

Therefore, it only remains the case where $r_0 \geq r/2$ and $r_1 \geq r/2$, for which

$$I \lesssim \int_{r/2}^\infty \int_{-\pi}^{+\pi} \frac{1}{r^\alpha} \frac{1}{r_0^\sigma} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} r_0 \mathrm{d}\theta_0 \mathrm{d}r_0 \lesssim \frac{1}{r^\alpha} \int_{r/2}^\infty \frac{1}{r_0^{\sigma-1/2}} \mathrm{d}r_0 \lesssim \frac{1}{r^{\alpha+\sigma-3/2}} . \quad \square$$

Using these two last lemmas, we can deduce the following bound on the map $\mathcal{L}$:

Proposition 12 *(properties of $\mathcal{L}$). For $\varepsilon \in \left(0, \frac{1}{2}\right)$ the operator $\mathcal{L}$ maps $\mathcal{R}_\varepsilon(\Omega)$ to $\mathcal{B}(\Omega)$ and there exists a constant $C_\varepsilon > 0$ independent of Ω such that $\left\|\mathcal{L}R\right\|_{\mathcal{B}(\Omega)} \le C_\varepsilon \left\|R\right\|_{\mathcal{R}_\varepsilon(\Omega)}$.*

Proof. We define $a = \mathrm{e}^r \mathcal{L}R$ and $S = \mathrm{e}^{-r} R$ so by hypothesis $|S| \lesssim r^{-3/2-\varepsilon}$. In view of the definition (27) of W, we have

$$\boldsymbol{\nabla}_{\boldsymbol{x}} W(\boldsymbol{x}, \boldsymbol{x}_0) = \mathrm{e}^{r-r_0}\left[\boldsymbol{\nabla} G(\boldsymbol{x} - \boldsymbol{x}_0) + G(\boldsymbol{x} - \boldsymbol{x}_0)\boldsymbol{e}_r\right],$$

and the representation formula (22) for $\mathcal{L}R$ and its gradient becomes

$$a(\boldsymbol{x}) = \int_\Omega W(\boldsymbol{x}, \boldsymbol{x}_0) S(\boldsymbol{x}_0)\,, \qquad \boldsymbol{\nabla} a(\boldsymbol{x}) = \int_\Omega \boldsymbol{\nabla}_{\boldsymbol{x}} W(\boldsymbol{x}, \boldsymbol{x}_0) S(\boldsymbol{x}_0)\,.$$

Since $|S| \lesssim r^{-3/2-\varepsilon}$, by using Lemma 10 and Lemma 11 we obtain that $|a| \lesssim r^{-1/2}$, $|\boldsymbol{e}_r \cdot \boldsymbol{\nabla} a| \lesssim r^{-3/2}$, and $|\boldsymbol{e}_\theta \cdot \boldsymbol{\nabla} a| \lesssim r^{-1}$. More precisely, we have $\|a\|_{\mathcal{A}(\Omega)} \le C_\varepsilon \|R\|_{\mathcal{R}_\varepsilon(\Omega)}$, where $C_\varepsilon > 0$ is a constant depending only on ε and not on Ω. □

In order to extract the asymptote of $\mathcal{L}R$, we will need the following two lemmas:

Lemma 13. *For all $r > 0$ and $\theta \in (-\pi, \pi]$, we have*

$$\int_{\mathbb{R}^2} \left| \frac{1}{r_1^{1/2}} \mathrm{e}^{r-r_0-r_1} - \frac{1}{r^{1/2}} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \right| \frac{1}{r_0^{3/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 \lesssim \frac{1}{r^{1/2+\varepsilon}}\,.$$

Proof. Let I denote the left-hand side of the claimed bound. For $r_0 \ge r/2$, by using (29), we have

$$\int_{\mathbb{R}^2} \frac{1}{r^{1/2}} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{1}{r_0^{3/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 \lesssim \frac{1}{r^{1/2}} \int_{r/2}^{\infty} \frac{1}{r_0^{1+\varepsilon}} \mathrm{d}r_0 \lesssim \frac{1}{r^{1/2+\varepsilon}}\,,$$

and in the same way as in the proof of Lemma 11, we obtain still for $r_0 \ge r/2$,

$$\int_{\mathbb{R}^2} \frac{1}{r_1^{1/2}} \mathrm{e}^{r-r_0-r_1} \frac{1}{r_0^{3/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 \lesssim \frac{1}{r^{1/2+\varepsilon}}\,,$$

so that I is bounded by $r^{-1/2-\varepsilon}$ in this region. For $r_0 \le r/2$, we have

$$\left| \frac{1}{r_1^{1/2}} - \frac{1}{r^{1/2}} \right| = \frac{1}{r_1^{1/2} r^{1/2}} \frac{r_0}{r^{1/2} + r_1^{1/2}} \frac{|r_0 - 2r\cos(\theta-\theta_0)|}{r + r_1} \lesssim \frac{r_0}{r_1^{1/2} r}\,,$$

and since

$$\big(r_1 - r + r_0\cos(\theta-\theta_0)\big)\big(r_1 + r - r_0\cos(\theta-\theta_0)\big) = r_0^2 \sin^2(\theta-\theta_0)\,,$$

we have

$$\begin{aligned}
\left|\mathrm{e}^{r-r_0-r_1} - \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))}\right| &\lesssim \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \left|1 - \mathrm{e}^{-r_1+r-r_0\cos(\theta-\theta_0)}\right| \\
&\lesssim \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \big(r_1 - r + r_0\cos(\theta-\theta_0)\big) \\
&\lesssim \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{r_0^2 \sin^2(\theta-\theta_0)}{r_1 + r - r_0\cos(\theta-\theta_0)}
\end{aligned}$$

$$\lesssim \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{r_0^2 \left|\theta-\theta_0\right|^2}{r} .$$

Therefore, still for $r_0 \le r/2$, we have

$$\begin{aligned} I &\lesssim \int_{\mathbb{R}^2} \left| \frac{1}{r_1^{1/2}} - \frac{1}{r^{1/2}} \right| \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{1}{r_0^{3/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 + \int_{\mathbb{R}^2} \frac{1}{r_1^{1/2}} \left| \mathrm{e}^{r-r_0-r_1} - \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \right| \frac{1}{r_0^{3/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 \\ &\lesssim \frac{1}{r} \int_{\mathbb{R}^2} \frac{1}{r_1^{1/2}} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{1}{r_0^{1/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 + \int_{\mathbb{R}^2} \frac{1}{r_1^{1/2}} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{r_0 \left|\theta-\theta_0\right|^2}{r_0^{1/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 \\ &\lesssim \frac{1}{r} \int_{\mathbb{R}^2} \frac{1}{r_1^{1/2}} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} \frac{1 + r_0 \left|\theta-\theta_0\right|^2}{r_0^{1/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 \\ &\lesssim \frac{1}{r} \int_{\mathbb{R}^2} \frac{1}{r_1^{1/2}} \mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))/2} \frac{1}{r_0^{1/2+\varepsilon}} \mathrm{d}\boldsymbol{x}_0 \\ &\lesssim \frac{1}{r^{3/2}} \int_0^{r/2} \frac{1}{r_0^{\varepsilon}} \mathrm{d}r_0 \lesssim \frac{1}{r^{1/2+\varepsilon}} . \quad \square \end{aligned}$$

Lemma 14. *For $0 < \alpha < 1$, $r > 0$, and $\theta_1, \theta_2 \in \mathbb{R}$, we have*

$$\left|\mathrm{e}^{-r(1-\cos\theta_1)} - \mathrm{e}^{-r(1-\cos\theta_2)}\right| \lesssim \left|\theta_1-\theta_2\right|^\alpha r^{\alpha/2} \left[\mathrm{e}^{-r(1-\cos\theta_1)/2} + \mathrm{e}^{-r(1-\cos\theta_2)/2}\right].$$

Proof. First of all, we remark that for $a_1, a_2 \ge 0$, we have for any $0 < \alpha < 1$,

$$\left|\mathrm{e}^{-a_1} - \mathrm{e}^{-a_2}\right| \le \left|a_1 - a_2\right|^\alpha ,$$

because, if $a_1 \ge a_2$,

$$\left|\mathrm{e}^{a_1} - \mathrm{e}^{a_2}\right| \le \mathrm{e}^{a_1} \left|1 - \mathrm{e}^{-(a_1-a_2)}\right| \le \mathrm{e}^{a_1} \left|a_1 - a_2\right|^\alpha ,$$

and the claimed bound follows by interchanging a_1 and a_2. We define $\delta = \frac{\theta_1-\theta_2}{2}$ and for $\cos\theta_1 \ge \cos\theta_2$, we have

$$\begin{aligned} \left|\mathrm{e}^{r\cos\theta_1} - \mathrm{e}^{r\cos\theta_2}\right| &\le \mathrm{e}^{r\cos\theta_1} \left|1 - \mathrm{e}^{-2r\sin^2\delta}\right| + \left|\mathrm{e}^{r\cos\theta_1 - 2r\sin^2\delta} - \mathrm{e}^{r\cos\theta_2}\right| \\ &\le \mathrm{e}^{r\cos\theta_1} \left[\left|1 - \mathrm{e}^{-2r\sin^2\delta}\right| + \left|\mathrm{e}^{-2r\sin^2\delta} - \mathrm{e}^{-r(\cos\theta_1-\cos\theta_2)}\right|\right] \\ &\lesssim \mathrm{e}^{r\cos\theta_1} \left[r^{\alpha/2} \left|\delta\right|^\alpha + r^\alpha \left|\cos\theta_1 - \cos\theta_2 - 2r\sin^2\delta\right|^\alpha\right] \\ &\lesssim \mathrm{e}^{r\cos\theta_1} \left[r^{\alpha/2} \left|\delta\right|^\alpha + r^\alpha \left|\theta_1\right|^\alpha \left|\delta\right|^\alpha\right] , \end{aligned}$$

since

$$\cos\theta_1 - \cos\theta_2 - 2r\sin^2\delta = -4\cos\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right)\sin\delta .$$

Therefore, by using (10), we have

$$\left|\mathrm{e}^{-r(1-\cos\theta_1)} - \mathrm{e}^{-r(1-\cos\theta_2)}\right| \lesssim \mathrm{e}^{-r(1-\cos\theta_1)} \left[r^{\alpha/2} \left|\delta\right|^\alpha + r^\alpha \left|\theta_1\right|^\alpha \left|\delta\right|^\alpha\right] \lesssim \left|\delta\right|^\alpha r^{\alpha/2} \mathrm{e}^{-r(1-\cos\theta_1)/2} ,$$

provided $\cos\theta_1 \geq \cos\theta_2$. For $\cos\theta_1 \leq \cos\theta_2$, we have the same bound with θ_1 and θ_2 interchanged, so the lemma is proven. □

Proposition 15 *(asymptote of $\mathcal{L}$). If $R \in \mathcal{R}_\varepsilon(\Omega)$ for some $\varepsilon \in \left(0, \frac{1}{2}\right)$, then*

$$\mathcal{L}R = \frac{1}{r^{1/2}}\left(\mu_{\mathcal{L}}(\theta) + O(r^{-\varepsilon})\right),$$

where $\mu_{\mathcal{L}}$ is a 2π-periodic function depending on R which is α-Hölder continuous for $0 < \alpha < 2\varepsilon < 1$.

Proof. By using the asymptotic expansion of the Bessel function K_0, we have

$$\left|\sqrt{\frac{2}{\pi}}K_0(r) - \frac{\mathrm{e}^{-r}}{r^{1/2}}\right| \leq \frac{1}{r^{3/2}}\mathrm{e}^{-r},$$

for all $r > 0$, so

$$\left|\sqrt{8\pi}W - \frac{1}{r_1^{1/2}}\mathrm{e}^{r-r_1-r_0}\right| \leq \frac{1}{r_1^{3/2}}\mathrm{e}^{r-r_1-r_0},$$

and therefore

$$I = \int_{\mathbb{R}^2}\left|\sqrt{8\pi}W - \frac{1}{r^{1/2}}\mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))}\right|\frac{1}{r_0^{3/2+\varepsilon}}\mathrm{d}\boldsymbol{x}_0$$

is bounded by $I_1 + I_2$, where

$$I_1 = \int_{\mathbb{R}^2}\frac{1}{r_1^{3/2}}\mathrm{e}^{r-r_0-r_1}\frac{1}{r_0^{3/2+\varepsilon}}\mathrm{d}\boldsymbol{x}_0,$$

$$I_2 = \int_{\mathbb{R}^2}\left|\frac{1}{r_1^{1/2}}\mathrm{e}^{r-r_0-r_1} - \frac{1}{r^{1/2}}\mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))}\right|\frac{1}{r_0^{3/2+\varepsilon}}\mathrm{d}\boldsymbol{x}_0.$$

In view of Lemma 11, I_1 is bounded by $r^{-3/2}$ and by using Lemma 13, I_2 is bounded by $r^{-1/2-\varepsilon}$. Therefore I is bounded by $r^{-1/2-\varepsilon}$ and the asymptotic behavior of $a = \mathrm{e}^r\mathcal{L}R$ is given by

$$a(\boldsymbol{x}) = \frac{1}{r^{1/2}}\left(\mu_{\mathcal{L}}(\theta) + O(r^{-\varepsilon})\right),$$

where

$$\mu_{\mathcal{L}}(\theta) = \frac{1}{\sqrt{8\pi}}\int_{\Omega}\mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))}S(\boldsymbol{x}_0).$$

By using Lemma 14, we obtain the α-Hölder continuity of $\mu_{\mathcal{L}}$,

$$\begin{aligned}|\mu_{\mathcal{L}}(\theta) - \mu_{\mathcal{L}}(\theta')| &\lesssim \int_{\mathbb{R}^2}\left|\mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))} - \mathrm{e}^{-r_0(1-\cos(\theta'-\theta_0))}\right|\frac{1}{r_0^{3/2+\varepsilon}}\mathrm{d}\boldsymbol{x}_0\\ &\lesssim |\theta-\theta'|^{\alpha}\int_{\mathbb{R}^2}\left(\mathrm{e}^{-r_0(1-\cos(\theta-\theta_0))/2} + \mathrm{e}^{-r_0(1-\cos(\theta'-\theta_0))/2}\right)\frac{1}{r_0^{3/2+\varepsilon-\alpha/2}}\mathrm{d}\boldsymbol{x}_0\end{aligned}$$

$$\lesssim |\theta-\theta'|^{\alpha} \int_0^{\infty} \frac{1}{1+r_0^{1/2}} \frac{1}{r_0^{1/2+\varepsilon-\alpha/2}} \mathrm{d}r_0 \lesssim |\theta-\theta'|^{\alpha} \,,$$

for $0 < \alpha < 2\varepsilon < 1$, since the last integral converges. Therefore, the asymptotic behavior claimed on $\mathcal{L}R = \mathrm{e}^{-r} a$ is proven. □

6. Proof of the main result

This section is dedicated to the proof of Theorem 2. We start with the following basic lemma:

Lemma 16. *For any $\alpha > 0$ and $A_1, A_2 \in \mathbb{R}$, there exists $C > 0$, such that*

$$|\theta|^{\alpha} \, r^{A_1(1-\cos\theta)-A_2\sin\theta} \mathrm{e}^{-r(1-\cos\theta)} \leq \frac{C}{r^{\alpha/2}} \mathrm{e}^{-r(1-\cos\theta)/2} \,,$$

for all $r > 1$ and $\theta \in (-\pi, \pi]$.

Proof. First of all, for $|\theta| \geq 1$, the bound is trivial due to the exponential decay of $\mathrm{e}^{-r(1-\cos\theta)}$. For $|\theta| \leq 1$, we have for any $\beta > 0$,

$$r^{A_1(1-\cos\theta)-A_2\sin\theta} = \mathrm{e}^{[A_1(1-\cos\theta)-A_2\sin\theta]\log r} \leq \mathrm{e}^{(|A_1|+|A_2|)|\theta|\log r} \leq \mathrm{e}^{(|A_1|+|A_2|)z} \,,$$

where $z = r^{1/2}\,|\theta|$. Therefore,

$$|\theta|^{\alpha} \, r^{A_1(1-\cos\theta)-A_2\sin\theta} \mathrm{e}^{-r(1-\cos\theta)/2} \lesssim \frac{z^{\alpha}}{r^{\alpha/2}} \mathrm{e}^{(|A_1|+|A_2|)z} \mathrm{e}^{-z^2/4} \lesssim \frac{1}{r^{\alpha/2}} \,,$$

and the claimed bound follows by multiplying by $\mathrm{e}^{-r(1-\cos\theta)/2}$. □

The proof of the main Theorem 2 will be deduced as a corollary of Theorem 1 and the following result:

Theorem 17. *Let $\boldsymbol{u} \in \dot{H}^1(\Omega)$ be a solution of (1). If there exists $c, \nu > 0$ and $(A_1, A_2) \in \mathbb{R}^2$ such that*

$$\begin{aligned} \left|(\boldsymbol{u}-\boldsymbol{u}_\infty-\boldsymbol{u}_h)\cdot\boldsymbol{e}_r\right| &\leq \frac{c}{r^{1/2}} \mathrm{e}^{-r(1-\cos\theta)/2} + \frac{c}{r^{1+\nu}} \,, \\ \left|(\boldsymbol{u}-\boldsymbol{u}_\infty-\boldsymbol{u}_h)\cdot\boldsymbol{e}_\theta\right| &\leq \frac{c}{r} \mathrm{e}^{-r(1-\cos\theta)/2} + \frac{c}{r^{1+\nu}} \,, \end{aligned} \tag{30}$$

for all $\boldsymbol{x} \in \Omega$, where

$$\boldsymbol{u}_h = \frac{2A_1}{r}\boldsymbol{e}_r - \frac{2A_2}{r}\boldsymbol{e}_\theta \,, \tag{31}$$

then (16) holds. Moreover for all $\varepsilon < \min\left(\frac{1}{2}, \nu\right)$, the asymptotic expansion (15) is valid for some 2π-periodic and 2ε-Hölder continuous function μ.

Proof. For $\rho > 0$, let $\Omega_\rho = \mathbb{R}^2 \setminus \bar{B}_\rho$, where $\bar{B}_\rho$ is the closed ball of radius ρ centered at the origin. We only consider $\rho > \rho_0 > 1$, with ρ_0 large enough, such that $\Omega_{\rho_0} \subset \Omega$ and $\boldsymbol{f}$ vanishes on Ω_{ρ_0}. We note that by the standard elliptic regularity, the solution is smooth in Ω_ρ. Since we are only interested in the behavior of the solution near infinity, we consider the solution $\boldsymbol{u}$ restricted to Ω_ρ, with ρ to be fixed later on.

By hypothesis the solution $\boldsymbol{u}$ of (1) can be written as $\boldsymbol{u} = \boldsymbol{u}_\infty + \boldsymbol{u}_h + \bar{\boldsymbol{u}}$, with $\bar{\boldsymbol{u}}$ satisfying

$$|\bar{\boldsymbol{u}}\cdot\boldsymbol{e}_r| \le \frac{C}{r^{1/2}}\mathrm{e}^{-r(1-\cos\theta)/2} + \frac{C}{r^{1+\nu}}\,, \qquad |\bar{\boldsymbol{u}}\cdot\boldsymbol{e}_\theta| \le \frac{C}{r}\mathrm{e}^{-r(1-\cos\theta)/2} + \frac{C}{r^{1+\nu}}\,, \tag{32}$$

for all $r \ge \rho$ and $\theta \in (-\pi, \pi]$. We can assume without loss of generality that $\nu < \frac{1}{2}$. To prove the result, we consider the vorticity equation

$$\Delta\omega - 2\partial_1\omega - \boldsymbol{u}_h\cdot\boldsymbol{\nabla}\omega = \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\omega + \boldsymbol{\nabla}\wedge\boldsymbol{f}\,. \tag{33}$$

In order to show that the solution ω of this equation satisfies the optimal asymptotic properties claimed, a bootstrap argument from the hypothesis (30) is not sufficient. We will proceed in an unusual way by viewing the vorticity equation (33) as a linear equation for ω with $\bar{\boldsymbol{u}}$ fixed. More precisely we will not assume that ω is defined by $\omega = \boldsymbol{\nabla}\wedge\boldsymbol{u}$, but only that ω is a solution satisfying (33) in Ω_ρ together with the boundary condition $\omega = \boldsymbol{\nabla}\wedge\boldsymbol{u}$ on $\partial\Omega_\rho$. Since the boundary $\partial\Omega_\rho$ and $\boldsymbol{\nabla}\wedge\boldsymbol{u}$ are smooth, there exists an extension $\omega^* \in C_0^\infty(\bar{\Omega}_\rho)$ of $\boldsymbol{\nabla}\wedge\boldsymbol{u}$ having bounded support. So by defining $\omega_0 = \omega - \omega^*$, we obtain that ω_0 vanishes at the boundary and satisfies

$$\Delta\omega_0 - 2\partial_1\omega_0 - \boldsymbol{u}_h\cdot\boldsymbol{\nabla}\omega_0 = \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\omega_0 + Q\,, \tag{34}$$

for some $Q \in C_0^\infty(\bar{\Omega}_\rho)$. By using the results of Galdi [1, Lemmas XII.3.1 & XII.3.2], the field $\omega = \boldsymbol{\nabla}\wedge\boldsymbol{u}$ satisfies $\omega \in \dot{H}^1(\Omega_\rho)\cap L^4(\Omega_\rho)$, *i.e.* $\omega_0 \in \dot{H}_0^1(\Omega_\rho)\cap L^4(\Omega_\rho)$. In view of the Liouville-type result proven in Proposition 4, the solution of (34) is unique in the class $\omega_0 \in \dot{H}_0^1(\Omega_\rho)\cap L^4(\Omega_\rho)$. The rest of this proof is dedicated to the construction by some fixed point arguments of a solution $\omega_0 \in \dot{H}_0^1(\Omega_\rho)\cap L^4(\Omega_\rho)$ of (34) satisfying (16) and (15) in Ω_ρ, for ρ large enough. This construction will end the proof of the theorem.

First of all, we make a change of variables in order to formally transform (34) into an equation having an explicit dominant term at infinity. The change of variables,

$$\omega_0(r,\theta) = r^{A_1(1-\cos\theta)-A_2\sin\theta}\mathrm{e}^{r\cos\theta}b(r,\theta)\,, \tag{35}$$

transforms, after some straightforward calculations, the original equation (34) into

$$\Delta b - b = \mathcal{N}b + R\,, \qquad \mathcal{N}b = \boldsymbol{v}\cdot(\boldsymbol{\nabla}b + b\boldsymbol{e}_r) + hb\,, \tag{36}$$

where

$$\begin{aligned}
h = \frac{1}{r^2}\Big[&-\log^2 r\,(A_1\sin\theta - A_2\cos\theta)^2 - 2A_2\log r\,(A_1\sin\theta - A_2\cos\theta) \\
&- \log r\,(A_1\cos\theta + A_2\sin\theta) + \sin\theta\left(\left(A_1^2 - A_2^2\right)\sin\theta - 2A_1A_2\cos\theta\right)\Big] \\
&- (1-\cos\theta)\bar{u}_r - \sin\theta\,\bar{u}_\theta + \frac{\log r}{r}\,(A_1\sin\theta - A_2\cos\theta)\,\bar{u}_\theta + \frac{A_1(1-\cos\theta) - A_2\sin\theta}{r}\bar{u}_r\,, \\
\boldsymbol{v} = \bar{\boldsymbol{u}} &+ \frac{2}{r}\,(A_1\cos\theta + A_2\sin\theta)\,\boldsymbol{e}_r - \frac{2\log r}{r}\,(A_1\sin\theta + A_2\cos\theta)\,\boldsymbol{e}_\theta - \frac{2A_2}{r}\boldsymbol{e}_\theta\,, \\
R = r&^{-A_1(1-\cos\theta)+A_2\sin\theta}\mathrm{e}^{-r\cos\theta}Q\,.
\end{aligned}$$

We now sketch the two reasons behind the particular splitting of $\mathcal{N}b$ in (36). As we will see later, b decays exponentially fast, so by defining $a = \mathrm{e}^r b$, we expect a to decay algebraically. Therefore, we expect that $\boldsymbol{\nabla}a = \mathrm{e}^r\,(\boldsymbol{\nabla}b + b\boldsymbol{e}_r)$ will decay faster than a. We will see that there is indeed a cancellation in the term $\boldsymbol{\nabla}b + b\boldsymbol{e}_r$, which decays faster than b and $\boldsymbol{\nabla}b$. This is the first reason behind the arrangement of the terms in the definition of $\mathcal{N}b$ in (36), the other being that h decays faster than $\boldsymbol{v}\cdot\boldsymbol{e}_r$. More precisely, by using Lemma 16, we obtain the following estimates from (32),

$$|\boldsymbol{v}\cdot\boldsymbol{e}_r| \leq \frac{C}{r^{1/2}}\,, \qquad |\boldsymbol{v}\cdot\boldsymbol{e}_\theta| \leq \frac{C\log r}{r}\,, \qquad |h| \leq \frac{C}{r^{1+\nu}}\,, \tag{37}$$

for some $C > 0$ independent of ρ.

We first construct a solution $b \in H_0^1(\Omega_\rho)$ of (36) for ρ large enough. This solution will be used in a second step to prove that $b \in \mathcal{B}(\Omega_\rho)$. For any $b \in H_0^1(\Omega_\rho)$, we have

$$\|\mathcal{N}b\|_{L^2(\Omega_\rho)} \leq \Big(\|\boldsymbol{v}\|_\infty \|\boldsymbol{\nabla} b\|_{L^2(\Omega_\rho)} + (\|\boldsymbol{v}\|_\infty + \|h\|_\infty)\|b\|_{L^2(\Omega_\rho)}\Big) \leq C\rho^{-1/2}\|b\|_{H^1(\Omega_\rho)}\,, \tag{38}$$

and therefore in view of Lemma 5 for any $\rho > C^2$, a fixed point argument proves the existence of a unique solution $\tilde{b} \in H_0^1(\Omega_\rho)$ of (36). By elliptic regularity and since $\boldsymbol{v}$, h, and R are smooth, the solution $\tilde{b}$ is smooth.

Now the aim is to show the existence of a solution $b \in \mathcal{B}(\Omega_\rho)$ of (36) (see Definition 8 for the definition of the function spaces). In view of Proposition 6 and since $\tilde{b}$ is a solution of (36),

$$\tilde{b} = \mathcal{L}\mathcal{N}\tilde{b} + \mathcal{L}R + \mathcal{M}\tilde{b}\,. \tag{39}$$

The aim is now to prove the existence of a solution $b \in \mathcal{B}(\Omega_\rho)$ of

$$b = \mathcal{L}\mathcal{N}b + \mathcal{L}R + \mathcal{M}\tilde{b}\,. \tag{40}$$

Since R has bounded support, $\mathcal{L}R \in \mathcal{B}(\Omega_\rho)$ by Proposition 12 and Proposition 9 implies that $\mathcal{M}\tilde{b} \in \mathcal{B}(\Omega_\rho)$. Let $\varepsilon < \nu$, so in view of Proposition 12, if $a = \mathrm{e}^r b \in \mathcal{A}(\Omega_\rho)$, we have

$$\big\|\mathcal{N}b\big\|_{\mathcal{R}_\varepsilon(\Omega_\rho)} = \big\|\mathrm{e}^r\,(\boldsymbol{v}\cdot\boldsymbol{\nabla} a + ha)\big\|_{\mathcal{R}_\varepsilon(\Omega_\rho)} \leq C\rho^{-(\nu-\varepsilon)}\|b\|_{\mathcal{B}(\Omega_\rho)}\,.$$

Using Proposition 15, we therefore obtain

$$\big\|\mathcal{L}\mathcal{N}b\big\|_{\mathcal{B}(\Omega_\rho)} \leq C_\varepsilon\big\|\mathcal{N}b\big\|_{\mathcal{R}_\varepsilon(\Omega_\rho)} \leq C_\varepsilon C\rho^{-(\nu-\varepsilon)}\|b\|_{\mathcal{B}(\Omega_\rho)}\,,$$

so a fixed point argument proves the existence of a solution $b \in \mathcal{B}(\Omega_\rho)$ of (40) provided ρ is large enough. However, a priori the solution $b \in \mathcal{B}(\Omega_\rho)$ does not necessarily vanish at the boundary $\partial\Omega_\rho$. To prove this is indeed the case, we now show that $b = \tilde{b}$. Subtracting (40) from (39) leads to

$$\tilde{b} - b = \mathcal{L}\mathcal{N}(\tilde{b} - b)\,,$$

since $\mathcal{L}$ and $\mathcal{N}$ are linear. By using (38) and Lemma 7,

$$\big\|\tilde{b} - b\big\|_{H^1(\Omega_\rho)} \leq 2\big\|\mathcal{N}(\tilde{b} - b)\big\|_{L^2(\Omega_\rho)} \leq 2C\rho^{-1/2}\big\|\tilde{b} - b\big\|_{H^1(\Omega_\rho)}\,,$$

so $b = \tilde{b}$ for $\rho > 4C^2$. This ends the proof of the existence of a smooth $b \in H_0^1(\Omega_\rho) \cap \mathcal{B}(\Omega_\rho)$ satisfying (36) in the classical sense, provided ρ is large enough.

Therefore, ω_0 defined by (35) is a classical smooth solution of (34). By using Lemma 16, we have

$$|\omega_0| \lesssim r^{-1/2} r^{A_1(1-\cos\theta) - A_2\sin\theta}\mathrm{e}^{-r(1-\cos\theta)} \lesssim r^{-1/2}\mathrm{e}^{-r(1-\cos\theta)/2}\,,$$
$$|\boldsymbol{\nabla}\omega_0| \lesssim \Big(|\theta|\,r^{-1/2} + r^{-1}\Big)\,\mathrm{e}^{-r(1-\cos\theta)/2} \lesssim r^{-1}\mathrm{e}^{-r(1-\cos\theta)/4}\,,$$

so $\omega_0 \in L^4(\Omega_\rho)$ and $\boldsymbol{\nabla}\omega_0 \in L^2(\Omega_\rho)$. Since $b \in H_0^1(\Omega_\rho)$, it follows that $\omega_0 \in \dot{H}_0^1(\Omega_\rho)$ and this completes the proof of the existence of a solution $\omega_0 \in \dot{H}_0^1(\Omega_\rho) \cap L^4(\Omega_\rho)$ of (34) satisfying (16).

Finally, by using Propositions 9 and 15, there exists a 2π-periodic function $\mu = \mu_{\mathcal{L}} - \mu_{\mathcal{M}}$ such that

$$b(\boldsymbol{x}) = \frac{\mathrm{e}^{-r}}{r^{1/2}}\big(\mu(\theta) + O(r^{-\varepsilon})\big),$$

and μ is α-Hölder continuous for $0 < \alpha < 2\varepsilon < 1$. Therefore, (15) is proven by using the freedom in the choice of $\varepsilon < \nu$ in order to allow $\alpha = 2\varepsilon$. $\square$

Combining Theorem 1 and Theorem 17, we deduce Theorem 2 as a corollary:

Proof of Theorem 2. First of all, by using (6) in conjunction with Lemma 16, we obtain

$$\mathbf{E}(\boldsymbol{x})\boldsymbol{F} = \boldsymbol{u}_h + O\left(\left(\frac{\boldsymbol{e}_1}{r^{1/2}} + \frac{\boldsymbol{e}_2}{r}\right)\mathrm{e}^{-r(1-\cos\theta)/2}\right), \qquad \boldsymbol{A} = \frac{\boldsymbol{F}}{8\pi},$$

where $\boldsymbol{u}_h$ is the harmonic function defined by (31). Therefore, by applying again Lemma 16, the conclusion of Theorem 1 implies the hypothesis (30), so the theorem is proven by applying Theorem 17. $\square$

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