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DECAY ESTIMATES FOR LINEARIZED UNSTEADY INCOMPRESSIBLE VISCOUS FLOWS AROUND ROTATING AND TRANSLATING BODIES

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Abstract. We consider the time-dependent Oseen sytem with rotational terms. This system is a linearized model for the flow of a viscous incompressible fluid around a rigid body moving at a constant velocity and rotating with constant angular velocity. We present results on temporal and spatial decay of solutions to this system in the whole space. The spatial asymptotics we establish exhibit a wake.

1. Introduction

Consider the motion of a viscous incompressible fluid around a rigid body translating with constant velocity and rotating at constant angular velocity. Suppose the fluid flow is described with respect to a coordinate system in which the body is at rest. Then the flow in question is usually represented by a modified Navier-Stokes system which reads like this:

$$\partial_t v - \nu\,\Delta_z v + (v\cdot\nabla_z)v - (V+\omega\times z)\cdot\nabla_z v + \omega\times v + \nabla_z q = F, \quad \mathrm{div}_z v = 0 \tag{1.1}$$

in $(\mathbb{R}^3\backslash\overline{\mathfrak{D}})\times(0,T)$. Here $\mathfrak{D}\subset\mathbb{R}^3$ is a bounded domain representing the rigid body. The function v denotes the velocity field of the fluid, and the function q its pressure field. The vector V describes the constant translation of the body, and the vector ω its constant angular velocity. We suppose that V and ω are parallel. The function F stands for an exterior force exerted on the fluid, and the parameter $\nu\in(0,\infty)$ characterizes the viscosity of the fluid. By a suitable normalization and some changes of variables (see [16]), system (1.1) may be rewritten in the form

$$\partial_t u - \Delta_x u + \tau\,\partial_{x_1}u + \tau(u\cdot\nabla_x)u - (\varrho\, e_1\times x)\cdot\nabla_x u + \varrho\, e_1\times u + \nabla_x\sigma = f, \;\; \mathrm{div}_x u = 0, \tag{1.2}$$

where $\tau\in(0,\infty)$ is the Reynolds number and $\varrho\in\mathbb{R}\backslash\{0\}$ the Taylor number.

In recent years, many articles dealt with flows around a rotating body. As examples we mention [10] – [14], [17] – [18], [20] – [21]. In the present context, an article by Chen and Miyakava [1] is relevant. These authors proved existence of a global weak solution to (1.1) in the whole space $\mathbb{R}^n$ with $n=2$ and $n=3$, and derived algebraic decay rates (as $t\to\infty$) for the kinetic energy associated with this solution. They assumed $F=0$, $\nu=1$ but considered

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nonzero initial data and admitted the case that V and ω are functions depending on time, and need not be parallel. We will show results related to those in [1], but pertaining to the Oseen system with rotational terms, that is, to the following system obtained by dropping the nonlinearity in (1.2),

$$\partial_t u - \Delta_x u + \tau\, \partial_{x_1} u - (\varrho\, e_1 \times x) \cdot \nabla_x u + \varrho\, e_1 \times u + \nabla_x \sigma = f, \quad \operatorname{div}_x u = 0. \tag{1.3}$$

Under the assumption that f does not depend on time and decays in an appropriate way, we will study the asymptotics of $U(x)-u(x,t)$ and $\nabla_x\big(U(x)-u(x,t)\big)$ with respect to both the space variable x and the time variable t, where u is the velocity part of a solution to (1.3) with initial data zero, and U the velocity part of a solution to the stationary variant of (1.3), that is,

$$-\Delta U + \tau\, \partial_1 U - (\varrho\, e_1 \times x) \cdot \nabla_x U + \varrho\, e_1 \times U + \nabla \Pi = f, \quad \operatorname{div} U = 0. \tag{1.4}$$

The decay bounds we obtain for $U(x) - u(x,t)$ exhibit a wake. In addition, they imply optimal rates of spatial decay for $u(x,t)$ when $|x| \to \infty$. These rates are uniform with respect to t. Our estimates of $U(x)-u(x,t)$ further yield that $u(\,\cdot\,,t)$ converges to U with respect to a weighted $W^{1,\infty}$-norm, which we will denote by $\|\;\|_{1,\infty,w,\epsilon}$. The rate of this convergence is $t^{-\epsilon}$, where ϵ may be arbitrarily chosen in $(0,\,1/2)$ but enters into the definition of $\|\;\|_{1,\infty,w,\epsilon}$. This convergence result means in particular that U is unconditionally asymptotically stable with respect to the norm $\|\;\|_{1,\infty,w,\epsilon}$. For more details on our results we refer to Theorem 2, Corollary 1 and the comments in Section 4.

2. Notations, definitions and auxiliary results

If $A \subset \mathbb{R}^3$, we write A^c for the complement $\mathbb{R}^3 \backslash A$ of A. The symbol $|\;|$ denotes the Euclidean norm of $\mathbb{R}^3$ and also the length of a multiindex from $\mathbb{N}_0^3$, that is, $|\alpha| := \alpha_1+\alpha_2+\alpha_3$ for $\alpha \in \mathbb{N}_0^3$. The open ball centered at $x \in \mathbb{R}^3$ and with radius $r > 0$ is denoted by $B_r(x)$. If $x = 0$, we will write B_r instead of $B_r(0)$. Put $e_1 := (1,0,0)$. Let $x \times y$ denote the usual vector product of $x, y \in \mathbb{R}^3$.

The parameters $\tau \in (0,\infty)$ and $\varrho \in \mathbb{R}\backslash\{0\}$ will be kept fixed throughout. Put $s_\tau(x) := 1 + \tau\,(|x| - x_1)$ for $x \in \mathbb{R}^3$. Define the matrix $\Omega \in \mathbb{R}^{3\times 3}$ by

$$\Omega := \varrho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

so that $\varrho\, e_1 \times x = \Omega \cdot x$ for $x \in \mathbb{R}^3$. By the symbol $\mathfrak{C}$, we denote constants only depending on τ or ω. We write $\mathfrak{C}(\gamma_1, ..., \gamma_n)$ for constants that additionally depend on parameters γ_1, ..., $\gamma_n \in \mathbb{R}$, for some $n \in \mathbb{N}$.

For $p \in [1,\infty)$ and for open sets $A \subset \mathbb{R}^3$, we write $W^{1,p}(A)$ for the usual Sobolev space of order 1 and exponent p. If $B \subset \mathbb{R}^3$ is open, define $W^{1,p}_{loc}(B)$ as the set of all functions $g : B \mapsto \mathbb{R}$ such that $g|U \in W^{1,p}(U)$ for any open bounded set $U \subset \mathbb{R}^3$ with $\overline{U} \subset B$. If V is a normed space whose norm is denoted by $\|\;\|_V$, and if $n \in \mathbb{N}$, we equip the product space V^n with a norm $\|\;\|_V^{(n)}$ defined by $\|v\|_V^{(n)} := \Big(\sum_{j=1}^n \|v_j\|_V^2\Big)^{1/2}$ for $v \in V^n$. But for simplicity, we will write $\|\;\|_V$ instead of $\|\;\|_V^{(n)}$.

Let K denote the usual fundamental solution to the heat equation,

$$K(x,t) = (4\pi t)^{-3/2} e^{-|x|^2/(4t)} \text{ for } x \in \mathbb{R}^3,\ t \in (0,\infty). \tag{2.1}$$

Recall that the Kummer function ${}_1F_1(1, \cdot, \cdot)$ is given by

$$ {}_1F_1(1, c, u) := \sum_{n=0}^{\infty} (\Gamma(c)/\Gamma(u+c))u^n \text{ for all } u \in \mathbb{R}, c \in (0, \infty), $$

where Γ denotes the usual Gamma function. We put

$$ H_{jk}(x) := x_j x_k |x|^{-2} \text{ for } x \in \mathbb{R}^3 \setminus \{0\}, $$
$$ \Lambda_{jk}(x,t) := K(x,t)(\delta_{jk} - H_{jk}(x) - {}_1F_1(1, 5/2, |x|^2/(4t)))(\delta_{jk}/3 - H_{jk}(x)) $$

for $x \in \mathbb{R}^3 \setminus \{0\}$, $t \in (0,\infty)$, $j,k \in \{1,2,3\}$. In what follows, the letter Γ will always stand for a matrix-valued function defined by

$$ (\Gamma_{jk}(y,z,t))_{1 \le j,k \le 3} := (\Lambda_{rs}(y - \tau t e_1 - e^{-t\Omega} \cdot z, t))_{1 \le r,s \le 3} \cdot e^{-t\Omega} $$

for $y, z \in \mathbb{R}^3$, $t \in (0,\infty)$ with $y - \tau t e_1 - e^{-t\Omega} \cdot z \neq 0$.

This function is the velocity part of the fundamental solution to (1.3) introduced by Guenther, Thomann [22]. Our following lemma restates [3, Corollary 3.1].

Lemma 1. *The function Γ_{jk} may be extended continuously to a function belonging to $C^\infty\big(\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)\big)$.*

We will use the ensuing technical lemmas:

Lemma 2. *(see [4, Lemma 2.9]) Let $x \in \mathbb{R}^3$, $t \in \mathbb{R}$. Then*

$$ |e^{t\Omega} \cdot x| = |x|, \ (e^{t\Omega} \cdot x)_1 = x_1, \ e^{t\Omega} \cdot e_1 = e_1. $$

Lemma 3. *(see [2, Lemma 4.8]) For $x, y \in \mathbb{R}^3$ we have*

$$ s_\tau(x-y)^{-1} \le \mathfrak{C}\,(1+|y|) s_\tau(x)^{-1}. $$

Lemma 4. *(see [9, Lemma 4.3]) Let $\beta \in (1,\infty)$. Then*

$$ \int_{\partial B_r} s_\tau(x)^{-\beta} do_x \le \mathfrak{C}(\beta) r \text{ for } r \in (0,\infty). $$

Lemma 5. *(see [4], Lemma 2.4) Let $S \in (0,\infty)$, $x \in B_S^c$. Then*

$$ |x| \ge \mathfrak{C}(S)\, s_\tau(x). $$

Lemma 6. *(see [3, Lemma 3.2])*

$$ |\partial_y^\beta \Gamma_{jk}(y,z,t)| + |\partial_z^\beta \Gamma_{jk}(y,z,t)| \le \mathfrak{C}(|y - \tau t e_1 - e^{t\Omega} \cdot z|^2 + t)^{-3/2-|\beta|/2} \tag{2.2} $$

for $y, z \in \mathbb{R}^3$, $t \in (0,\infty)$, $\beta \in \mathbb{N}_0^3$ with $|\beta| \le 1$.

Lemma 7. *(see [3, Theorem 3.1]) Let $k \in \{0,1\}$, $R \in (0,\infty)$, $y, z \in B_R$ with $y \neq z$. Then*

$$ \int_0^\infty (|y - \tau t e_1 - e^{-t\Omega} \cdot z|^2 + t)^{-3/2-k/2} dt \le \mathfrak{C}(R)\, |y-z|^{-1-k}. \tag{2.3} $$

Due to the preceding lemma and by (2.2), we may define

$$ \mathfrak{Z}_{jk}(y,z,T) := \int_T^\infty \Gamma_{jk}(y,z,t) dt $$

for $T \in [0,\infty)$, $y, z \in \mathbb{R}^3$ with $y \neq z$, $1 \le j,k \le 3$. The function $\mathfrak{Z}(\cdot, \cdot, 0)$ is the velocity part of the fundamental solution of (1.3) proposed by Guenther, Thomann [22].

Lemma 8. *Let $j,k \in \{1,2,3\}$, $T \in [0,\infty)$. Then $\mathfrak{Z}_{jk}(\,\cdot\,,\,\cdot\,,T) \in C^1\big((\mathbb{R}^3 \times \mathbb{R}^3) \setminus \{(x,x) : x \in \mathbb{R}^3\}\big)$, and*

$$(2.4)\ \partial_{y_n}\mathfrak{Z}_{jk}(y,z,T) = \int_T^\infty \partial_{y_n}\Gamma_{jk}(y,z,t)dt,\ \partial_{z_n}\mathfrak{Z}_{jk}(y,z,T) = \int_T^\infty \partial z_n\Gamma_{jk}(y,z,t)dt$$

for $y,z \in \mathbb{R}^3$ with $y \neq z$, $n \in \{1,2,3\}$. If $R \in (0,\infty)$, $y,z \in B_R$ with $y \neq z$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, we have

$$|\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)| + |\partial_z^\alpha \mathfrak{Z}_{jk}(y,z,T)| \leq \mathfrak{C}(R)\,|y-z|^{-1-|\alpha|}. \tag{2.5}$$

Proof. The proof of [4, Lemma 2.15] carries over to the present situation ($T \in [0,\infty)$ instead of $T = 0$). Note that (2.5) follows from (2.4) and (2.3). ☐

Theorem 1. *Let $S,\delta \in (0,\infty)$, $\nu \in (1,\infty)$, $T \in (0,\infty)$ and $0 \leq \epsilon < \nu - 1$, or $T = 0$ and $\epsilon = 0$. Then*

$$\int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu}dt \leq \mathfrak{C}(S,\delta,\epsilon,\nu)T^{-\epsilon}(|y|s_\tau(y))^{-\nu+\epsilon+1/2} \tag{2.6}$$

for $y \in B^c_{(1+\delta)S}$, $z \in \overline{B_S}$, and

$$\int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu}dt \leq \mathfrak{C}(S,\delta,\epsilon,\nu)T^{-\epsilon}(|z|s_\tau(z))^{-\nu+\epsilon+1/2} \tag{2.7}$$

for $z \in B^c_{(1+\delta)S}$, $y \in \overline{B_S}$. Moreover,

$$|\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)| + |\partial_z^\alpha \mathfrak{Z}_{jk}(y,z,T)| \leq \mathfrak{C}(s,\delta,\epsilon)\,T^{-\epsilon}(|y|s_\tau(y))^{-1-|\alpha|/2+\epsilon} \tag{2.8}$$

for $j,k \in \{1,2,3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $y \in B^c_{(1+\delta)S}$, $z \in \overline{B_S}$,

$$|\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)| + |\partial_z^\alpha \mathfrak{Z}_{jk}(y,z,T)| \leq \mathfrak{C}(s,\delta,\epsilon)\,T^{-\epsilon}(|z|s_\tau(z))^{-1-|\alpha|/2+\epsilon} \tag{2.9}$$

for $y \in \overline{B_S}$, $z \in B^c_{(1+\delta)S}$ and j,k,α as above.

Proof. In the case $T = 0$, $\epsilon = 0$, Theorem 1 restates [4, Theorem 2.19]. Now suppose that $T > 0$ and $0 \leq \epsilon < \nu - 1$. Then inequalities (2.6) and (2.7) may be reduced to the preceding reference. In fact, take y,z as in (2.6) or (2.7). Then

$$\int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu}dt \leq \int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu+\epsilon}\,t^{-\epsilon}dt$$
$$\leq T^{-\epsilon}\int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu+\epsilon}dt \leq T^{-\epsilon}\int_0^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu+\epsilon}dt.$$

Since $-\nu + \epsilon < -1$, we may now use [4, Theorem 2.19] with ν replaced by $\nu - \epsilon$, obtaining (2.6) and (2.7). The estimates in (2.8) and (2.9) follow from (2.2), (2.4), (2.6) and (2.7). ☐

3. Volume potentials

We will study volume potentials involving the kernel $\mathfrak{Z}(\,\cdot\,,T)$.

Lemma 9. *Let $p \in (1,\infty)$, $q \in (1,2)$, $f \in L^p_{\mathrm{loc}}(\mathbb{R}^3)^3$ with $f|B^c_S \in L^q(B^c_S)^3$ for some $S \in (0,\infty)$. Then, for $j,k \in \{1,2,3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, we have*

$$\int_{\mathbb{R}^3}\int_0^\infty |\partial_y^\alpha \Gamma(y,z,t)|\,dt\,|f_k(z)|dz < \infty \text{ for a.e. } y \in \mathbb{R}^3. \tag{3.1}$$

Let $T \in [0,\infty)$. We define $\mathfrak{R}(f)(\,\cdot\,,T) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by putting

$$\mathfrak{R}_j(f)(y,T) := \int_{\mathbb{R}^3} \sum_{k=1}^{3} \int_T^\infty \Gamma_{jk}(y,z,t)dt\, f_k(z)dz = \int_{\mathbb{R}^3} \sum_{k=1}^{3} \mathfrak{Z}_{jk}(y,z,T)\, f_k(z)dz$$

for $y \in \mathbb{R}^3$ such that (3.1) holds; otherwise we set $\mathfrak{R}_j(t)(y,T) := 0$ $(1 \le j \le 3)$. Then $\mathfrak{R}(f)(\,\cdot\,,T) \in W^{1,1}_{\text{loc}}(\mathbb{R}^3)^3$ and

$$\partial_l \mathfrak{R}_j(f)(y,T) = \int_{\mathbb{R}^3} \sum_{k=1}^{3} \partial_{y_l} \mathfrak{Z}_{jk}(y,z,T)\, f_k(z)dz \tag{3.2}$$

for $j,l \in \{1,2,3\}$ and for a.e. $y \in \mathbb{R}^3$. Moreover, if $f \in L^1(\mathbb{R}^3)^3$, $T > 0$, we have

$$|\partial^\alpha \mathfrak{R}(f)(y,T)| \le \mathfrak{C}\, T^{-\frac{1}{2}-\frac{|\alpha|}{2}} \|f\|_1 \quad \text{for } \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \le 1,\ y \in \mathbb{R}^3. \tag{3.3}$$

Proof. In view of (2.8) and (2.9) with $\epsilon = 0$, and due to Lemma 8, all the statements of Lemma 9 except (3.3) may be proved in exactly the same way, without any modification, as analogous statements in [4, Lemma 3.1]. As for (3.3), we use (2.2) to obtain for $y \in \mathbb{R}^3$, $1 \le j,k \le 3$ that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \int_T^\infty |\partial_y^\alpha \Gamma_{jk}(y,z,t)|dt\, |f(z)|\, dz \\
&\le \mathfrak{C} \int_{\mathbb{R}^3} \int_T^\infty (|y - \tau t e_1 - e^{-t\Omega} \cdot z|^2 + t)^{-3/2-|\alpha|/2} dt\, |f(z)|\, dz \\
&\le \mathfrak{C} \int_{\mathbb{R}^3} \int_T^\infty t^{-3/2-|\alpha|/2} dt\, |f(z)|\, dz \le \mathfrak{C}\, T^{-1/2-|\alpha|/2} \|f\|_1.
\end{aligned}$$

Inequality (3.3) now follows with (3.2) and (2.4). □

Theorem 2. *Let $T \in (0,\infty)$, $S, S_1, \gamma \in (0,\infty)$ with $S_1 < S$, $p \in (1,\infty)$, $A \in [2,\infty)$, $B \in \mathbb{R}$, $0 \le \epsilon < 1/2 + |\alpha|/2$, $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ measurable with*

$$f|B_{S_1} \in L^p(B_{S_1})^3, \quad |f(z)| \le \gamma |z|^{-A} s_\tau(z)^{-B} \text{ for } z \in B^c_{S_1}, \quad A + \min\{1,B\} \ge 3.$$

Let $i,j \in \{1,2,3\}$, $y \in B^c_S$. Then

$$|\mathfrak{R}_j(f)(y,T)| \le \mathfrak{C}(S,S_1,A,B,\epsilon) T^{-\epsilon} (\|f|B_{S_1}\|_1 + \gamma) \big(|y|\, s_\tau(y)\big)^{-1+\epsilon} l_{A,B}(y), \tag{3.4}$$

$$|\partial_{y_i} \mathfrak{R}_j(f)(y,T)| \le \mathfrak{C}(S,S_1,A,B,\epsilon) T^{-\epsilon} (\|f|B_{S_1}\|_1 + \gamma) \big(|y|\, s_\tau(y)\big)^{-3/2+\epsilon} l_{A,B}(y)\, s_\tau(y)^{\max(0,\,7/2-A-B-2\epsilon)}, \tag{3.5}$$

where

$$l_{A,B}(y) = \begin{cases} 1 & \text{if } A + \min\{1,B\} > 3 \\ \max(1, \ln|y|) & \text{if } A + \min\{1,B\} = 3 \end{cases}$$

Proof. We modify the proof of [4, Theorem 3.1]. Since $A \ge 2$, we have $f|B^c_{S_1} \in L^q(B^c_{S_1})^3$ for any $q \in (3/2,\infty)$. But $f|B_{S_1} \in L^p(B_{S_1})^3$, so we get, say, $f \in L^{\min\{p,2\}}_{loc}(\mathbb{R}^3)^3$. Therefore f satisfies the assumptions of Lemma 9, hence $\mathfrak{R}(f)(\,\cdot\,,T)$ is well defined, belongs to $W^{1,1}_{loc}(\mathbb{R}^3)^3$ and verifies (3.2).

By (2.8), we find for $k \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$ that

$$\int_{B_{S_1}} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)|\,|f(z)|\,dz \le \mathfrak{C}(S,S_1,\epsilon)T^{-\epsilon}\big(\,|y|\,s_\tau(y)\,\big)^{-1-|\alpha|/2+\epsilon}\|f|B_{S_1}\|_1. \tag{3.6}$$

We further get with (2.4), (2.2), a change of variables and Lemma 2 that

$$\begin{aligned}
\mathfrak{A}_\alpha &:= \int_{B_{S_1}^c} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)|\,|f(z)|\,dz \\
&\le \mathfrak{C}\gamma \int_T^\infty \int_{B_{S_1}^c} (|y-\tau t e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-3/2-|\alpha|/2}|z|^{-A}s_\tau(z)^{-B}\,dz\,dt \\
&= \mathfrak{C}\gamma \int_T^\infty \int_{B_{S_1}^c} (|y-\tau t e_1 - x|^2 + t)^{-3/2-|\alpha|/2}|x|^{-A}s_\tau(e^{t\Omega}\cdot x)^{-B}\,dx\,dt \\
&= \mathfrak{C}\gamma T^{-\epsilon} \int_{B_{S_1}^c}\int_T^\infty (|y-\tau t e_1 - x|^2 + t)^{-3/2-|\alpha|/2+\epsilon}\,dt\,|x|^{-A}s_\tau(x)^{-B}\,dx.
\end{aligned} \tag{3.7}$$

The preceding integral over $B_{S_1}^c$ is split into a sum of integrals over $B_{S_1}^c \cap B_{S/2}(y)$ and $B_{S_1}^c \backslash B_{S/2}(y)$, respectively. In order to estimate the integral over $B_{S_1}^c \cap B_{S/2}(y)$, we observe that for $x \in \mathbb{R}^3$, the term $(|y-\tau t e_1-x|^2+t)^{-3/2-|\alpha|/2+\epsilon}$ is bounded by $(|y-\tau t e_1-x|^2+t)^{-2}$ if $|y-\tau t e_1 - x|^2 + t \le 1$. Else it may be bounded by $\min\{1,\, t^{-3/2-|\alpha|/2+\epsilon}\}$. Thus we get by (2.3) with $y-z$ in the place of y and with $z=0$ that

$$\begin{aligned}
&\int_{B_{S_1}^c \cap B_{S/2}(y)}\int_T^\infty (|y-\tau t e_1 - x|^2 + t)^{-3/2-|\alpha|/2+\epsilon}\,dt\,|x|^{-A}s_\tau(x)^{-B}\,dx \\
&\le \int_{B_{S_1}^c \cap B_{S/2}(y)}\int_T^\infty \big(\,(|y-\tau t e_1 - x|^2 + t)^{-2} + \min\{1,\, t^{-3/2-|\alpha|/2+\epsilon}\}\,\big)\,dt \\
&\qquad\qquad |x|^{-A}s_\tau(x)^{-B}\,dx \\
&\le \mathfrak{C}(S)\int_{B_{S_1}^c \cap B_{S/2}(y)}\Big(\,|y-x|^{-2} + \int_0^\infty \min\{1,\, t^{-3/2-|\alpha|/2+\epsilon}\}\,)\,dt\Big)\,|x|^{-A}s_\tau(x)^{-B}\,dx \\
&\le \mathfrak{C}(S,\epsilon)\int_{B_{S_1}^c \cap B_{S/2}(y)} (|y-x|^{-2}+1)\,|x|^{-A}s_\tau(x)^{-B}\,dx,
\end{aligned}$$

where we used the assumption $\epsilon < 1/2 + |\alpha|/2$ in the last inequality. On the other hand, we apply (2.6) with y, ν replaced by $y-x$, $-3/2-|\alpha|/2+\epsilon$, respectively, and with $z=0$, $\epsilon=0$, to obtain

$$\int_T^\infty (|y-\tau t e_1 - x|^2 + t)^{-3/2-|\alpha|/2+\epsilon}\,dt \le \mathfrak{C}(S,\epsilon)\big(\,|y-x|\,s_\tau(y-x)\,\big)^{-1-|\alpha|/2+\epsilon}$$

for $x \in B_{S_1}^c \backslash B_{S/2}(y)$. Here the assumption $\epsilon < 1/2 + |\alpha|/2$ is again relevant. Now we may deduce from (3.7),

$$\begin{aligned}
\mathfrak{A}_\alpha \le \mathfrak{C}(S,\epsilon)\gamma T^{-\epsilon}\Big(&\int_{B_{S_1}^c \cap B_{S/2}(y)} (|y-x|^{-2}+1)\,|x|^{-A}s_\tau(x)^{-B}\,dx \\
&+ \int_{B_{S_1}^c \backslash B_{S/2}(y)} \big(\,|y-x|\,s_\tau(y-x)\,\big)^{-1-|\alpha|/2+\epsilon}|x|^{-A}s_\tau(x)^{-B}\,dx\Big).
\end{aligned} \tag{3.8}$$

Next we observe that for $x \in B_{S/2}(y)$, we have $|x| \geq |y| - |y-x| \geq |y| - S/2 \geq |y|/2$, and by Lemma 3,

$$s_\tau(x)^{-1} \leq \mathfrak{C}(1+|y-x|)\, s_\tau(y)^{-1} \leq \mathfrak{C}(S)\, s_\tau(y)^{-1}.$$

For $x \in B_{S/2}(y)^c$, we find

$$\begin{aligned}|y-x| &= |y-x|/2 + |y-x|/2 \geq S/4 + |y-x|/2 \\ &\geq \min\{S/4,\, 1/2\}\,(1+|y-x|),\end{aligned}$$

and for $x \in B^c_{S_1}$ we get $|x| \geq \mathfrak{C}(S_1)\,(1+|x|)$. Therefore from (3.8),

$$\begin{aligned}\mathfrak{A}_\alpha &\leq \mathfrak{C}(S,S_1,A,B,\epsilon)\, T^{-\epsilon}\gamma\Big(|y|^{-A}\, s_\tau(y)^{-B} \int_{B_{S/2}(y)} (|y-x|^{-2}+1)\,dx \\ &\quad + \int_{B^c_{S_1}\setminus B_{S/2}(y)} \big(\,(1+|y-x|)\, s_\tau(y-x)\,\big)^{-1-|\alpha|/2+\epsilon} (1+|x|)^{-A}\, s_\tau(x)^{-B}\; dx\Big) \\ &\leq \mathfrak{C}(S,S_1,A,B,\epsilon)\,\gamma T^{-\epsilon}\Big(|y|^{-A}\, s_\tau(y)^{-B} \\ &\quad + \int_{\mathbb{R}^3} \big(\,(1+|y-x|)\, s_\tau(y-x)\,\big)^{-1-|\alpha|/2+\epsilon} (1+|x|)^{-A}\, s_\tau(x)^{-B}\; dx\Big).\end{aligned} \tag{3.9}$$

By Lemma 5 and because $y \in B^c_S$, $A - 3/2 > 0$, $A + B \geq A + \min\{1, B\} \geq 3$, we further observe that

$$|y|^{-A}\, s_\tau(y)^{-B} \leq \mathfrak{C}(S,A)\, |y|^{-3/2}\, s_\tau(y)^{-A+3/2-B} \leq \mathfrak{C}(S,A)\, |y|^{-3/2}\, s_\tau(y)^{-3/2}. \tag{3.10}$$

Moreover, by the proof of [19, Theorem 3.1] we get

$$\int_{\mathbb{R}^3} \big(\,(1+|y-x|)\, s_\tau(y-x)\,\big)^{-1+\epsilon} (1+|x|)^{-A}\, s_\tau(x)^{-B}\; dx \leq \mathfrak{C}(\epsilon,A,B)\big(\,|y|\, s_\tau(y)\,\big)^{-1+\epsilon}\, l_{A,B}(y).$$

Similarly, the proof of [19, Theorem 3.2] yields

$$\begin{aligned}&\int_{\mathbb{R}^3} \big(\,(1+|y-x|)\, s_\tau(y-x)\,\big)^{-3/2+\epsilon} (1+|x|)^{-A}\, s_\tau(x)^{-B}\; dx \\ &\leq \mathfrak{C}(\epsilon,A,B)\,\big(\,|y|\, s_\tau(y)\,\big)^{-3/2+\epsilon}\, l_{A,B}(y)\, s_\tau(y)^{\max(0,\,7/2-A-B-2\epsilon)}.\end{aligned}$$

The two preceding estimates together with (3.6), (3.9) and (3.10) imply (3.4) and (3.5). □

Corollary 1. *Consider the situation of Theorem 2. As additional assumptions, suppose that* $A + \min\{1, B\} > 3$, $T > 0$ *and* $\epsilon < 1/2$. *Then* $f \in L^1(\mathbb{R}^3)^3$.

For $v \in W^{1,1}_{loc}(\mathbb{R}^3)^3$, *define*

$$\begin{aligned}\|v\|_{1,\infty,w,\epsilon} &:= \sup\{|v(x)|\,[(1+|x|)\, s_\tau(x)]^{1-\epsilon} \,:\, x \in \mathbb{R}^3\} \\ &\quad + \sup\{|\nabla v(x)|\,[(1+|x|)\, s_\tau(x)]^{3/2-\epsilon}\, s_\tau(x)^{-\max(0,\,7/2-A-B-2\epsilon)} \,:\, x \in \mathbb{R}^3\}.\end{aligned}$$

Then

$$\|\mathfrak{R}(f)(\,\cdot\,,T)\|_{1,\infty,w,\epsilon} \leq \mathfrak{C}(S,S_1,A,B,\epsilon)\,(\|f\|_1 + \gamma)\, \max\{T^{-\epsilon}, T^{-1}\}.$$

Proof. Put $B^* := \min\{1, B\}$, $\delta := \min\{(A-2)/2,\, (A+B^*-3)/2\}$. The assumption $A + B^* > 3$ implies $A > 2$, so $\delta > 0$ and $-A + 2 - \delta < 0$. Thus we get with Lemma 5 that

$$|x|^{-A} s_\tau(x)^{-B} = |x|^{-2-\delta}|x|^{-A+2+\delta} s_\tau(x)^{-B} \leq \mathfrak{C}(S_1,A)\, |x|^{-2-\delta} s_\tau(x)^{-A-B+2+\delta} \tag{3.11}$$

for $x \in B^c_{S_1}$. We further observe that $-A-B+2+\delta \le -A-B^*+2+\delta < -1$, where the last inequality follows from the choice of δ and the assumption $A+B^* > 3$. Now Lemma 4 and (3.11) yield $\int_{B^c_{S_1}} |f|\,dx < \infty$, so we may conclude $f \in L^1(\mathbb{R}^3)^3$ in view of the assumption $f|B_{S_1} \in L^p(B_{S_1})^3$. At this point (3.3) implies

$$|\partial_y^\alpha \mathfrak{R}(f)(y,T)| \le \mathfrak{C}(S_1)\, T^{-1/2-|\alpha|/2} \|f\|_1 \quad \text{for } y \in B_{S_1},\ \alpha \in \mathbb{N}_0 \text{ with } |\alpha| \le 1. \tag{3.12}$$

Obviously $1 \ge \mathfrak{C}(S_1)\,(1+|y|)\, s_\tau(y)$ for $y \in B_{S_1}$ and $|y| \ge \mathfrak{C}(S_1)\,(1+|y|)$ for $y \in B^c_{S_1}$. Thus Corollary 1 follows from Theorem 2 (in the case $|y| \ge 1$) and inequality (3.12) (in the case $|y| \le 1$). □

4. Comments

Let $f \in C_0^\infty(\mathbb{R}^3)^3$. It is implicit in the proof of [3, Theorem 4.2] that the function $U := \mathfrak{R}(f)(\,\cdot\,,0)$ is the velocity part of a classical solution to the stationary problem (1.4) in the whole space $\mathbb{R}^3$. On the other hand, according to [22, Theorem 1.2], the velocity part u of a solution to (1.3) in $\mathbb{R}^3 \times (0,\infty)$ with initial data zero is given by $u(x,t) := \int_0^t \int_{\mathbb{R}^3} \Gamma(x,z,t-s)\, f(z)\, dz\, ds$.

But $U - u(\,\cdot\,,T) = \mathfrak{R}(f)(\,\cdot\,,T)$ for $T > 0$, so Theorem 2 yields a decay estimate of $U(x) - u(x,t)$ with respect to the space variable x and the time variable t. In addition, the function $\mathfrak{R}(f)(\,\cdot\,,0)$ is known to satisfy all the statements of Theorem 2 with $\epsilon = 0$ ([4, Theorem 3.1]). Therefore these statements with $\epsilon = 0$ carry over to $u(\,\cdot\,,t)$, yielding pointwise spatial decay estimates of $u(\,\cdot\,,t)$ which are uniform with respect to $t \in (0,\infty)$. These estimates are optimal in the sense that the fundamental solution of the stationary Oseen system (without rotational terms) decays with those same rates ([19]). The powers of s_τ appearing in the estimates stated in Theorem 2 should be considered as a mathematical manifestation of the wake extending behind a body which moves in a viscous incompressible fluid.

Corollary 1 means that $U - u(\,\cdot\,,t)$ converges to zero for $t \to \infty$ with respect to the weighted $W^{1,\infty}$-norm $\|\ \|_{1,\infty,w,\epsilon}$. As already mentioned in Section 1, this convergence result means in particular that U is unconditionally asymptotically stable with respect to this norm. The notion of stability which we refer to here is the one introduced in [15, Definition 5.2] in a Hilbert space setting. Obviously it may also be used in the context of Banach spaces.

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