Weak-strong uniqueness of solutions to the exterior Navier-Stokes flow in a half space

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Abstract

We consider the problem of a body moving within an incompressible fluid at constant speed parallel to a wall, in an otherwise unbounded domain. This situation is modeled by the incompressible Navier-Stokes equations in an exterior domain in a half space, with appropriate boundary conditions on the wall, the body, and at infinity. In our previous work, we have shown the existence of stationary solutions for this problem for a simplified situation where the body is replaced by a source term with compact support. Furthermore, the leading order asymptotic behavior of velocity field was also shown. Here, we prove in a very general setup that the solution of the auxiliary problem is unique and coincides outside a finite region with solutions of the problem of the moving body. The uniqueness of weak

*Supported by the SSSTC Follow-up Project (No. EG19-032009) and National Science Foundation of China (Grant No. 11301394).

†Supported by the Swiss National Science Foundation (Grant No. ******).

solutions for the body problem follows as a by-product of our method of proof.

Contents

1 Introduction **2**

2 Weak solutions for the body problem **6**
- 2.1 Main results 6
- 2.2 Definition of weak solutions 6
- 2.3 Existence of weak solutions 9
- 2.4 Limit of weak solution as the obstacle vanishes 12

3 Asymptotic behavior of weak solution **15**
- 3.1 Truncation procedure 16
- 3.2 Existence of α-solutions 18

4 Uniqueness of weak solutions for the body problem **21**

A Technical lemmas **22**

Mathematics Subject Classification (2000): 35Q30, 76D05, 35B40, 35B42

Keywords: Stationary Navier-Stokes equation; Fluid-structure interaction; Weak solution

1 Introduction

In this paper we discuss solutions of the Navier-Stokes equations for the stationary flow around a body which moves with constant speed parallel to a wall in an otherwise unbounded space filled with incompressible fluid. In the previous work [11] we considered the simplified case where the body is replaced by a force term with compact support. We showed the existence of solutions in a functional framework that provides basic information on the asymptotic behavior at infinity. Using this result, we now show in a very general setup, that the solution of the simplified problem coincides outside a finite region with solutions of the problem of the moving body. This information is then used to prove the uniqueness of weak solutions.

The mathematical formulation of this problem is as follows. Let $\Omega_+ = \{(x,y,z) \in \mathbb{R}^3 \,|\, z > 1\}$, $\partial\Omega_+ = \{(x,y,z) \in \mathbb{R}^3 \,|\, z = 1\}$, $B \subset \Omega_+$, $\mathbf{e}_1 = (1,0,0)$ and $B_t = \{\mathbf{X} \in \mathbb{R}^3 \,|\, \mathbf{X} + t\mathbf{e}_1 \in B\}$. As a function of $t \geq 0$ the set B_t corresponds to a body which is immersed in a fluid and moves at constant speed from right to left parallel to a wall $\partial\Omega_+$. The flow around this body is modeled by the Navier-Stokes equations

$$\begin{aligned} \partial_t \mathbf{U} + (\mathbf{U} \cdot \boldsymbol{\nabla})\, \mathbf{U} + \boldsymbol{\nabla} P &= \Delta \mathbf{U}, \\ \boldsymbol{\nabla} \cdot \mathbf{U} &= 0, \end{aligned} \tag{1.1}$$

in $\Omega_t = \Omega_+ \backslash B_t$ with the boundary conditions (the fluid is at rest and we choose no slip boundary conditions at the surface of the body),

$$\mathbf{U}|_{\partial\Omega_+} = 0, \qquad \lim_{\substack{\mathbf{X} \in \Omega_t \\ \mathbf{X} \to \infty}} \mathbf{U}(\mathbf{X}, t) = 0, \quad \mathbf{U}|_{\partial B_t} = -\mathbf{e}_1. \tag{1.2}$$

Note that we have set without restriction of generality all physical constants and the speed of the moving body equal to one. This can always be achieved by an appropriate scaling. With this choice of normalization the Reynolds number of the moving body corresponds to the diameter of B. The problem contains a second length-scale, which is the distance h of (the center of) B from the wall $\partial\Omega_+$.

In what follows we are interested in the case of a small body moving at a distance of order one from the wall. For this purpose we set $B = S_\epsilon$, and $\Omega_\epsilon = \Omega_+ \backslash \overline{\mathcal{S}_\epsilon}$, where $\mathcal{S}_\epsilon$ is defined as follows: let $\mathcal{S} \subset \mathbb{R}^3$, with $0 \in \mathcal{S}$, an open relatively compact set with smooth boundary $\partial\mathcal{S}$. Then, $\mathcal{S}_\epsilon = (0,0,1+h) + \epsilon\mathcal{S}$, with $h > 0$, fixed. Let $B(h,\lambda) := \{(x,y,z) \in \Omega_+ \,|\, |(x,y,z) - (0,0,1+h)| < \lambda\}$. Since $\mathcal{S}_\epsilon$ is relatively compact, there exists $\epsilon_0 > 0$, such that $\mathcal{S}_\epsilon \subset B(h, h/3)$ for all $0 \leq \epsilon < \epsilon_0$, and we will always assume from now on that ϵ has been chosen smaller than ϵ_0. As indicated above we are interested in the construction of solutions of equation (1.1) and (1.2) that are stationary when viewed in a reference frame attached to the moving body. To this end, we therefore set

$$\begin{aligned} \mathbf{U}(\mathbf{X}, t) &= \mathbf{u}(\mathbf{x}), \\ P(\mathbf{X}, t) &= p(\mathbf{x}), \end{aligned}$$

where $\mathbf{x} = \mathbf{X} + t\mathbf{e}_1$, and get the following stationary problem

$$\begin{aligned} -(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \partial_x \mathbf{u} + \Delta \mathbf{u} - \boldsymbol{\nabla} p &= 0, \\ \boldsymbol{\nabla} \cdot \mathbf{u} &= 0. \end{aligned} \tag{1.3}$$

in Ω_ϵ, with the boundary conditions

$$\mathbf{u}|_{\partial\Omega_\epsilon} = 0, \quad \lim_{\substack{\mathbf{x}\in\Omega_\epsilon \\ \mathbf{x}\to\infty}} \mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{u}|_{\partial\mathcal{S}_\epsilon} = -\mathbf{e}_1. \tag{1.4}$$

For convenience of sequent presentation, we call equation (1.3) and (1.4) the body problem. A standard technique for solving the body problem is to prove the existence of weak solutions. Such solutions are constructed by considering a nested sequence of finite domains that converges to Ω_+. Existence then follows by a compactness argument. See for example [8, 9, 21], for the case of an exterior domain in $\mathbb{R}^3$, and [13, 16] for the case of an exterior domain in the half space. The weak solutions constructed in this way are smooth, the only shortcoming of this method is that only little information is obtained about the behavior of solutions at infinity. The techniques of the present paper allow us to obtain such information.

Consider the function space for which (a precise definition will be given below)

$$\int_{\Omega_\epsilon} |\nabla\mathbf{u}(\mathbf{x})|^2 d\mathbf{x} < \infty.$$

We state our main result as following.

Theorem 1.1 *For all sufficiently small $\epsilon > 0$, there exists a unique weak solution $\mathbf{u}$ of the body problem with $\int_{\Omega_\epsilon} |\boldsymbol{\nabla}\mathbf{u}(\mathbf{x})|^2 d\mathbf{x} < \infty$. Furthermore, there exists a constant $C_\epsilon < \infty$ such that, for all $(x, y, z) \in \Omega_\epsilon$,*

$$|\mathbf{u}(x, y, z)| \leq \frac{C_\epsilon}{z^2}. \tag{1.5}$$

In Theorem 1.1, we restrict to solid bodies $\mathcal{S}_\epsilon = (0, 0, 1 + h) + \epsilon\mathcal{S}$ just for simplicity.This enables us to apply an explicit estimate on the constant in a Poincaré inequality. However, it is straightforward to extend our method to other solid bodies.

In order to prove Theorem 1.1 we proceed as follows, see [5], by truncating a weak solution for the body problem. We can define $\tilde{\mathbf{u}}$ and $\tilde{p}$ to be zero in the interior of $B(h, h/3)$ and by the equations (3.1) and (3.2) in the exterior of $B(h, h/3)$. These details will be presented in Section 3.1. By construction $\tilde{\mathbf{u}}$ and $\tilde{p}$ are smooth and satisfy the following equations in Ω_+

$$\begin{aligned} -\partial_x\tilde{\mathbf{u}} - (\tilde{\mathbf{u}}\cdot\boldsymbol{\nabla})\,\tilde{\mathbf{u}} + \Delta\tilde{\mathbf{u}} - \boldsymbol{\nabla}\tilde{p} &= \mathbf{F}, \\ \boldsymbol{\nabla}\cdot\tilde{\mathbf{u}} &= 0, \end{aligned} \tag{1.6}$$

subject to the boundary conditions

$$\tilde{\mathbf{u}}|_{\partial\Omega_+} = 0, \qquad \lim_{\substack{\mathbf{x}\in\Omega_+ \\ \mathbf{x}\to\infty}} \tilde{\mathbf{u}}(\mathbf{x}) = 0, \tag{1.7}$$

and with $\mathbf{F}$ a smooth vector field of compact support in Ω_+, *i.e.*, $\mathbf{F} \in C_c^\infty(\Omega_+)$. In what follows, we call equations (1.6) and (1.7) the source term problem. We use the following steps to obtain detailed information on weak solutions of the body problem at infinity:

1. We show the existence of weak solutions for the body problem by the invading method of Leray.
2. We use the cut-off procedure described above to obtain a weak solution for the source term problem from a weak solution for the body problem with a certain vector field F.
3. We show that, for ϵ small enough, the function F satisfies the smallness condition formulated in our previous work [11], so that there exists at least one α-solution $(\mathbf{u}_\alpha, p_\alpha)$ for the source term problem. We recall that α-solutions are a class of strong solutions that satisfy particularly also the bound (1.5).
4. For ϵ small enough we prove a weak-strong uniqueness result for the source term problem.
5. For ϵ small enough we prove uniqueness of weak solutions for the body problem.

Remark 1.2 *The weak-strong uniqueness result for the source term problem implies that for ϵ small enough any weak solution of the source term problem is equal to the strong solution of the source term problem constructed in our previous paper [11]. Since, by construction, $(\tilde{\mathbf{u}}, \tilde{p}) = (\mathbf{u}, p)$ in the complement of $B(h, 2h/3)$ where $(\tilde{\mathbf{u}}, \tilde{p})$ and $(\mathbf{u}, p)$ are solutions for the source term problem and body problem respectively, we find that $(\mathbf{u}, p) = (\mathbf{u}_\alpha, p_\alpha)$ in the complement of $B(h, 2h/3)$.*

Remark 1.3 *The uniqueness of solutions for the source term problem does not directly imply the uniqueness of solutions for the body problem, because different solutions of the body problem may lead to different functions F. However, once we have shown that $(\mathbf{u}, p) = (\mathbf{u}_\alpha, p_\alpha)$ in the complement of $B(h, 2h/3)$, the same arguments that we use to prove the weak-strong uniqueness for the source term problem can be repeated in order to obtain the uniqueness of weak solutions for the body problem.*

Remark 1.4 *The reason for the introduction of the cut-off procedure is that the techniques that are used for the construction of α-solutions require the problem is formulated on Ω_+.*

2 Weak solutions for the body problem

2.1 Main results

In this section, we discuss the existence of weak solutions for the body problem. The main result reads

Theorem 2.1 *There exists a family $(\mathbf{u}_\epsilon)_{\epsilon>0}$ which is defined for ϵ sufficiently small and such that*

(i). for all $\epsilon > 0$, $\mathbf{u}_\epsilon$ is a weak solution of body problem for $\mathcal{S}_\epsilon$.

(ii). $\lim_{\epsilon\to 0} ||\mathbf{u}_\epsilon, D|| = 0$.

(iii). there exists a pressure p_ϵ such that $(\mathbf{u}_\epsilon, p_\epsilon)$ satisfies (1.6) in Ω_ϵ and there holds for given $m \in \mathbb{N}$

$$||\mathbf{u}_\epsilon; \mathcal{C}^{m+1}(\mathcal{A}_2)|| + ||p_\epsilon; \mathcal{C}^m(\mathcal{A}_2)/\mathbb{R}|| \le C_m ||\mathbf{u}_\epsilon, D||,$$

for some universal constant C_m depending only on m.

The definitions of weak solution, the norm and the function space are given in the subsequent sections. The proof of this result is given below by three steps. We first recall the Leray's method for the construction of weak solutions, then obtain a family of weak solutions which satisfy a particular uniform bound with respect to the size of obstacle. Eventually, we show this family of solutions goes to 0 as the obstacle vanishes.

2.2 Definition of weak solutions

We first introduce the concept of weak solutions. Let $(\mathbf{u}, p)$ be a smooth solution of the body problem. We extend $\mathbf{u}$ from Ω_ϵ to the whole Ω_+ by setting $\mathbf{u} = -\mathbf{e}_1$ on $\bar{\mathcal{S}}_\epsilon$. Let φ be a smooth divergence free vector field with compact support in Ω_+ which is equal to a given constant vector field ϕ on $\mathcal{S}_\epsilon$. Then, if we multiply equation (1.3) by φ and integrate on Ω_ϵ we get

$$\int_{\Omega_\epsilon} (\Delta\mathbf{u} - \boldsymbol{\nabla} p) \cdot \varphi d\mathbf{x} = \int_{\Omega_\epsilon} [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \partial_x \mathbf{u}] \cdot \varphi d\mathbf{x}. \tag{2.1}$$

Applying integration by parts to the left-hand side of (2.1) yields

$$\int_{\Omega_\epsilon} (\Delta \mathbf{u} - \boldsymbol{\nabla} p) \cdot \varphi d\mathbf{x} = - \int_{\Omega_\epsilon} \boldsymbol{\nabla}\mathbf{u} \cdot \boldsymbol{\nabla}\varphi d\mathbf{x} + \int_{\partial\Omega_\epsilon} (\boldsymbol{\nabla}\mathbf{u} \cdot \mathbf{n}) \cdot \varphi d\boldsymbol{\sigma} - \int_{\partial\Omega_\epsilon} (p\mathbf{I} \cdot \mathbf{n}) \cdot \varphi d\boldsymbol{\sigma},$$

where $\mathbf{n}$ is the outward normal vector on $\partial\Omega_\epsilon$. Using the boundary conditions for $\mathbf{u}$ and φ, we obtain

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u} \cdot \boldsymbol{\nabla}\varphi d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \partial_x \mathbf{u}] \cdot \varphi d\mathbf{x} = \Sigma \cdot \phi, \tag{2.2}$$

with

$$\Sigma = \int_{\partial\mathcal{S}_\epsilon} (\boldsymbol{\nabla}\mathbf{u} - p\mathbf{I})\, \mathbf{n} d\boldsymbol{\sigma}. \tag{2.3}$$

The vector Σ is the force which the fluid exerts on $\mathcal{S}_\epsilon$. If we replace, on a formal level, φ by $\mathbf{u}$ in (2.2) we get (using that $\varphi = \mathbf{e}_1$ in this case),

$$\int_{\Omega_+} |\boldsymbol{\nabla}\mathbf{u}|^2 d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \partial_x \mathbf{u}] \cdot \mathbf{u} d\mathbf{x} = \Sigma \cdot \mathbf{e}_1. \tag{2.4}$$

Since $\mathbf{u}$ is divergence free, after integration by parts for the second term on the left-hand side of (2.4) we get

$$\int_{\Omega_+} [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \partial_x \mathbf{u}] \cdot \mathbf{u} d\mathbf{x} = 0,$$

and therefore (2.4) reduces to

$$\int_{\Omega_+} |\boldsymbol{\nabla}\mathbf{u}|^2 d\mathbf{x} = \Sigma \cdot \mathbf{e}_1. \tag{2.5}$$

We conclude that if $(\mathbf{u}, p)$ is a classical solution of the body problem which decays sufficiently rapidly at infinity, the $\mathbf{u}$ satisfies the integral equation (2.2) and we have the identity (2.5). Equation (2.5) implies in particular that $\boldsymbol{\nabla}\mathbf{u} \in L^2(\Omega_+)$.

Motivated by the above discussion, we have the following functional setting for weak solutions of the body problem. Let $\mathcal{D}$ be the vector space of divergence-free vector fields with compact support in Ω_+. We equip $\mathcal{D}$ with the scalar product

$$(\mathbf{w}_1, \mathbf{w}_2) = \int_{\Omega_+} \boldsymbol{\nabla}\mathbf{w}_1 \cdot \boldsymbol{\nabla}\mathbf{w}_2 d\mathbf{x}, \tag{2.6}$$

and the norm

$$\|\mathbf{w}, \mathcal{D}\| = (\mathbf{w}, \mathbf{w})^{\frac{1}{2}}. \tag{2.7}$$

Let D be the completion of $\mathcal{D}$ with respect to the norm (2.7), and let $\|\mathbf{w},D\|$ be the norm of an element $\mathbf{w}$ of D. As a consequence of Poincaré's inequality we have that $D \subset L^2_{loc}(\Omega_+)$. Furthermore, given $\epsilon < \epsilon_0$, we denote by $\mathcal{D}^\epsilon \subset \mathcal{D}$ the vector-fields $\mathbf{w} \in \mathcal{D}$ which are constant on $\mathcal{S}_\epsilon$, and by D^ϵ the closure of $\mathcal{D}^\epsilon$ in D. We define the function Γ on D^ϵ by

$$\begin{aligned} \Gamma: \quad & D^\epsilon \longrightarrow \mathbb{R}^3 \\ & \mathbf{w} \longmapsto \mathbf{W} = \frac{1}{|\mathcal{S}_\epsilon|}\int_{\mathcal{S}_\epsilon} \mathbf{w}(\mathbf{x})d\mathbf{x}. \end{aligned} \tag{2.8}$$

Poincaré's inequality implies that Γ is bounded. For convenience later on we define, for given $\mathbf{w} \in \mathbb{R}^3$,

$$\mathcal{D}^\epsilon_{\mathbf{W}} = \{\mathbf{w} \in \mathcal{D}^\epsilon |\ \mathbf{w}|_{\mathcal{S}_\epsilon} = \mathbf{W}\}, \tag{2.9}$$

$$D^\epsilon_{\mathbf{W}} = \{\mathbf{w} \in D^\epsilon |\ \mathbf{w}|_{\mathcal{S}_\epsilon} = \mathbf{W}\}. \tag{2.10}$$

We note that $D^\epsilon_{\mathbf{W}}$ is the closure of $\mathcal{D}^\epsilon_{\mathbf{W}}$. Such spaces have been extensively studied in Galdi's book [9]

We now define weak solutions of the body problem by following the work of Leary.

Definition 2.2 *A vector-field* $\mathbf{u}$ *is called a weak solution, if*

(i) $\mathbf{u} \in D^\epsilon_{-\mathbf{e}_1}$,

(ii) there exists a vector $\Sigma \in \mathbb{R}^3$, *such that for all* $\mathbf{w} \in \mathcal{D}^\epsilon$

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{w}d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \partial_x\mathbf{u}]\cdot\mathbf{w}d\mathbf{x} = \Sigma\cdot\Gamma(\mathbf{w}), \tag{2.11}$$

and

$$\int_{\Omega_+} |\boldsymbol{\nabla}\mathbf{u}|^2\, d\mathbf{x} \le \Sigma\cdot\mathbf{e}_1. \tag{2.12}$$

The following lemma shows that the weak solutions are well defined.

Lemma 2.3 *Let* $\mathcal{O} \subset\subset \Omega_+$. *Let* $(\mathbf{u},\mathbf{v}) \in D^2$ *and let* $\mathbf{w} \in D$ *with Supp*$(\mathbf{w}) \subset \mathcal{O}$. *Then,*

$$\int_{\Omega_+} [(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v} + \partial_x\mathbf{v}]\cdot\mathbf{w}d\mathbf{x} = -\int_{\Omega_+} [(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w} + \partial_x\mathbf{w}]\cdot\mathbf{v}d\mathbf{x}, \tag{2.13}$$

and

$$\int_{\Omega_+} [(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v} + \partial_x\mathbf{v}]\cdot\mathbf{w}d\mathbf{x} \le C(\mathcal{O})[||\mathbf{u}||_{L^6(\mathcal{O})}||\mathbf{v},D||\cdot||\mathbf{w}||_{L^3(\mathcal{O})} + ||\mathbf{v},D||\cdot||\mathbf{w}||_{L^2(\mathcal{O})}]. \tag{2.14}$$

(2.13) can be proved in component form and (2.14) follows using the Höder's inequality.

2.3 Existence of weak solutions

In this part, we show the existence of weak solutions for the body problem. Namely, we prove

Theorem 2.4 *There exist constants* $\varkappa$ *and* $\epsilon_1 > 0$ *such that if* $\epsilon < \epsilon_1$*, there exists at least a weak solution* $\mathbf{u}$ *for the body problem, satisfying the further bound* $||\mathbf{u};D|| \leq \varkappa$*.*

The proof is based on the exhaustion method of Leray. Namely, we consider a nested sequence of finite domains that converges to Ω_+ and, for any domain of this sequence, we prove existence of one approximate weak solution having support in this domain and satisfying a suitable estimate. Our result then follows by a compactness argument. Many aspects of the proof are standard, but the uniform bound is new to our knowledge. In the proof the size of ϵ of the obstacle is fixed such that $\mathcal{S}_\epsilon \subset B(h, h/3)$. We mention further assumptions on ϵ when needed. We consider the sequence $(\triangle_n)_{n\geq 1}$ given in the introduction. This sequence satisfies, for all $n \in \mathbb{N}$:

(I) $\triangle_n$ is a bounded open set having a smooth boundary

(II) $\mathcal{S}_\epsilon \subset\subset \triangle_n \subset \triangle_{n+1}$

(III) $\cup_{n\in\mathbb{N}} \triangle_n = \Omega_+$.

Given $\triangle_n$, we define $D^{\epsilon,n}$ and $D^{\epsilon,n}_{\mathbf{W}}$ by

$$D^{\epsilon,n} = \{\mathbf{w} \in D^\epsilon | \mathbf{w}|_{\Omega_+ \setminus \bar{\triangle}_n} = 0\}, \quad D^{\epsilon,n}_{\mathbf{W}} = \{\mathbf{w} \in D^\epsilon_{\mathbf{W}} | \mathbf{w}|_{\Omega_+ \setminus \bar{\triangle}_n} = 0\}.$$

The proof of Theorem 2.4 is a direct consequence of the following lemmas.

Lemma 2.5 *There exists a constant* $\epsilon_1 > 0$ *such that if* $\epsilon < \epsilon_1$ *there exists at least one approximate weak solution (defined below) on* $\triangle_n$*, for all* $n \in \mathbb{N}$*.*

Lemma 2.6 *Let* ϵ *be as in Lemma 2.5 and* $n \in \mathbb{N}$*. There exists a constant* $\varkappa < \infty$ *such that* $||\mathbf{u}; D|| + |\Sigma| \leq \varkappa$ *for any approximate weak solution* $\mathbf{u}$ *on* $\triangle_n$*with associated force* Σ*.*

The definition of approximate weak solution is the following.

Definition 2.7 *Let* $n \in \mathbb{N}$*. A vector-field* $\mathbf{u}$ *is called an approximate weak solution on* $\triangle_n$*if* *(i).* $\mathbf{u} \in D^{\epsilon,n}_{-\mathbf{e}_1}$,

(ii). for all $\mathbf{w} \in D^{\epsilon,n}_0$,

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{w} d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \partial_x \mathbf{u}] \cdot \mathbf{w} d\mathbf{x} = 0.$$

Now we give the proofs of the two lemmas.

Proof: In these proofs $n \in \mathbb{N}$ is fixed. Since $D^{\epsilon,n}$ and $D^{\epsilon,n}_0$ are closed subspace of D^ϵ, they are Hilbert spaces with respect to the scalar product (2.6). The space $D^{\epsilon,n}_{-\mathbf{e}_1}$ is not empty which is guaranteed by Proposition A.1. We note that $D^{\epsilon,n}_{-\mathbf{e}_1}$ is an affine subspace of $D^{\epsilon,n}$ with direction $D^{\epsilon,n}_0$. We also introduce $\mathbf{U}_{-\mathbf{e}_1}$, the unique minimizer of the D-norm among the velocity fields in $D^{\epsilon,n}_{-\mathbf{e}_1}$ for later use. This velocity field possesses

1. $D^{\epsilon,n}_{-\mathbf{e}_1} = \mathbf{U}_{-\mathbf{e}_1} + D^{\epsilon,n}_0$
2. $((\mathbf{U}_{-\mathbf{e}_1}, \mathbf{w})) = 0$ for all velocity fields $\mathbf{w} \in D^{\epsilon,n}_0$.

We now reformulate the existence of an approximate weak solution on $\triangle_n$ as a fixed point problem for a functional equation. First, we note that Lemma 2.3 implies that for all $\mathbf{u} \in D^{\epsilon,n}_{-\mathbf{e}_1}$, the map

$$\begin{array}{rcl} D^{\epsilon,n}_0 & \rightarrow & \mathbb{R} \\ \mathbf{w} & \longmapsto & -\int_{\Omega_+} [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \partial_x \mathbf{u}] \cdot \mathbf{w} d\mathbf{x}, \end{array}$$

is a continuous linear form. By the Riesz-Fréchet theorem we can define a continuous map $b^\star_n$ by the formula

$$((b^\star_n(\mathbf{u}), \mathbf{w})) = \int_{\Omega_+} [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \partial_x \mathbf{u}] \cdot \mathbf{w} d\mathbf{x}, \text{ for } \mathbf{u} \in D^{\epsilon,n}_{-\mathbf{e}_1} \text{ and } \mathbf{w} \in D^{\epsilon,n}_0. \quad (2.15)$$

With these definitions in hand, $\mathbf{u}$ is an approximate weak solution on $\triangle_n$ if and only if $\mathbf{u} = \mathbf{U}_{-\mathbf{e}_1} + \mathbf{v}$, with $\mathbf{v}$ a solution of the functional equation

$$\mathbf{v} = b^\star_n(\mathbf{U}_{-\mathbf{e}_1} + \mathbf{v}), \quad \mathbf{v} \in D^{\epsilon,n}_0. \quad (2.16)$$

On the other hand, Lemma 2.3 implies that $b^\star_n$ is continuous on $D^{\epsilon,n}_{-\mathbf{e}_1}$ equipped with the $L^6(\triangle_n)$-norm. Using that $H^1_0(\triangle_n)$ is compactly compactly imbedded in $L^6(\triangle_n)$ yields that $b^\star_n$ is completely continuous, i.e., for any given bounded sequence $(\mathbf{v}_i)_{i\geq 1}$ in $D^{\epsilon,n}_0$, there exists a subsequence $(\mathbf{v}_{i_j})_{j\geq 1}$ such that the sequence

$b_n^\star(\mathbf{U}_{-\mathbf{e}_1} + \mathbf{v}_{i_j})_{j\geq 1}$ converges strongly in $D_0^{\epsilon,n}$. Therefore, the Leary-Schauder fixed point theorem (See [10]) guarantees the existence of a solution of (2.16) by proving a suitable estimate on a priori solutions to an auxiliary problem. This estimate is the content of the following proposition. $\square$

Proposition 2.8 *There exist constants* $\epsilon_1 > 0$ *and* $C < \infty$ *such that, for all* $\epsilon < \epsilon_1$, $\lambda \in [0,1]$ *and all* $(\mathbf{u},\Sigma) \in D_{-\mathbf{e}_1}^{\epsilon,n} \times \mathbb{R}^3$ *which satisfy*

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{w} d\mathbf{x} + \lambda \int_{\Omega_+} [(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \partial_x \mathbf{u}]\cdot\mathbf{w} d\mathbf{x} = \Sigma\cdot\Gamma(\mathbf{w}), \quad \forall \mathbf{w} \in D^{\epsilon,n}, \tag{2.17}$$

we have the bound $||\mathbf{u};D|| + |\Sigma| \leq C$.

Proof of Proposition: Note that given $(\mathbf{u},\Sigma,n,\lambda)$ as in Proposition 2.8 we can set $\mathbf{w} = \mathbf{u}$ in (2.17), then we obtain (2.5). Hence, it suffices to find a bound on Σ. To this end, we introduce an additional family of cut-off functions χ_δ as follows.

$$\chi_\delta(x,y,z) = \zeta\left(\frac{|(x,y,z-1-h)|}{\delta} - 1\right), \quad (x,y,z) \in \Omega_+ \text{ and } 0 < \delta < h/3,$$

where smooth function $\zeta : \mathbb{R} \to \mathbb{R}$ truncates in balls around the point $(0,0,1+h)$. Precisely,

$$\zeta(s) = 1, \quad \forall s < 0, \quad \zeta(s) = 0, \quad \forall s > 1.$$

With this definition, we easily have $\chi_\delta = 1$ in $B(\delta)$ while $\chi_\delta = 0$ outside of $B(2\delta)$. Now given $(\mathbf{u},\Sigma,n,\lambda)$ and an obstacle $\mathcal{S}_\epsilon$, we set $\delta(\epsilon) = \lambda_0\epsilon$, with $\lambda_0 = \sup\{|(x,y,z)|, (x,y,z) \in \mathcal{S}\}$, and define a test function for arbitrary $\mathbf{W} \in \mathbb{R}^3$

$$\mathbf{w}_\epsilon(x,y,z) = \frac{1}{2}\nabla \times (\chi_{\delta(\epsilon)}(x,y,z)\mathbf{W} \times ((x,y,z) - (0,0,1+h)))$$

Since $\mathcal{S}_\epsilon$ tends homothetically to a point when $\epsilon \to 0$, we can always choose ϵ_0 such that $\mathbf{w}_\epsilon = \mathbf{W}$ on $\mathcal{S}_\epsilon$ and $\mathbf{w}_\epsilon = \mathbf{0}$ in the exterior of $B(2h/3)$ for $\epsilon < \epsilon_0$. Thus, we can use $\mathbf{w}_\epsilon$ as a test-function in (2.17). By direct computation of $\mathbf{w}_\epsilon$, there exists a universal constant C_1 such that

$$||\mathbf{w}_\epsilon, D|| + ||\mathbf{w}_\epsilon||_{L^\infty(\mathbb{R}^3)} \leq C_1|\mathbf{W}|. \tag{2.18}$$

Then we can obtain the inequality from (2.17) that

$$\begin{aligned} |\Sigma\cdot\mathbf{W}| &\leq ||\mathbf{u};D||||\mathbf{w}_\epsilon, D|| + \lambda||\mathbf{u}+\mathbf{e}_1||_{L^2(B(2\delta(\epsilon)))}||\mathbf{u};D||||\mathbf{w}_\epsilon||_{L^\infty(\mathbb{R}^3)} \\ &\leq C_1|\mathbf{W}|\left(||\mathbf{u};D|| + ||\mathbf{u}+\mathbf{e}_1||_{L^2(B(2\delta(\epsilon)))}||\mathbf{u};D||\right). \end{aligned}$$

Since $\mathbf{u}|_{\mathcal{S}_\epsilon} = -\mathbf{e}_1$, the Poincaré inequality implies that there exists a constant $\hat{C}_2$ such that

$$||\mathbf{u} + \mathbf{e}_1||_{L^2(B(2\delta(\epsilon)))} \leq \hat{C}_2 ||\mathbf{u}; D||. \tag{2.19}$$

A scaling argument [8] shows that $\hat{C}_2 = \epsilon C_2$ with a constant C_2 independent of ϵ and $\mathbf{u}$. Therefore,

$$|\Sigma| \ \leq C_1 \left(||\mathbf{u}; D|| + \epsilon C_2 ||\mathbf{u}; D||^2 \right). \tag{2.20}$$

From (2.20) and (2.5) we find that if ϵ satisfies moreover $\epsilon < 1/(2C_1C_2)$, we have the bound on $|\Sigma|$ and $||\mathbf{u}; D||$ as following which is independent of n, λ and ϵ

$$||\mathbf{u}; D|| \leq |\Sigma|^{1/2} \leq 2C_1.$$

Therefore, Lemma 2.6 is then proved. □

2.4 Limit of weak solution as the obstacle vanishes

In this section we prove that weak solutions converge to zero when the size of the obstacle tends to zero. This convergence is measured in the $\mathcal{C}^m_{loc}$-topology. We use the following notation. Let X be an arbitrary Banach space and $p \in X$. Then

$$||p; X/\mathbb{R}|| := \inf\{||p + c; X||, c \in \mathbb{R}\}. \tag{2.21}$$

Furthermore, we define, for $n \in \mathbb{N}$, the set $\mathcal{A}_n$ by

$$\mathcal{A}_n = \Delta_n \backslash \overline{B(h, 2^{-n}h)},$$

where Δ_n satisfies $B(h, 2^n h) \subset \Delta_n \subset B(h, (2^n + 1)\, h)$ and has a smooth boundary. Therefore, $\mathcal{A}_n$ has a smooth boundary, too. In order to come to our result, we need two lemmas. The first one shows that weak solutions converge to zero in the sense of their energy.

Lemma 2.9 *Given $\eta > 0$ there exists $\epsilon_\eta > 0$ such that any weak solution $\mathbf{u}_\epsilon$ of the body problem with $\epsilon < \epsilon_\eta$ satisfies the inequality $||\mathbf{u}_\epsilon; D|| \leq \eta$.*

Proof: We prove this lemma by contradiction. Indeed, suppose not, there would exist η_0, sequences $(\epsilon_n)_{n\in\mathbb{N}} \in (0, \epsilon_0)^{\mathbb{N}}$ and $(\mathbf{u}_n, \Sigma_n)_{n\in\mathbb{N}} \in (D \times \mathbb{R}^3)^{\mathbb{N}}$ such that $\lim_{n\to\infty} \epsilon_n = 0$, the velocity-field $\mathbf{u}_n$ is a weak solution of the body problem associated to $\mathcal{S}_{\epsilon_n}$ with exerted force Σ_n, and $||\mathbf{u}_n, D|| \geq \eta_0$ for all $n \in \mathbb{N}$. However, since $\lim_{n\to\infty} \epsilon_n = 0$, there exists $N_0 \in \mathbb{N}$ such that point *(ii)* in Theorem 2.1 holds for $\mathbf{u}_n$. In particular, the sequence $(\mathbf{u}_n, \Sigma_n)$ is bounded. Consequently, up to extract a subsequence, there exists $(\mathbf{u}, \Sigma) \in D \times \mathbb{R}^3$ such that $\mathbf{u}_n$ converges weakly to $\mathbf{u}$ and Σ_n converges strongly to Σ in $\mathbb{R}^3$. Given $\delta > 0$ and $\mathbf{W} \in \mathbb{R}^3$ we construct a test-function $\mathbf{w}_\delta$ similarly as in the previous section: $\mathbf{w}_\delta(x,y,z) = \frac{1}{2}\nabla \times (\chi_\delta(x,y,z)\mathbf{W} \times ((x,y,z) - (0,0,1+h)))$. We recall there exists $C_1 < \infty$ for which, for $0 < \delta < h/3$

$$||\mathbf{w}_\delta; D|| + ||\mathbf{w}_\delta||_{L^\infty(\mathbb{R}^3)} \leq C_1|\mathbf{W}|,$$

and $\mathrm{Supp}(\mathbf{w}_\delta) \subset B(h, 2h/3)$. As $\lim_{n\to\infty} \epsilon_n = 0$, there exists N_δ such that $\mathbf{w}_\delta$ is an admissible test-function for $N \geq N_\delta$

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u}_n : \boldsymbol{\nabla}\mathbf{w}_\delta d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}_n \cdot \boldsymbol{\nabla})\mathbf{u}_n + \partial_x \mathbf{u}_n] \cdot \mathbf{w}_\delta d\mathbf{x} = \Sigma_n \cdot \mathbf{W},$$

As previously, when n goes to ∞, we obtain

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{w}_\delta d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \partial_x \mathbf{u}] \cdot \mathbf{w}_\delta d\mathbf{x} = \Sigma \cdot \mathbf{W}.$$

Consequently, we obtain

$$|\Sigma| \leq C_1 ||\boldsymbol{\nabla}\mathbf{u}||_{L^2(B(h,2\delta/3))} \left(||\mathbf{u} + \mathbf{e}_1||_{L^2(B(h,2\delta/3))} + 1 \right) .$$

Letting δ go to 0, this yields $\Sigma = 0$. Finally, we have $\lim_{n\to\infty} \Sigma_n = 0$ and the energy estimate (2.12) implies $\lim ||\mathbf{u}_n, D|| = 0$. We obtain a contradiction. □

Lemma 2.10 *Let $(n,m) \in \mathbb{N}^2$ and $\epsilon < \epsilon_1$ such that $\mathcal{S}_\epsilon \subset\subset \mathbb{R}^3_+ \setminus \overline{\mathcal{A}_{n+m+1}}$. Then, there exists a constant $C_{m,n}$ such that any weak solution $\mathbf{v}$ of the body problem such that $||\mathbf{v}; D|| \leq 1$ with associated pressure-field p, satisfies the following inequality*

$$||\mathbf{v}; H^{m+1}(\mathcal{A}_n)|| + ||p; H^m(\mathcal{A}_n)/\mathbb{R}|| \leq C_{m,n} ||\mathbf{v}; D||. \tag{2.22}$$

Proof: Given $\mathbf{v}$ a weak solution, one can test (2.11) with any smooth solenoidal vector-field $\mathbf{w}$ having compact support in Ω_ϵ. Hence, $\mathbf{v}$ is a generalized solution to the Stokes equation on Ω_ϵ in the sense of [8, Definition 1.1, Page 185] with source term:

$$\left\{ \begin{array}{c} \mathbf{f} = (\mathbf{v} + \mathbf{e}_1) \cdot \boldsymbol{\nabla}\mathbf{v} \in H^{-1}(\Omega'), \\ ||\mathbf{f}||_{H^{-1}(\Omega')} \leq C(\Omega')[||\mathbf{v}||_{H^1(\Omega')} + ||\mathbf{v}; D||], \end{array} \right. \quad \forall \Omega' \subset\subset \Omega_\epsilon. \tag{2.23}$$

Consequently, we apply the method in [8, Lemma 1.1, Page 186] to construct a pressure field $p \in L^2_{loc}(\Omega_\epsilon)$ such that, in the sense of distributions, there holds

$$\left\{ \begin{array}{c} \Delta \mathbf{v} - \boldsymbol{\nabla} p = \mathbf{f}, \\ \boldsymbol{\nabla} \cdot \mathbf{v} = 0, \end{array} \right. \quad \text{in } \Omega_\epsilon. \tag{2.24}$$

Moreover, the pressure p is unique up to a finite number of constants, we usually call p the pressure associated with $\mathbf{v}$, and ($\mathbf{v}$,p) satisfies (1.3). In the following, we prove Lemma 2.10 by induction.

Given $n \in \mathbb{N}$, by restriction, ($\mathbf{v}$,p) is a solution to Stokes system on $\mathcal{A}_n$ with data $\mathbf{f} = (\mathbf{v} + \mathbf{e}_1) \cdot \boldsymbol{\nabla}\mathbf{v}$ and boundary values $\mathbf{v} = \mathbf{v}|_{\partial \mathcal{A}_n}$. Hence, combining [8, Theorem 1.1, Page 188] and (2.23), there holds

$$||\mathbf{v}||_{H^1(\mathcal{A}_n)} + ||p||_{L^2(\mathcal{A}_n)/\mathbb{R}} \leq C_n \left[||\mathbf{v}; D|| + ||\mathbf{v}; D||^2 \right]. \tag{2.25}$$

Since $||\mathbf{v}; D|| \leq K$ for $\epsilon < \epsilon_1$ (See Theorem 2.4), the right-hand side of (2.25) is dominated by $C_n||\mathbf{v}; D||$. The induction assertion holds true for $m = 0$. We will not recall this last argument in what follows.

For all $n \in \mathbb{N}$, by construction we have $\mathcal{A}_n \subset \overline{\mathcal{A}_n} \subset \mathcal{A}_{n+1}$. Consequently, there exists a smooth truncation function χ_n such that $\chi_n = 1$ on $\mathcal{A}_n$ and $\chi_n = 0$ outside $\mathcal{A}_{n+1}$. Setting now $\tilde{\mathbf{u}} = \chi_n \mathbf{v}$ and

$\tilde{p} = \chi_n p$. We obtain ($\tilde{\mathbf{u}}$,$\tilde{p}$) is a solution to the Stokes system

$$\begin{cases} \Delta \tilde{\mathbf{u}} - \boldsymbol{\nabla}\tilde{p} = \tilde{\mathbf{f}}, & \text{on } \mathcal{A}_{n+1} \\ \boldsymbol{\nabla} \cdot \tilde{\mathbf{u}} = \tilde{g}, & \text{on } \mathcal{A}_{n+1} \end{cases} \qquad \tilde{\mathbf{u}} = 0, \quad \text{on } \partial\mathcal{A}_{n+1}, \tag{2.26}$$

with source term

$$\tilde{\mathbf{f}} := \chi(\mathbf{v} + \mathbf{e}_1) \cdot \boldsymbol{\nabla}\mathbf{v} + 2\boldsymbol{\nabla}\boldsymbol{\chi} \cdot \boldsymbol{\nabla}\mathbf{v} + \Delta\boldsymbol{\chi}\mathbf{v} - p\boldsymbol{\nabla}\boldsymbol{\chi}, \quad \tilde{g} := \mathbf{v} \cdot \boldsymbol{\nabla}\boldsymbol{\chi}.$$

For simplicity, we drop indices n in the truncated functions. As a preliminary, we observe that, with the above control on $\mathbf{v}$ and p (2.25), there holds $\tilde{\mathbf{f}} \in L^q(\mathcal{A}_{n+1})$ and $\tilde{g} \in W^{1,q}(\mathcal{A}_{n+1})$, for given $q < 2$ with

$$||\tilde{\mathbf{f}}||_{L^q(\mathcal{A}_{n+1})} + ||\tilde{g}||_{W^{1,q}(\mathcal{A}_{n+1})} \leq C_q[||\mathbf{v}||_{H^1(\mathcal{A}_{n+1})} + ||\mathbf{v}||^2_{H^1(\mathcal{A}_{n+1})} + ||p||_{L^2(\mathcal{A}_{n+1})}].$$

Then we apply [8, Theorem 6.1, Page 231] to get $\tilde{\mathbf{u}} \in W^{2,q}(\mathcal{A}_{n+1})$, $\tilde{p} \in W^{1,q}(\mathcal{A}_{n+1})$ with

$$||\tilde{\mathbf{u}}|_{W^{2,q}(\mathcal{A}_{n+1})} + ||\tilde{p}||_{W^{1,q}(\mathcal{A}_{n+1})/\mathbb{R}} \leq C_q[||\mathbf{v}||_{H^1(\mathcal{A}_{n+1})} + ||\mathbf{v}||^2_{H^1(\mathcal{A}_{n+1})} + ||p||_{L^2(\mathcal{A}_{n+1})}].$$

We emphasize here we can always change p with $p + c$ before truncation. So, on the right-hand side of the last inequality, we can change $||p||_{L^2(\mathcal{A}_{n+1})}$ with $||p||_{L^2(\mathcal{A}_{n+1})/\mathbb{R}}$.This argument will be repeated implicitly in the following. Combined with (2.25), this yields

$$||\tilde{\mathbf{u}}|_{W^{2,q}(\mathcal{A}_{n+1})} + ||\tilde{p}||_{W^{1,q}(\mathcal{A}_{n+1})/\mathbb{R}} \leq C_q||\mathbf{v}; D||.$$

In particular, $\mathbf{v} \in W^{2,q}(\mathcal{A}_n) \subset L^\infty(\mathcal{A}_n)$ with $||\mathbf{v}|_{L^\infty(\mathcal{A}_n)} \le K_n||\mathbf{v}; D||$.

Assuming that we have the property for all $m' \le m$, as $(\tilde{\mathbf{u}},\tilde{p})$ is a solution to (2.26), we apply ellipticity of the Stokes operator to obtain

$$||\tilde{\mathbf{u}}|_{H^{m+2}(\mathcal{A}_{n+1})} + ||\tilde{p}||_{H^{m+1}(\mathcal{A}_{n+1})/\mathbb{R}} \le C_m[||\tilde{\mathbf{f}}||_{H^m(\mathcal{A}_{n+1})} + ||\tilde{g}||_{H^{m+1}(\mathcal{A}_{n+1})}], \quad (2.27)$$

where standard computations yield

$$||\tilde{\mathbf{f}}||_{H^m(\mathcal{A}_{n+1})} + ||\tilde{g}||_{H^{m+1}(\mathcal{A}_{n+1})} \le C_m[||\mathbf{v}||_{L^\infty(\mathcal{A}_{n+1})}||\mathbf{v}||_{H^{m+1}(\mathcal{A}_{n+1})} + ||\mathbf{v}||_{H^{m+1}(\mathcal{A}_{n+1})} + ||\mathbf{v}||^2_{H^{m+1}(\mathcal{A}_{n+1})} + ||p||_{H^m(\mathcal{A}_{n+1})/\mathbb{R}}].$$

Applying the induction assumption, there exists a constant $C_{m,n+1}$ such that for $\epsilon < \epsilon_0/2^{m+n+1}$ there holds

$$||\mathbf{v}||_{H^{m+1}(\mathcal{A}_{n+1})} + ||p||_{H^m(\mathcal{A}_{n+1})/\mathbb{R}} \le C_{m,n+1}||\mathbf{v}; D||.$$

A fortiori, for $\epsilon < \epsilon_0/2^{m+n+1}$, there exists a constant $C_{m+1,n}$ depending on m and n such that

$$||\mathbf{v}||_{H^{m+2}(\mathcal{A}_n)} + ||p||_{H^{m+1}(\mathcal{A}_n)/\mathbb{R}} \le C_{m+1,n}||\mathbf{v}; D||.$$

This ends the induction argument. We complete the proof of Lemma 2.10. □

With Lemma 2.9 and Lemma 2.10 in hand, Theorem 2.1 is a straightforward consequence of them.

3 Asymptotic behavior of weak solution

As mentioned in the introduction, the only drawback of the weak solution theory is that it does not give sufficient information on the way the condition at infinity is satisfied. In this section, we show that the weak solutions of body problem constructed above possesses the expected decay rate. That is to prove

Theorem 3.1 *There exists $\epsilon_e > 0$, such that, for all $\epsilon < \epsilon_e$, the weak solution $\mathbf{u}_\epsilon$ satisfies the following decay estimate*

$$|\mathbf{u}_\epsilon(x,y,z)| \le \frac{C_\epsilon}{z^2}, \quad \forall (x,y,z) \in \Omega_+ \backslash \overline{\mathcal{S}_\epsilon},$$

for some $C_\epsilon < \infty$.

The proof is divided into three steps by comparing weak solutions with the α-solutions. First, we construct the weak solution for the source term problem with the help of truncation method applied to the body problem. We prove that, when the solid is sufficiently small, weak solutions to body problem provided by Theorem 2.1 produce weak solutions to the source term problem where the source term is arbitrary small, so that we can obtain the α-solution. We conclude by proving that any weak solution coincides with the α-solution when the source term is sufficiently small.

3.1 Truncation procedure

In the following we show how to construct a weak solution for a source term by truncating a weak solution for the body problem. Let $\chi(|\mathbf{x}|) = \varrho(|\mathbf{x}_h|/h)$, where $\mathbf{x}_h = \mathbf{x} - (0, 0, 1 + h)$ with $0 < h \ll 1$ and ϱ is a strictly increasing smooth function from $\mathbb{R}^+$ to $[0, 1]$ such that, $\varrho(x) = 0$, for $0 \leq x \leq 1/3$ and $\varrho(x) = 1$, for $x \geq 2/3$. Then for any $(\mathbf{v},q)$, a weak solution for the body problem, satisfying

$$\boldsymbol{\nabla}\mathbf{v} \in L^2 \text{ in } \Omega_\epsilon, \;\; \boldsymbol{\nabla}\cdot\mathbf{v} = 0, \;\; \mathbf{v}|_{\partial S_\epsilon} = -e_1, \;\; \mathbf{v}|_{\partial\Omega_+} = 0, \;\; \text{and} \;\; \lim_{|\mathbf{x}|\to\infty} \mathbf{v}(\mathbf{x}) = 0.$$

We define in Ω_+

$$\mathbf{u}(\mathbf{x}) = \chi(|\mathbf{x}|)\mathbf{v}(\mathbf{x}) + \mathbf{T}[\mathbf{e}_1](|\mathbf{x}|)\,(1 - \chi(|\mathbf{x}|)) + \boldsymbol{\nabla}\chi(|\mathbf{x}|) \times \boldsymbol{\psi}(\mathbf{x}), \tag{3.1}$$

$$p = \chi q, \tag{3.2}$$

where

$$\boldsymbol{\psi}(\mathbf{x}) = \frac{1}{4\pi}\int_{\Omega_\epsilon} \boldsymbol{\nabla}\times(\mathbf{v}(\mathbf{y}) - \mathbf{T}[\mathbf{e}_1](\mathbf{y}))\left(\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{y}|}\right) d^3\mathbf{y},$$

$$\mathbf{T}[\mathbf{e}_1](\mathbf{x}) = \mathbf{e}_1\,(\chi(|\mathbf{x}|) - 1) + \frac{1}{2}|\mathbf{x}|\chi'(|\mathbf{x}|)\mathbf{e}_1 - \mathbf{x}\left(\frac{(\mathbf{e}_1\cdot\mathbf{x})\chi'(|\mathbf{x}|)}{2|\mathbf{x}|}\right).$$

Let

$$F = (\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} + \partial_x\mathbf{u} - \Delta\mathbf{u} + \boldsymbol{\nabla}p = TNS[v, q],$$

then F has compact support in $\overline{B(h, 2h/3}\backslash B(h, h/3)$. We will show in the Appendix the formulas for $\mathbf{u}$ and p given by (3.1) and (3.2) are all well defined in Ω_+. Now we prove a weak solution of the body problem also provides a weak solution for the source problem with certain source F.

Proposition 3.2 *Given ϵ such that $\mathcal{S}_\epsilon \subset\subset B(h/4)$ and a weak solution $\mathbf{u}$ of the body problem for $\mathcal{S}_\epsilon$ with associated pressure p, the vector-field $\tilde{\mathbf{u}}$ given by 3.1 satisfies* *(i)* $\tilde{\mathbf{u}} \in D$,

(ii) there holds

$$\int_{\Omega_+} \nabla\tilde{\mathbf{u}} : \nabla\mathbf{w} d\mathbf{x} + \int_{\Omega_+} [(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} + \partial_x \mathbf{u}] \cdot \mathbf{w} d\mathbf{x} = \int_{\Omega_+} \mathbf{F} \cdot \mathbf{w} d\mathbf{x}, \qquad \forall \mathbf{w} \in \mathcal{D} \quad (3.3)$$

and

$$\int_{\Omega_+} |\nabla\tilde{\mathbf{u}}|^2 \, d\mathbf{x} \leq \int_{\Omega_+} \mathbf{F} \cdot \tilde{\mathbf{u}} d\mathbf{x}. \quad (3.4)$$

with $\mathbf{F} = TNS[u,p]$.

As in the obstacle case, this definition makes sense because source term $\mathbf{f}$ and the test function $\mathbf{w}$ have compact support. Hence, the integrals in (3.3) and (3.4) are well-defined.

Theorem 3.3 *Given* $\epsilon < \epsilon_0$ *and* $\mathbf{v}$ *is a weak solution of the body problem associated to some pressure* q, *then the vector-field* $\mathbf{u}$ *given by (3.1) is a weak solution for the source term problem with source term* $F = (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \partial_x \mathbf{u} - \Delta\mathbf{u} + \nabla p$, p *is given by (3.2).*

Proof: First, we emphasize this lemma makes sense for $\epsilon < \epsilon_0$. Indeed, for such an ϵ, there holds $\mathcal{S}_\epsilon \subset B(h, h/3)$ and any weak solution $\mathbf{v}$ of the body problem with associated pressure q satisfy $(\mathbf{v},q) \in (D \cap C^\infty(\overline{\Omega_+} \backslash B(h,h/3))) \times C^\infty(\overline{\Omega_+} \backslash B(h,h/3)$. Hence formula (3.1) on $\mathbf{u}$ and $F = (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \partial_x \mathbf{u} - \Delta\mathbf{u} + \nabla p$ are well- defined. Moreover, $(\mathbf{v},q)$ is a classical solution to Navier-Stokes system outside of $\mathcal{S}_\epsilon$ and in particular in $\Omega_+ \backslash B(h,h/3)$.

To prove $\mathbf{u}$ is a weak solution for some source term, by the above argument, we obtain $\mathbf{u} \in D$. Then, as $(\mathbf{v},q)$ is a classical solution to the body problem in $\Omega_+ \backslash B(h,h/3)$, previous computations in the truncation process imply that

$$F = (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \partial_x \mathbf{u} - \Delta\mathbf{u} + \nabla p, \quad \text{in } \Omega_+. \quad (3.5)$$

Multiplying this equality by $\mathbf{w} \in \mathcal{D}$ and integrating by parts, we obtain (3.3) for $\mathbf{u}$ and F. Next we manage to obtain the energy estimates (3.4) for $\mathbf{u}$. To this end, we multiply Navier-Stokes equations satisfied by $(\mathbf{u}, p)$ on $B(h, 7h/4) := \mathcal{O}$. Integrating by parts, it yields

$$\int_{\mathcal{O}} |\nabla\mathbf{u}|^2 d\mathbf{x} = \int_{\mathcal{O}} F \cdot \mathbf{u} d\mathbf{x} + \int_{\partial\mathcal{O}} [T(\mathbf{u},p) \cdot \mathbf{u} - \frac{|\mathbf{u}|^2}{2}(\mathbf{u} + \mathbf{e}_1)] \cdot \mathbf{n} d\boldsymbol{\sigma}, \quad (3.6)$$

where n is the outward normal vector on $\partial\mathcal{O}$, and $T(\mathbf{u},p) = (\boldsymbol{\nabla}\mathbf{u} - p\mathbf{I})$. Then, multiplying equation (1.3) in the body problem by $\mathbf{v}$ on $\mathcal{O}\backslash\overline{\mathcal{S}_\epsilon}$, we obtain, with Σ the associated force applied on $\mathcal{S}_\epsilon$

$$\int_{\Omega_+} |\boldsymbol{\nabla}\mathbf{v}|^2 \, d\mathbf{x} = \Sigma \cdot e_1 + \int_{\partial\mathcal{O}} [T(\mathbf{v},q)\cdot\mathbf{v} - \frac{|\mathbf{v}|^2}{2}(\mathbf{v}+\mathbf{e}_1)]\cdot\mathbf{n}d\boldsymbol{\sigma}, \tag{3.7}$$

On the other hand, by definition, we have

$$\int_{\Omega_+} |\boldsymbol{\nabla}\mathbf{v}|^2 \, d\mathbf{x} \leq \Sigma \cdot \mathbf{e}_1. \tag{3.8}$$

The substraction of (3.7) and (3.8) gives

$$\int_{\Omega_+\backslash\overline{\mathcal{O}}} |\boldsymbol{\nabla}\mathbf{v}|^2 \, d\mathbf{x} \leq -\int_{\partial\mathcal{O}} [T(\mathbf{v},q)\cdot\mathbf{v} - \frac{|\mathbf{v}|^2}{2}(\mathbf{v}+\mathbf{e}_1)]\cdot\mathbf{n}d\boldsymbol{\sigma} \tag{3.9}$$

However, outside $B(h,2h/3)$, by the truncation procedure, we find that both velocity-fields coincide, i.e., $\mathbf{u} = \mathbf{v}$ and $p = q$, then combining (3.6) and (3.9) yields

$$\int_{\Omega_+} |\boldsymbol{\nabla}\mathbf{u}|^2 \, d\mathbf{x} \leq \int_{\mathcal{O}} F\cdot\mathbf{u}d\mathbf{x} = \int_{\Omega_+} F\cdot\mathbf{u}d\mathbf{x}.$$

The last equality holds in the above since F has compact support in $\overline{B(h,2h/3}\backslash B(h,h/3)$. Therefore, we complete the proof of the existence of weak solutions for the source term problem. □

3.2 Existence of α-solutions

Firstly, we recall the definition and main properties of α-solutions. Let first introduce the functional framework:

Definition 3.4 *Let $\mathbb{R}^2_0 = \mathbb{R}^2\backslash\{0\}$. We define, for fixed $\alpha \geq 0$ and $p \geq 0$, $\mathcal{B}_{\alpha,p}$ to be the Banach space of functions $f \in C(\mathbb{R}^2_0 \times [1,\infty), \mathbb{C})$, for which the norm*

$$||f;\mathcal{B}_{\alpha,p}|| = \sup_{t\geq 1}\sup_{\mathbf{k}\in\mathbb{R}^2_0} \frac{|f(\mathbf{k},t)|}{\frac{1}{t^p}\bar{\mu}_\alpha(\mathbf{k},t)}$$

is finite, where

$$\bar{\mu}_\alpha(\mathbf{k},t) = \frac{1}{1+(|\mathbf{k}|t)^\alpha},$$

with

$$|\mathbf{k}| = \sqrt{k_1^2+k_2^2}.$$

Furthermore, we set

$$\mathcal{B}^n_{\alpha,p} = \underbrace{\mathcal{B}_{\alpha,p} \times \ldots \times \mathcal{B}_{\alpha,p}}_{n \ times} \ ,$$

and

$$\mathcal{W}_\alpha = \mathcal{B}^3_{\alpha,3} \ , \quad \mathcal{V}_\alpha = \mathcal{B}^3_{\alpha,1} \times \mathcal{B}^3_{\alpha,0} \ .$$

In [11], we proved the following existence theorem:

Theorem 3.5 *(Existence) Let* $\alpha > 2$, $\mathbf{F} = (F_1, F_2, F_3) \in C^\infty_c(\Omega_+)$, *and let* $\hat{\mathbf{F}}$ *be the Fourier transform of* $\mathbf{F}$*. If* $\|\hat{\mathbf{F}}; \mathcal{W}_\alpha\|$ *is sufficiently small, then there exists an* α*-solution* $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}})$ *for* $\hat{\mathbf{F}}$ *in* $\mathcal{V}_\alpha$*, with* $\|(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}); \mathcal{V}_\alpha\| \leq C_\alpha \|\hat{\mathbf{F}}; \mathcal{W}_\alpha\|$*, for some constant* C_α *depending only on the choice of* α*.*

We let the readers refer to the mentioned paper for more details. This α-solution possesses the following properties:

1. $\mathbf{u} \in H^1_0(\Omega_+)$, where $\mathbf{u}$ is the inverse Fourier transform corresponding to $\hat{\mathbf{u}}$ in Theorem 3.5.

2. There exists an absolute constant C such that:

$$||\mathbf{u}||_{H^1_0(\Omega_+)} \leq C||\hat{\mathbf{u}}; \mathcal{B}^3_{\alpha,0}|| \qquad \text{and} \qquad |u_i(x,y,z)| \leq \frac{C||\hat{\mathbf{u}}; \mathcal{B}^3_{\alpha,0}||}{z^2}, \qquad \forall (x,y,z) \in \Omega_+.$$

Further details on the asymptotic behavior of α-solutions is given in [12]. The explicit computation of source term given by (3.5) enables us to construct α-solutions when the obstacle is small. Indeed, we have the following technical lemma.

Lemma 3.6 *Given* $(m,p) \in \mathbb{N} \times [0, \infty)$*, there exists a constant* K^m_p *such that, for given* $f \in C^\infty_c(\Omega_+)$ *with supp*$(f) \subset B(h, 2h/3)$*, there holds:*

$$||\hat{f}; \mathcal{B}_{m,p}|| \leq K^m_p ||f||_{\mathcal{C}^m(\Omega_+)}.$$

Proof: Given $f \in C^\infty_c(\Omega_+)$ with $\text{supp}(f) \subset B(h, 2h/3)$, the Fourier transform $\hat{f}$ of f is well-defined and continuous on Ω_+. Moreover, there holds

$$\hat{f}(k_1, k_2, z) = \int_{\mathbb{B}(0,2h/3)} e^{ik_1x} e^{ik_2y} f(x,y,z) dx dy,$$

where $\mathbb{B}(0, 2h/3)$ is a ball centered at zero in $\mathbb{R}^2$. Consequently, by integration by parts we have

$$\begin{aligned} \left|\hat{f}(k_1,k_2,z)\right| &\leq \text{Const.} \, \|f\|_{\mathcal{C}^0(\Omega_+)} \ , \\ \left|\hat{f}(k_1,k_2,z)\right| &\leq \text{Const.} \min\{\frac{1}{|k_1|^m}, \frac{1}{|k_2|^m}, \frac{1}{|k_1|^\beta \, |k_2|^{m-\beta}}\} \, \|f\|_{\mathcal{C}^m(\Omega_+)} \\ &\leq \text{Const.} \frac{\|f\|_{\mathcal{C}^m(\Omega_+)}}{|\mathbf{k}|^m}. \end{aligned}$$

As f has compact support in z, we have

$$\left|\hat{f}(k_1,k_2,z)\right| \leq \mathrm{C(m,p)}\frac{\|f\|_{\mathcal{C}^m(\Omega_+)}}{z^p\left(1+(|\mathbf{k}|z)^m\right)},$$

where the constant C(m,p) depending on m and p. We complete the proof by combining Definition 3.4. □

By applying this property to the source term F in (3.5) and combining the results for α-solutions recalled above, we obtain

Theorem 3.7 *Given $\alpha > 3$, there exists $\epsilon_\alpha > 0$ such that, for all $\epsilon < \epsilon_\alpha$ and weak solution $\mathbf{v}_\epsilon$ for the body problem with associated pressure p_ϵ is such that the source term problem admits an α-solution $\mathbf{u}_\alpha$ for some source term F given by (3.5). Moreover, there exists $C_\alpha < \infty$ depending only on α such that this α-solution satisfies $||\hat{\mathbf{u}}_\alpha;\mathcal{B}^3_{\alpha,0}|| \leq C_\alpha||\mathbf{v}_\epsilon;D||$, where $\hat{\mathbf{u}}_\alpha$ is the Fourier transform of $\mathbf{u}_\alpha$ with respect to x and y.*

Proof: The proof of this theorem is a straightforward consequence of Lemma 3.6, Theorem 2.1, Theorem 3.5 and the direct properties of source term F given by (3.5). □

In what follows, we discuss weak-strong uniqueness. In previous sections, we showed a weak solution $\mathbf{v}$ for the body problem provides a weak solution $\mathbf{u}$ for source term F by truncation. As a consequence of Theorem 3.7, we also have that, if the obstacle is sufficiently small, there exists an α-solution $\mathbf{u}_\alpha$ for the same source term. Now we will prove that, up to taking a smaller obstacle if needed, the weak solution $\mathbf{u}$ and the α-solution $\mathbf{u}_\alpha$ coincide. Precisely, we prove

Theorem 3.8 *Given $\alpha > 2$, there exists $\eta_\alpha > 0$ satisfying the following property. Given $\mathbf{f} \in C_c^\infty(\Omega_+)$, such that $|\hat{\mathbf{u}}_\alpha;\mathcal{B}^3_{\alpha,0}|| < \eta_\alpha$, and $\mathbf{u}$, $\mathbf{u}_\alpha$ mentioned above for source term $\mathbf{f}$, then there holds $\mathbf{u} = \mathbf{u}_\alpha$.*

Proof: It is obvious that α-solution is also a weak solution with data $\mathbf{f}$. Hence, we have (3.3) and (3.4) for $\mathbf{u}$ and $\mathbf{u}_\alpha$. we use D-norm to estimate the difference of $\mathbf{u}$ and $\mathbf{u}_\alpha$.

$$||\mathbf{u}-\mathbf{u}_\alpha;D||^2 = ||\mathbf{u};D||^2 + ||\mathbf{u}_\alpha;D||^2 - 2\int_{\Omega_+}\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}_\alpha d\mathbf{x},$$

In view of (3.4)

$$||\mathbf{u};D||^2 \le \int_{\Omega_+} \mathbf{f}\cdot\mathbf{u}d\mathbf{x}, \quad ||\mathbf{u}_\alpha;D||^2 \le \int_{\Omega_+} \mathbf{f}\cdot\mathbf{u}_\alpha d\mathbf{x}. \tag{3.10}$$

Moreover, by (3.3) and (A.8) in Proposition A.4 in Appendix we have

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}_\alpha d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}]\cdot\mathbf{u}_\alpha d\mathbf{x} = \int_{\Omega_+} \mathbf{f}\cdot\mathbf{u}_\alpha d\mathbf{x} \tag{3.11}$$

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u}_\alpha : \boldsymbol{\nabla}\mathbf{u} d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}_\alpha+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}_\alpha]\cdot\mathbf{u} d\mathbf{x} = \int_{\Omega_+} \mathbf{f}\cdot\mathbf{u} d\mathbf{x} \tag{3.12}$$

Combining (3.10)-(3.12) yields

$$||\mathbf{u}-\mathbf{u}_\alpha;D||^2 \le \int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}]\cdot\mathbf{u}_\alpha d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}_\alpha+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}_\alpha]\cdot\mathbf{u} d\mathbf{x}.$$

We can integrate by parts on the right-hand side of the previous inequality by Proposition A.6 in Appendix. This yields

$$\begin{aligned} ||\mathbf{u}-\mathbf{u}_\alpha;D||^2 &\le -\int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}_{\boldsymbol{\alpha}}]\cdot\mathbf{u} d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}_\alpha+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}_\alpha]\cdot\mathbf{u} d\mathbf{x} \\ &\le \int_{\Omega_+} [(\mathbf{u}_\alpha-\mathbf{u})\cdot\boldsymbol{\nabla}\mathbf{u}_\alpha]\cdot\mathbf{u} d\mathbf{x} \le \int_{\Omega_+} [(\mathbf{u}_\alpha-\mathbf{u})\cdot\boldsymbol{\nabla}\mathbf{u}_\alpha]\cdot(\mathbf{u}-\mathbf{u}_\alpha)\, d\mathbf{x} \end{aligned}$$

Finally, by (A.9) in Proposition A.4 in the Appendix, we obtain

$$||\mathbf{u}-\mathbf{u}_\alpha;D||^2 \le C||\hat{\mathbf{u}}_\alpha;\mathcal{B}^3_{\alpha,0}||||\mathbf{u}-\mathbf{u}_\alpha;D||^2.$$

By construction, for η_α sufficiently small we have $C||\hat{\mathbf{u}}_\alpha;\mathcal{B}^3_{\alpha,0}|| < 1/2$. This completes the proof. □

4 Uniqueness of weak solutions for the body problem

Finally, we sketch the proof that weak solutions for small obstacles are also unique.

Theorem 4.1 *There exists $\epsilon^u > 0$, such that, for all $\epsilon < \epsilon^u$, if $\mathbf{u}$ is a weak solution to the body problem for S_ϵ, then $\mathbf{u} = \mathbf{u}_\epsilon$.*

Proof of Proposition: The proof is very close to that of Theorem 3.8. Here, we only give the main ideas. First, we fix $\alpha > 3$ and choose ϵ_0^u such that, for all $\epsilon < \epsilon_0^u$, any weak solution is equal to the α-solution outside $B(h, 2h/3)$. Furthermore, there holds

$$||\hat{\mathbf{u}}_\alpha;\mathcal{B}^3_{\alpha,0}|| \leq C_\alpha||\mathbf{u}_\epsilon; D||.$$

Now, let $\epsilon < \epsilon_0^u$ and $\mathbf{u}$ be a weak solution to the body problem for S_ϵ. Following the sketch of Theorem 3.8, we obtain

$$||\mathbf{u} - \mathbf{u}_\epsilon; D||^2 \leq \int_{\Omega_+} [(\mathbf{u}_\epsilon - \mathbf{u}) \cdot \boldsymbol{\nabla}\mathbf{u}_\epsilon] \cdot (\mathbf{u} - \mathbf{u}_\epsilon)\, d\mathbf{x}.$$

The technicalities which arise here are analogous to (3.11),(3.12), Proposition A.6) and (A.9), and are justified by splitting integrals as follows:

$$I(\mathbf{v}, \mathbf{w}, \mathbf{z}) = \int_{\Omega_+} [(\mathbf{v} + \mathbf{e}_1) \cdot \boldsymbol{\nabla}\mathbf{w}] \cdot \mathbf{z} d\mathbf{x} = I_{int} + I_{ext},$$

where

$$I_{ext} = \int_{\Omega_+\setminus B(2h/3)} [(\mathbf{v} + \mathbf{e}_1) \cdot \boldsymbol{\nabla}\mathbf{w}] \cdot \mathbf{z} d\mathbf{x},$$

$$I_{int} = \int_{B(2h/3)} [(\mathbf{v} + \mathbf{e}_1) \cdot \boldsymbol{\nabla}\mathbf{w}] \cdot \mathbf{z} d\mathbf{x}.$$

Therefore, as discussed in the proof of Theorem 3.8, noticing the fact that in I_{ext} the weak solution $\mathbf{u}_\epsilon$ coincide with the α-solution $\mathbf{u}_\alpha$, this yields

$$\begin{aligned} ||\mathbf{u} - \mathbf{u}_\epsilon; D||^2 &\leq C\left[||\hat{\mathbf{u}}_\alpha;\mathcal{B}^3_{\alpha,0}|| + ||\mathbf{u}_\epsilon; D||\right] ||\mathbf{u} - \mathbf{u}_\epsilon; D||^2 \\ &\leq C_\alpha||\mathbf{u}_\epsilon; D||||\mathbf{u} - \mathbf{u}_\epsilon; D||^2. \end{aligned}$$

According to Theorem 2.4, there exists ϵ_1^u such that if $\epsilon < \epsilon_1^u$, we have $||\mathbf{u}_\epsilon; D|| < 1/(2C_\alpha)$, this completes the proof. □

A Technical lemmas

In this section, we give some technical details. The first task is to show (3.1) is well-defined.

Proposition A.1 *Suppose that* $\mathbf{v}$ *satisfy*

$$\boldsymbol{\nabla}\mathbf{v} \in L^2 \text{ in } \Omega_\epsilon,\ \boldsymbol{\nabla} \cdot \mathbf{v} = 0,\ \ \mathbf{v}|_{\partial S_\epsilon} = -e_1,\ \ \mathbf{v}|_{\partial\Omega_+} = 0,\ \ \textit{and} \ \lim_{|\mathbf{x}|\to\infty} \mathbf{v}(\mathbf{x}) = 0.$$

then

$$\mathbf{u}(\mathbf{x}) = \chi(|\mathbf{x}|)\mathbf{v}(\mathbf{x}) + \mathbf{T}[\mathbf{e}_1](\mathbf{x})\,(1-\chi(|\mathbf{x}|)) + \boldsymbol{\nabla}\chi(|\mathbf{x}|)\times\boldsymbol{\psi}(\mathbf{x}),$$

with

$$\boldsymbol{\psi}(\mathbf{x}) = \frac{1}{4\pi}\int_{\Omega_\epsilon}\boldsymbol{\nabla}\times(\mathbf{v}(\mathbf{y})-\mathbf{T}[\mathbf{e}_1](\mathbf{y}))\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}-\frac{1}{|\mathbf{y}|}\right)d^3\mathbf{y},$$
$$\mathbf{T}[\mathbf{e}_1](\mathbf{x}) = \mathbf{e}_1\,(\chi(|\mathbf{x}|)-1) + \frac{1}{2}|\mathbf{x}|\chi'(|\mathbf{x}|)\mathbf{e}_1 - \mathbf{x}\left(\frac{(\mathbf{e}_1\cdot\mathbf{x})\chi'(|\mathbf{x}|)}{2|\mathbf{x}|}\right).$$

is well-defined for $\mathbf{x}\in\Omega_+$*, satisfies*

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \mathbf{v}(\mathbf{x}), & \text{if } |\mathbf{x}|\geq B(h,2h/3) \\ -e_1, & \text{if } |\mathbf{x}|\leq B(h,h/3) \end{cases} \tag{A.1}$$

and $\mathbf{u}(\mathbf{x})$ *is divergence free.*

Note that both $\mathbf{T}[\mathbf{e}_1](\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$ are divergence-free functions over Ω_+, vanishing identically inside a sphere containing S_{ϵ_0} and identically equal to their argument outside of a bigger sphere containing S_{ϵ_0}. $\mathbf{u}(\mathbf{x})$ is the main object of interest here, $\mathbf{T}[\mathbf{e}_1](\mathbf{x})$ is a convenient way of avoiding boundary terms when using the divergence theorem to study $\boldsymbol{\psi}(\mathbf{x})$ below.

Proof: The verification of (A.1) is trivial. In particular, $\mathbf{u}(\mathbf{x})$ is well defined outside of the annulus $\mathcal{A}_h = B(h,2h/3)\backslash B(h,h/3)$. We next show that $\boldsymbol{\psi}(\mathbf{x})$ is well defined for $\mathbf{x}\in\mathcal{A}_h$. For convenience, we denote $\mathbf{v}(\mathbf{y})-\mathbf{T}[\mathbf{e}_1](\mathbf{y})$ by $\tilde{\mathbf{v}}(\mathbf{y})$, and note that $||\boldsymbol{\nabla}\times\tilde{\mathbf{v}}||_{L^2}<\infty$. Then there exists a compactly supported function $\rho:\mathbb{R}^+\to\mathbb{R}$ with $\rho(0)=1$, and a constant C such that for all $\mathbf{x}\in\mathcal{A}_h$, there holds

$$\sup_{\mathbf{x}\in\mathcal{A}_h}|\boldsymbol{\psi}(\mathbf{x})| \leq C\int_{\Omega_\epsilon}|\boldsymbol{\nabla}\times\tilde{\mathbf{v}}(\mathbf{y})|\left(\frac{\rho(|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}+\frac{|\mathbf{x}|}{|\mathbf{y}|^2}\right)d^3\mathbf{y} \leq C||\boldsymbol{\nabla}\times\tilde{\mathbf{v}}||_{L^2}<\infty,$$

which shows that $\mathbf{u}(\mathbf{x})$ is well-defined for all $\mathbf{x}\in\Omega_+$.

It only remains to show that $\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{x})=0$. Since $\boldsymbol{\nabla}\cdot\mathbf{T}[\mathbf{e}_1](\mathbf{x})=0$ and $\boldsymbol{\nabla}\cdot\mathbf{v}=0$, we find

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{x}) &= \boldsymbol{\nabla}\chi(|\mathbf{x}|)\cdot\tilde{\mathbf{v}}(\mathbf{x}) - \boldsymbol{\nabla}\chi(|\mathbf{x}|)\cdot(\boldsymbol{\nabla}\times\boldsymbol{\psi}(\mathbf{x})) \\ &= \boldsymbol{\nabla}\chi(|\mathbf{x}|)\cdot(\tilde{\mathbf{v}}(\mathbf{x})-\boldsymbol{\nabla}\times\boldsymbol{\psi}(\mathbf{x})),\end{aligned}$$

where we have used the identity

$$\boldsymbol{\nabla}\cdot(\mathbf{f}\times\mathbf{g}) = (\boldsymbol{\nabla}\times\mathbf{f})\cdot\mathbf{g} - \mathbf{f}\cdot(\boldsymbol{\nabla}\times\mathbf{g}).$$

We thus only need to show that $\tilde{\mathbf{v}}(\mathbf{x}) = \boldsymbol{\nabla}\times\boldsymbol{\psi}(\mathbf{x})$ for all $\mathbf{x}\in\mathcal{A}_h$ since $\boldsymbol{\nabla}\chi(|\mathbf{x}|) = 0$ if $\mathbf{x}\notin\mathcal{A}_h$. Using the completely antisymmetric tensor $\epsilon_{i,j,k}$ with $\epsilon_{1,2,3} = 1$ so that $[\boldsymbol{\nabla}\times\mathbf{u}]_i = \sum_{j,k}\epsilon_{i,j,k}\partial_{x_j}\mathbf{u}_k$, therefore

$$\begin{aligned}[\boldsymbol{\nabla}\times\boldsymbol{\psi}(\mathbf{x})]_i &= \frac{1}{4\pi}\sum_{j,k,l,m=1}^{3}\epsilon_{i,j,k}\epsilon_{k,l,m}\int_{\Omega_\epsilon}(\partial_{y_l}\tilde{\mathbf{v}}_m(\mathbf{y}))\partial_{x_j}(|\mathbf{x}-\mathbf{y}|^{-1})d^3\mathbf{y}\\ &= \lim_{R\to\infty}(\mathbf{I}_i(R,\mathbf{x}) + \mathbf{J}_i(R,\mathbf{x})),\end{aligned}$$

in the sense of distribution, where

$$\begin{aligned}\mathbf{I}_i(R,\mathbf{x}) &= \int_{\Omega_R}\sum_{l=1}^{3}\partial_{y_l}\left(\sum_{j,k,m=1}^{3}\frac{\epsilon_{i,j,k}\epsilon_{k,l,m}}{4\pi}\tilde{\mathbf{v}}_m(\mathbf{y})\partial_{x_j}(|\mathbf{x}-\mathbf{y}|^{-1}\right)d^3\mathbf{y},\\ \mathbf{J}_i(R,\mathbf{x}) &= \sum_{j,k,l,m=1}^{3}\frac{\epsilon_{i,j,k}\epsilon_{k,l,m}}{4\pi}\int_{\Omega_R}\tilde{\mathbf{v}}_m(\mathbf{y})\partial_{x_l}\partial_{x_j}(|\mathbf{x}-\mathbf{y}|^{-1})d^3\mathbf{y},\end{aligned}$$

with $\Omega_R = B_R\cap\Omega_\epsilon$, where B_R denotes a ball centered at $(0,0,1+h)$ with radius R. We claim that for all $\mathbf{x}\in\mathcal{A}_h$, we have

$$\lim_{R\to\infty}\mathbf{I}(R,\mathbf{x}) = 0 \quad\text{and}\quad \lim_{R\to\infty}\mathbf{J}(R,\mathbf{x}) = \tilde{\mathbf{v}}(\mathbf{x}).$$

We first consider the function $\mathbf{J}(R,\mathbf{x})$. Since $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, using the symbols $\boldsymbol{\nabla}_\mathbf{x}$, respectively $\boldsymbol{\nabla}_\mathbf{y}$ to denote the Nabla operators in the variables $\mathbf{x}$ respectively $\mathbf{y}$, we find that

$$\begin{aligned}\mathbf{J}(R,\mathbf{x}) &= \frac{1}{4\pi}\int_{\Omega_R}\boldsymbol{\nabla}_\mathbf{x}\times\boldsymbol{\nabla}_\mathbf{x}\times\left(\frac{\tilde{\mathbf{v}}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right)d^3\mathbf{y}\\ &= \tilde{\mathbf{v}}(\mathbf{x}) + \frac{1}{4\pi}\int_{\Omega_R}\boldsymbol{\nabla}_\mathbf{x}\boldsymbol{\nabla}_\mathbf{x}\cdot\left(\frac{\tilde{\mathbf{v}}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right)d^3\mathbf{y}\\ &= \tilde{\mathbf{v}}(\mathbf{x}) - \frac{1}{4\pi}\int_{\Omega_R}\boldsymbol{\nabla}_\mathbf{x}\boldsymbol{\nabla}_\mathbf{y}\cdot\left(\frac{\tilde{\mathbf{v}}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right)d^3\mathbf{y} = \tilde{\mathbf{v}}(\mathbf{x}) + \mathbf{K}(R,x),\end{aligned}$$

where we have used the following vector identity

$$\Delta\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})$$

and

$$\Delta(-\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}) = \delta(\mathbf{x}-\mathbf{y}),\ \text{where } \delta \text{ is the Dirac delta function.}$$

Assume that $R > 2h/3$, and apply the divergence theorem to $\mathbf{K}$ and $\mathbf{I}_i$. Since $\mathcal{A}_h \cap \partial\Omega_R = \phi$, $|\mathbf{x}-\mathbf{y}|^{-1}$ is nonsingular for $\mathbf{y} \in \partial\Omega_R$ and $\mathbf{x} \in \mathcal{A}_h$, we thus obtain

$$\mathbf{K}(R,x) = \int_{\partial B_R} \frac{(\mathbf{x}-\mathbf{y})(\tilde{\mathbf{v}}(\mathbf{y})\cdot(\mathbf{y}-(0,0,1+h)))}{4\pi|\mathbf{x}-\mathbf{y}|^3\,|\mathbf{y}-(0,0,1+h)|}d\sigma,$$

$$\mathbf{I}_i(R,\mathbf{x}) = \int_{\partial B_R} \sum_{l=1}^{3}\left(\sum_{j,k,m=1}^{3} \frac{\epsilon_{i,j,k}\epsilon_{k,l,m}}{4\pi}\tilde{\mathbf{v}}_m(\mathbf{y})\frac{(\mathbf{x}-\mathbf{y})_j}{|\mathbf{x}-\mathbf{y}|}\right)\frac{(\mathbf{y}-(0,0,1+h))_l}{|\mathbf{y}-(0,0,1+h)|}d^3\mathbf{y},$$

where we noticed that $\tilde{\mathbf{v}}(\mathbf{y}) = 0$ for $\partial\Omega_\epsilon$. Therefore

$$|\mathbf{K}(R,x)|^2 + |\mathbf{I}(R,\mathbf{x})|^2 \le C\left(\int_{\partial B_R}\frac{|\tilde{\mathbf{v}}(\mathbf{y})|}{|\mathbf{y}|^2}d\sigma\right)^2 \le C\int_{\partial B_R}\frac{|\tilde{\mathbf{v}}(\mathbf{y})|^2}{|\mathbf{y}|^2}d\sigma. \tag{A.2}$$

In order to complete the proof of this proposition, we only need to show the right-hand side of (A.2) goes to zero as $R \to \infty$. This can be achieved as follows.

Let

$$F(R) = \int_{\Omega_R} \boldsymbol{\nabla}\cdot\left(\frac{|\boldsymbol{\Phi}(\mathbf{x})|^2\,(\mathbf{x}-(0,0,1+h))}{|\mathbf{x}-(0,0,1+h)|}\right)d^3\mathbf{x}, \ \text{ with } \boldsymbol{\Phi}(\mathbf{x}) = \frac{\tilde{\mathbf{v}}(\mathbf{x})}{|\mathbf{x}|}.$$

By Hardy's inequality, we know that $||\boldsymbol{\Phi}||_{L^2(\Omega_+)} \le ||\nabla\tilde{\mathbf{v}}(\mathbf{x})||_{L^2(\Omega_+)} < \infty$, furthermore, $||\nabla\boldsymbol{\Phi}||_{L^2(\Omega_+)} \le 2\rho^{-1}||\nabla\tilde{\mathbf{v}}(\mathbf{x})||_{L^2(\Omega_+)}$, where ρ is the diameter of the largest sphere contained in S_ϵ. A straightforward computation gives

$$\sup_{R>2h/3}(F(R)) \le \frac{2}{\rho}||\boldsymbol{\Phi}||_{L^2}^2 + ||\boldsymbol{\Phi}||_{L^2}||\nabla\boldsymbol{\Phi}||_{L^2},$$

$$\lim_{R\to\infty} F(R) = \int_{\Omega_+}\boldsymbol{\nabla}\cdot\left(\frac{|\boldsymbol{\Phi}(\mathbf{x})|^2\,(\mathbf{x}-(0,0,1+h))}{|\mathbf{x}-(0,0,1+h)|}\right)d^3\mathbf{x},$$

which implies that $F(R)$ has a finite limit as $R \to \infty$. Now, by divergence theorem we have

$$F(R) = \int_{\Omega_R}\boldsymbol{\nabla}\cdot\left(\frac{|\boldsymbol{\Phi}(\mathbf{x})|^2\,(\mathbf{x}-(0,0,1+h))}{|\mathbf{x}-(0,0,1+h)|}\right)d^3\mathbf{x} = \int_{\partial B_R}|\boldsymbol{\Phi}(\mathbf{x})|^2 d\sigma,$$

Since for any finite $r > h/3$,

$$\int_r^\infty F(R)dR = \int_r^\infty\int_{\partial B_R}\frac{|\tilde{\mathbf{v}}(\mathbf{y})|^2}{|\mathbf{y}|^2}d\sigma dR = \int_{\Omega_+\setminus B_R}\frac{|\tilde{\mathbf{v}}(\mathbf{y})|^2}{|\mathbf{y}|^2}d^3\mathbf{y}$$
$$\le \int_{\Omega_+}\frac{|\tilde{\mathbf{v}}(\mathbf{y})|^2}{|\mathbf{y}|^2}d^3\mathbf{y} \le \int_{\Omega_+}|\nabla\tilde{\mathbf{v}}(\mathbf{y})|^2 d^3\mathbf{y} < \infty.$$

Finally, we get $F(R) \to 0$ as $R \to \infty$. This completes the proof of our proposition. $\square$

The second is to compute more general sufficient conditions for the trilinear form:

$$\int_{\Omega_+} [(\mathbf{u} + \mathbf{e}_1) \cdot \boldsymbol{\nabla} \mathbf{v}] \cdot \mathbf{w} d\mathbf{x}$$

to be well-defined. To this end, a major ingredient is the Hardy's inequality for D functions which reads

Proposition A.2 *There exists a constant C such that, for any $\mathbf{w} \in D$, we have*

$$\int_{\Omega_+} \frac{\mathbf{w}^2(x, y, z)}{z^2} dxdydz \le C||\mathbf{w}; D||^2.$$

Consequently, we have the following continuity result for the trilinear form:

Proposition A.3 *There exists a universal constant $C < \infty$ such that, for any $(\mathbf{v}, \mathbf{w}) \in D^2$ and $\bar{\mathbf{u}} \in H^1_{loc}(\Omega_+)$, if the right-hand side in these inequalities are finite, there holds*

$$\left| \int_{\Omega_+} [(\mathbf{v} + \mathbf{e}_1) \cdot \boldsymbol{\nabla} \mathbf{w}] \cdot \bar{\mathbf{u}} d\mathbf{x} \right| \le C\,[1 + ||\mathbf{v}; D||]\, ||\mathbf{w}; D|| \left[||\bar{\mathbf{u}}||_{L^2(\Omega_+)} + ||z\bar{\mathbf{u}}||_{L^\infty(\Omega_+)} \right] \tag{A.3}$$

and

$$\left| \int_{\Omega_+} [(\mathbf{v} + \mathbf{e}_1) \cdot \boldsymbol{\nabla} \bar{\mathbf{u}}] \cdot \mathbf{w} d\mathbf{x} \right| \le C\,[1 + ||\mathbf{v}; D||]\, ||\mathbf{w}; D|| \left[||z^2 \boldsymbol{\nabla} \bar{\mathbf{u}}||_{L^\infty(\Omega_+)} + ||z \boldsymbol{\nabla} \bar{\mathbf{u}}||_{L^2(\Omega_+)} \right] \tag{A.4}$$

Proof: We denote by I_1 and I_2 the two integrals to be computed. If $||\bar{\mathbf{u}}||_{L^2(\Omega_+)} + ||z\bar{\mathbf{u}}||_{L^\infty(\Omega_+)} < \infty$, we have

$$\begin{aligned} I_1 &= \int_{\Omega_+} [(\mathbf{v} + \mathbf{e}_1) \cdot \boldsymbol{\nabla} \mathbf{w}] \cdot \bar{\mathbf{u}} d\mathbf{x} \\ &= \int_{\Omega_+} [\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{w}] \cdot \bar{\mathbf{u}} d\mathbf{x} + \int_{\Omega_+} \partial_x \mathbf{w} \cdot \bar{\mathbf{u}} d\mathbf{x}. \end{aligned}$$

On the right-hand side, we dominate the second integral by the Cauchy-Schwarz inequality, i.e.,

$$\int_{\Omega_+} \partial_x \mathbf{w} \cdot \bar{\mathbf{u}} d\mathbf{x} \le ||\mathbf{w}; D|| ||\bar{\mathbf{u}}||_{L^2(\Omega_+)}. \tag{A.5}$$

Concerning the first integral we have

$$\begin{aligned}\left|\int_{\Omega_+} [\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{w}] \cdot \bar{\mathbf{u}} d\mathbf{x}\right| &= \left|\int_{\Omega_+} \left[\frac{\mathbf{v}}{z} \cdot \boldsymbol{\nabla} \mathbf{w}\right] \cdot z\bar{\mathbf{u}} d\mathbf{x}\right| \\ &\leq \left\|\frac{\mathbf{v}}{z}\right\|_{L^2(\Omega_+)} \|\boldsymbol{\nabla} \mathbf{w}\|_{L^2(\Omega_+)} \|z\bar{\mathbf{u}}\|_{L^\infty(\Omega_+)} \end{aligned} \tag{A.6}$$

where we have used the Hölder's inequality. Then the estimate for I_1 is a direct consequence of (A.5), (A.6) and the Hardy's inequality. The second integral I_2 is dominated similarly. □

Proposition A.4 *If f is the inverse Fourier transform of $\hat{f} \in \mathcal{B}_{\alpha,p}$ for $p \geq 0$, given $s \in [2, \infty]$ there exists a constant C depending only on α ($\alpha \geq 3$) and s such that, for any $z > 1$, there holds $f(., z) \in L^s(\mathbb{R}^2)$ with*

$$||f(., z)||_{L^s(\mathbb{R}^2)} \leq \frac{C||\hat{f}; \mathcal{B}_{\alpha,p}||}{z^l},$$

with $l = 2\,(1 - 1/s) + p$.

Proof: By definition of the space $\mathcal{B}_{\alpha,p}$, we get

$$\left|\hat{f}(\mathbf{k}, z)\right| \leq \left\|\hat{f}; \mathcal{B}_{\alpha,p}\right\| \frac{1}{z^p\,(1 + (|\mathbf{k}|z)^\alpha)}, \qquad \forall \mathbf{k} \in \mathbb{R}^2_0.$$

Consequently $\hat{f}(., z) \in L^r(\mathbb{R}^2)$ for any $r \in [1, 2]$. In view of Hausdorff-Young inequality, we know that $f(., z) \in L^s(\mathbb{R}^2)$ with $s \in [2, \infty]$. Moreover, given $s \geq 2$ and $r \leq 2$ its conjugate exponent, the norm $\|f(., z)\|_{L^s}$ can be controlled by $\left\|\hat{f}(., z)\right\|_{L^r}$. By a scaling argument, we have

$$\int_{\mathbb{R}^2} \frac{1}{(1 + (|\mathbf{k}|z)^\alpha)^r} d\mathbf{k} \leq \frac{C_{\alpha,r}}{z^2}.$$

Therefore we get

$$||f(., z)||_{L^s(\mathbb{R}^2)} \leq C_{\alpha,r} \left\|\hat{f}; \mathcal{B}_{\alpha,p}\right\| \frac{1}{z^{p+\frac{2}{r}}}.$$

This completes the proof. □

We note that the result of this proposition implies that, if $\hat{f} \in \mathcal{B}_{\alpha,p}$ for $p > 0$ and $\alpha \geq 3$, there holds $f \in L^2(\Omega_+)$ and

$$|f(x, y, z)| \leq \frac{C||\hat{f}; \mathcal{B}_{\alpha,p}||}{z^l}, \quad \forall (x, y, z) \in \Omega_+, \quad \text{with } l = p + 2.$$

In particular, for our α-solution $\mathbf{u}_\alpha = (u_{\alpha 1}, u_{\alpha 2}, u_{\alpha 3})$, there holds $\mathbf{u}_\alpha \in L^2(\Omega_+)$, $z\boldsymbol{\nabla}\mathbf{u}_\alpha \in L^2(\Omega_+)$ and

$$|u_{\alpha 1}| + |u_{\alpha 2}| + |u_{\alpha 3}| \leq \frac{C}{z^2}, \qquad |\boldsymbol{\nabla} u_{\alpha 1}| + |\boldsymbol{\nabla} u_{\alpha 2}| + |\boldsymbol{\nabla} u_{\alpha 3}| \leq \frac{C}{z^3}. \tag{A.7}$$

Consequently, for any $\mathbf{w} \in \mathcal{D}$, we apply (A.4) so that

$$\begin{aligned}
&\left| \int_{\Omega_+} [(\mathbf{u}_\alpha + \mathbf{e}_1) \cdot \boldsymbol{\nabla}\mathbf{u}_\alpha] \cdot \mathbf{w} d\mathbf{x} \right| \\
\leq\ & C\left[1 + ||\mathbf{u}_\alpha; D||\right] ||\mathbf{w}; D|| \left[||z^2 \boldsymbol{\nabla}\mathbf{u}_\alpha||_{L^\infty(\Omega_+)} + ||z\boldsymbol{\nabla}\mathbf{u}_\alpha||_{L^2(\Omega_+)} \right].
\end{aligned}$$

Thus, all linear forms of $\mathbf{w}$ in (A.4) for $\mathbf{u}_\alpha$ are continuous on D, so that the weak formulation (3.3) for $\mathbf{u}_\alpha$ extends to the whole D by density. Then we obtain

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u}_\alpha : \boldsymbol{\nabla}\mathbf{u} d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}_\alpha + \mathbf{e}_1) \cdot \boldsymbol{\nabla}\mathbf{u}_\alpha] \cdot \mathbf{u} d\mathbf{x} = \int_{\Omega_+} \mathbf{f} \cdot \mathbf{u} d\mathbf{x}. \tag{A.8}$$

With similar arguments, we obtain, for any $(\mathbf{v}, \mathbf{w}) \in D^2$:

$$\begin{aligned}
\left| \int_{\Omega_+} [\mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{u}_\alpha] \cdot \mathbf{w} d\mathbf{x} \right| &= \left| \int_{\Omega_+} \left[\frac{\mathbf{v}}{z} \cdot z^2 \boldsymbol{\nabla}\mathbf{u}_\alpha \right] \cdot \frac{\mathbf{w}}{z} d\mathbf{x} \right| \\
&\leq K \left\| z^2 \boldsymbol{\nabla}\mathbf{u}_\alpha \right\|_{L^\infty(\Omega_+)} \|\mathbf{v}; D\| \, \|\mathbf{w}; D\| \\
&\leq K \left\| \hat{\mathbf{u}}_\alpha; \mathcal{B}^3_{\alpha,0} \right\| \|\mathbf{v}; D\| \, \|\mathbf{w}; D\|.
\end{aligned} \tag{A.9}$$

We proceed with a proposition on approximation of α-solutions by velocity field of compact support.

Proposition A.5 *Given $\mathbf{u}_\alpha := (u_{\alpha 1}, u_{\alpha 2}, u_{\alpha 3})$ an α-solution with $\alpha > 3$, there exists a sequence $(\mathbf{w}_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$, such that $\mathbf{w}_n = \mathbf{u}_\alpha$ in $B(0,n)$, $\mathbf{w}_n = \mathbf{0}$ outside $B(0,n) \cap \Omega_+$ and there exists a constant C depending on $\mathbf{u}_\alpha$ for which there holds for all $n \in \mathbb{N}$*

$$\begin{aligned}
&||\mathbf{u}_\alpha - \mathbf{w}_n||_{L^\infty(\Omega_+)} + ||z\boldsymbol{\nabla}(\mathbf{u}_{\boldsymbol{\alpha}} - \mathbf{w}_{\mathbf{n}})||_{L^2(\Omega_+)} \\
&+ ||z(\mathbf{u}_{\boldsymbol{\alpha}} - \mathbf{w}_{\mathbf{n}})||_{L^\infty(\Omega_+)} + ||z^2 \boldsymbol{\nabla}(\mathbf{u}_{\boldsymbol{\alpha}} - \mathbf{w}_{\mathbf{n}})||_{L^\infty(\Omega_+)} \\
&\leq C.
\end{aligned}$$

Proof: We notice that $\mathbf{u}_\alpha$ can be given in Proposition A.1 by the formula of $\mathbf{u}(\mathbf{x})$, $\psi(x)$ is well-defined and has L^∞ bound in an annular region. Thus we introduce

$$\zeta_n(x, y, z) = \zeta\left(\frac{|(x, y, z)|}{n} - 1\right), \qquad \mathbf{w}_n = \boldsymbol{\nabla} \times [\zeta_n \psi].$$

Obviously, $\mathbf{w}_n \in \mathcal{D}$ and $\mathbf{w}_n = \mathbf{u}_\alpha$ in $B(0,n)$. Moreover, there holds

$$\mathbf{w}_n - \mathbf{u}_\alpha = (\zeta_n - 1)\mathbf{u}_\alpha + \psi \boldsymbol{\nabla} \times \zeta_n.$$

Via a scaling argument, one can show that $||\boldsymbol{\nabla}\zeta_n||_{L^2(\Omega_+)}$ is uniformly bounded for $n \in \mathbb{N}$. This yields a uniform bound for $||\mathbf{u}_\alpha - \mathbf{w}_n||_{L^\infty(\Omega_+)}$:

$$||\mathbf{u}_\alpha - \mathbf{w}_n||_{L^2(\Omega_+)} \leq ||\mathbf{u}_\alpha||_{L^2(\Omega_+)} + C_\zeta ||\psi||_{L^\infty(\Omega_+)}.$$

Similarly, $||z\boldsymbol{\nabla}\zeta_n||_{L^\infty(\Omega_+)}$ is uniformly bounded for $n \in \mathbb{N}$. Consequently, there holds

$$|z\left(\mathbf{u}_\alpha - \mathbf{w}_n\right)| \leq |z\mathbf{u}_\alpha| + C_\zeta|\psi|.$$

Concerning the derivatives of $\mathbf{w}_n$ and $\mathbf{u}_\alpha$, we have

$$|\boldsymbol{\nabla}\left(\mathbf{u}_\alpha - \mathbf{w}_n\right)| \leq |\boldsymbol{\nabla}\mathbf{u}_\alpha| + C|\boldsymbol{\nabla}\zeta_n||\mathbf{u}_\alpha| + |\boldsymbol{\nabla}^2\zeta_n||\psi|.$$

Applying similar scaling technique as previously, one can prove

$$||\boldsymbol{\nabla}^2\zeta_n||_{L^2(\Omega_+)} \leq \frac{C_\eta}{n}, \quad |z\boldsymbol{\nabla}^2\zeta_n| \leq C_\zeta, \qquad \forall (x,y,z,n) \in \Omega_+ \times \mathbb{N}.$$

Introducing this control in the domination of $\boldsymbol{\nabla}\left(\mathbf{u}_\alpha - \mathbf{w}_n\right)$, we obtain

$$\begin{aligned} |\,|z\boldsymbol{\nabla}\left(\mathbf{u}_\alpha - \mathbf{w}_n\right)|\,|_{L^2(\Omega_+)} \leq{}& ||z\boldsymbol{\nabla}\mathbf{u}_\alpha||_{L^2(\Omega_+)} + C||\boldsymbol{\nabla}\zeta_n||_{L^2(\Omega_+)}||z\mathbf{u}_\alpha||_{L^\infty(\Omega_+)} \\ &+ ||z^2\boldsymbol{\nabla}^2\zeta_n||_{L^2(\Omega_+)}||\psi||_{L^\infty(\Omega_+)}, \end{aligned}$$

which yields a uniform bound with respect to n. Finally, we have

$$z^2|\boldsymbol{\nabla}\left(\mathbf{u}_\alpha - \mathbf{w}_n\right)| \leq |z^2\boldsymbol{\nabla}\mathbf{u}_\alpha| + C|z\boldsymbol{\nabla}\zeta_n||z\mathbf{u}_\alpha| + |z^2\boldsymbol{\nabla}^2\zeta_n||\psi|.$$

Consequently, the above pointwise control on $\boldsymbol{\nabla}\mathbf{u}_\alpha$ implies $\|z^2\boldsymbol{\nabla}\left(\mathbf{u}_\alpha - \mathbf{w}_n\right)\|_{L^\infty(\Omega_+)}$ is finite and remains uniformly bounded for $n \in \mathbb{N}$. This completes the proof. □

Combing Proposition A.3 and Proposition A.5 we are able to obtain

$$\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}_\alpha d\mathbf{x} + \int_{\Omega_+} \left[(\mathbf{u} + \mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}\right]\cdot\mathbf{u}_\alpha d\mathbf{x} = \int_{\Omega_+} \mathbf{f}\cdot\mathbf{u}_\alpha d\mathbf{x} \tag{A.10}$$

which implies that (3.3) for $\mathbf{u}$ extends to $\mathbf{w} = \mathbf{u}_\alpha$. Indeed, let $\mathbf{u} \in D$ be a weak solution for source term $\mathbf{f}$ and $\mathbf{u}_\alpha$ be an α-solution. Applying Proposition A.5, we have a sequence $(\mathbf{w}_n)_{n\in\mathbb{N}} \in \mathcal{D}^\mathbb{N}$ approximating $\mathbf{u}_\alpha$. As $\mathbf{w}_n \in \mathcal{D}$, (3.3) holds with $\mathbf{w}_n$. According to properties satisfied by $\mathbf{u}$ and $\mathbf{u}_\alpha$, we have

$$\left|\int_{\Omega_+} \boldsymbol{\nabla}\mathbf{u} : (\boldsymbol{\nabla}\mathbf{u}_{\boldsymbol{\alpha}} - \boldsymbol{\nabla}\mathbf{w}_\mathbf{n})d\mathbf{x}\right| \leq C(\mathbf{u}_\alpha)\left(\int_{\Omega_+\setminus B(0,n)} |\boldsymbol{\nabla}\mathbf{u}|^2\, d\mathbf{x}\right)^{1/2},$$

so that

$$\lim_{n\to\infty}\int_{\Omega_+}\boldsymbol{\nabla}\mathbf{u}:\boldsymbol{\nabla}\mathbf{w_n})d\mathbf{x}=\int_{\Omega_+}\boldsymbol{\nabla}\mathbf{u}:\boldsymbol{\nabla}\mathbf{u}_{\boldsymbol{\alpha}}d\mathbf{x}.$$

Concerning the trilinear form, we apply (A.3) again to obtain

$$\begin{aligned}&\left|\int_{\Omega_+}\left[(\mathbf{u}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}\right]\cdot\left(\mathbf{u}_{\boldsymbol{\alpha}}-\mathbf{w}_n\right)d\mathbf{x}\right|\\&\le C\left[1+||\mathbf{u};D||\right]||\boldsymbol{\nabla}\mathbf{u}||_{L^2(\Omega_+\setminus B(0,n))}\left[||\mathbf{u}_{\boldsymbol{\alpha}}-\mathbf{w}_n||_{L^2(\Omega_+)}+||z\left(\mathbf{u}_{\boldsymbol{\alpha}}-\mathbf{w}_n\right)||_{L^\infty(\Omega_+)}\right]\\&\le C(\mathbf{u}_\alpha)\left[1+||\mathbf{u};D||\right]||\boldsymbol{\nabla}\mathbf{u}||_{L^2(\Omega_+\setminus B(0,n))}\end{aligned}$$

Then passing to the limit in n, this yields

$$\lim_{n\to\infty}\int_{\Omega_+}\left[(\mathbf{u}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}\right]\cdot\mathbf{w}_nd\mathbf{x}=\int_{\Omega_+}\left[(\mathbf{u}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}\right]\cdot\mathbf{u}_{\boldsymbol{\alpha}}d\mathbf{x}.\qquad\text{(A.11)}$$

Since $\mathbf{f}$ has compact support in Ω_+, for n sufficiently large, there holds

$$\int_{\Omega_+}\mathbf{f}\cdot\mathbf{w}_nd\mathbf{x}=\int_{\Omega_+}\mathbf{f}\cdot\mathbf{u}_{\boldsymbol{\alpha}}d\mathbf{x}.$$

Taking the limit in (3.3) with $\mathbf{w}_n$, we obtain (3.3) with $\mathbf{u}_\alpha$.

Proposition A.6 *For any α-solution $\mathbf{u}_{\boldsymbol{\alpha}}$ there holds*

$$\int_{\Omega_+}\left[(\mathbf{v}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{w}\right]\cdot\mathbf{u}_{\boldsymbol{\alpha}}d\mathbf{x}=-\int_{\Omega_+}\left[(\mathbf{v}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}_{\boldsymbol{\alpha}}\right]\cdot\mathbf{w}d\mathbf{x},\qquad\forall(\mathbf{v},\mathbf{w})\in D^2$$

Proof: Performing similar argument as in (A.11), for any $(\mathbf{v},\mathbf{w})\in D^2$ and $\mathbf{u}_{\boldsymbol{\alpha}}$, we have

$$\lim_{n\to\infty}\int_{\Omega_+}\left[(\mathbf{v}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{w}\right]\cdot\mathbf{w}_nd\mathbf{x}=\int_{\Omega_+}\left[(\mathbf{v}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{w}\right]\cdot\mathbf{u}_{\boldsymbol{\alpha}}d\mathbf{x}$$

Using (A.4) we have

$$\begin{aligned}&\left|\int_{\Omega_+}\left[(\mathbf{v}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\left(\mathbf{u}_{\boldsymbol{\alpha}}-\mathbf{w}_n\right)\right]\cdot\mathbf{w}d\mathbf{x}\right|\\&\le C\left[1+||\mathbf{v};D||\right]||\mathbf{w}/z||_{L^2(\Omega_+\setminus B(0,n))}\left[||z^2\boldsymbol{\nabla}\left(\mathbf{u}_{\boldsymbol{\alpha}}-\mathbf{w}_n\right)||_{L^\infty(\Omega_+)}+||\mathbf{u}_{\boldsymbol{\alpha}}-\mathbf{w}_n;D||\right]\\&\le C(\mathbf{w})\left[1+||\mathbf{v};D||\right]||\mathbf{w}/z||_{L^2(\Omega_+\setminus B(0,n))}.\end{aligned}$$

The Hardy inequality implies that

$$\lim_{n\to\infty}\int_{\Omega_+}\left[(\mathbf{v}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{w}_n\right]\cdot\mathbf{w}d\mathbf{x}=\int_{\Omega_+}\left[(\mathbf{v}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{u}_{\boldsymbol{\alpha}}\right]\cdot\mathbf{w}d\mathbf{x}$$

As, for any fixed n, we have $\mathbf{w}_n \in \mathcal{D}$, the following identity holds true

$$\int_{\Omega_+} [(\mathbf{v}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{w}]\cdot\mathbf{w}_n d\mathbf{x} = -\int_{\Omega_+} [(\mathbf{v}+\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{w}_n]\cdot\mathbf{w} d\mathbf{x}$$

Finally, the same identity holds for $\mathbf{w}$. □

Acknowledgments Guo will express his gratitude to Professor Peter Wittwer for his hospitality during the stay of University of Geneva.

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