Computer-Assisted Proofs in Analysis

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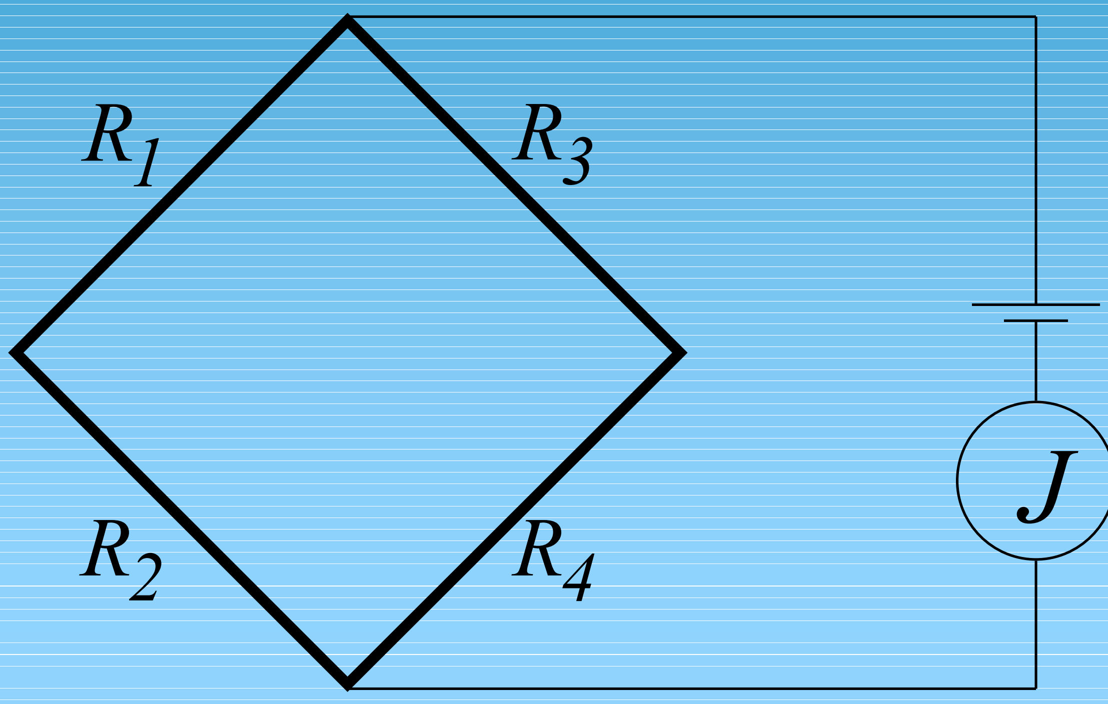
Overview

- Example - Theorem
- History
- What kind of problems?
- How to proceed
- Contraction mapping principle: the need for set-maps
- Decomposition into “Building Blocks”
- Bounds = set-maps
- Standard sets
- Bound on convolution
- Bound on derivative
- Summary

Example



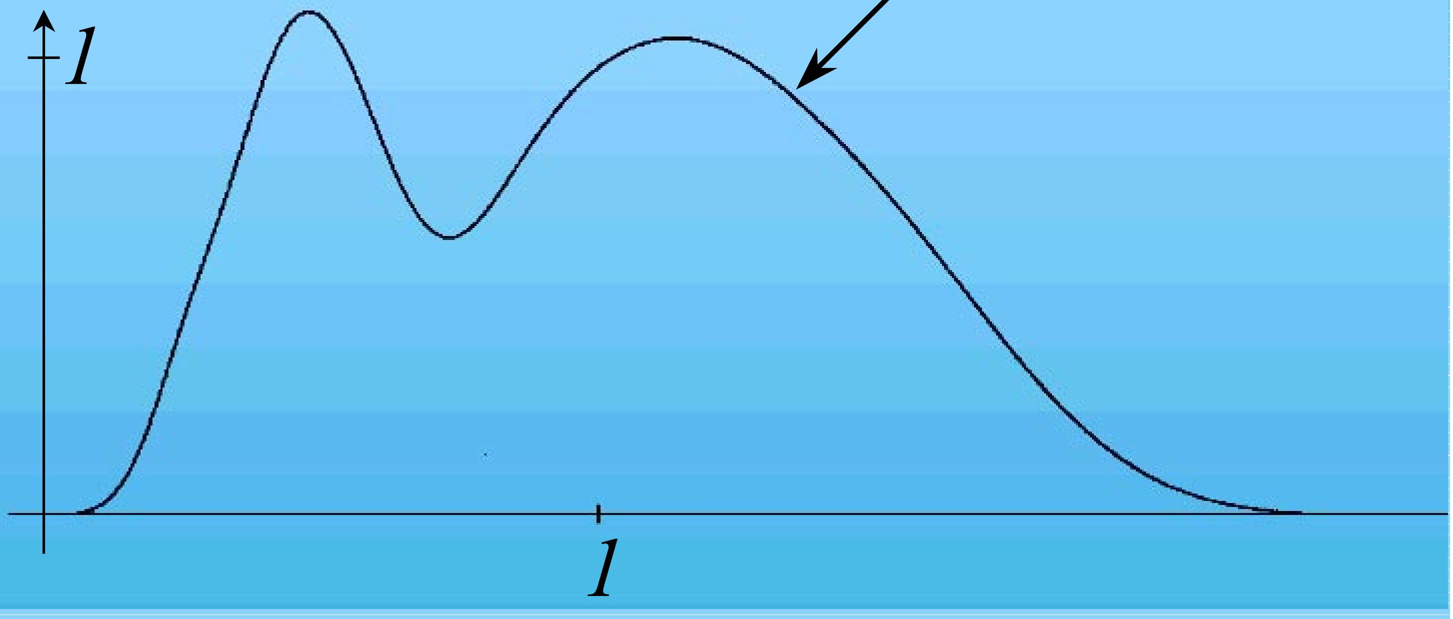
<u>Kirchhoff ´s laws:</u>

$$\mu R = \frac{1}{\frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4}}$$

Theorem

$$\exists \quad \mu \in [1.756203, 1.756205]$$

and a function g



such that $f(x)dx$

$$f(x) = a \cdot \delta_\infty(x) + (1-a)g(x)$$

$$a = (\sqrt{5} - 1)/2$$

solution of the problem.

History

1982 O.E. Lanford III

Analytic Functions

:

1996 H.Koch, A.Schenkel, P.W.

1996 A.Schenkel (Ph.D. thesis)

1997 . . .

Any Type of Functions

What Kind of Problems?

Prove existence of solutions for:

$$N(f) = f$$

$f \in B$ Banach space

$DN(f)$ reasonable

C.M.P. can be put into practice near the solution

How to Proceed

- f_0 approximate solution

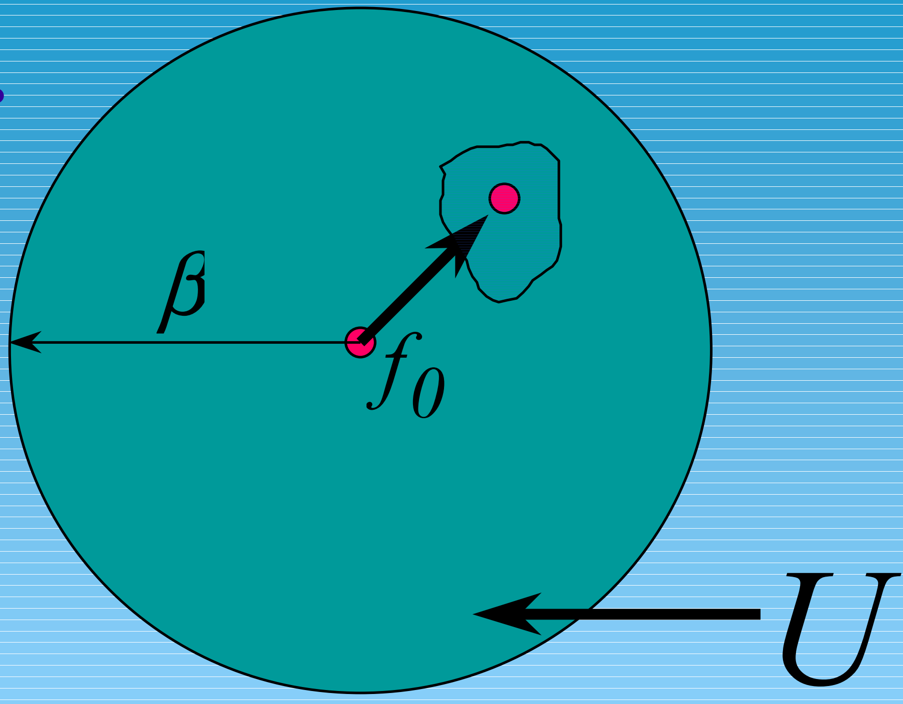
- $M(f) = f - L(N(f) - f)$

$$L \approx (DN(f_0) - 1)^{-1}$$

$$DM(f) \approx 0$$

- hypothesis of the C.M.P. satisfied

C.M.P.



$$\| M(f_0) - f_0 \| \le \varepsilon$$

$$\varepsilon < (1-\rho)\beta$$

$$\| DM(f)h \| \le \rho < 1$$

$$\forall \quad f \in U, \ \|h\| \le 1$$

Compute with Sets

"Building Blocks"

- Decompose

$$M = M_1 \circ M_2 \cdots$$

- Bound "Factors"

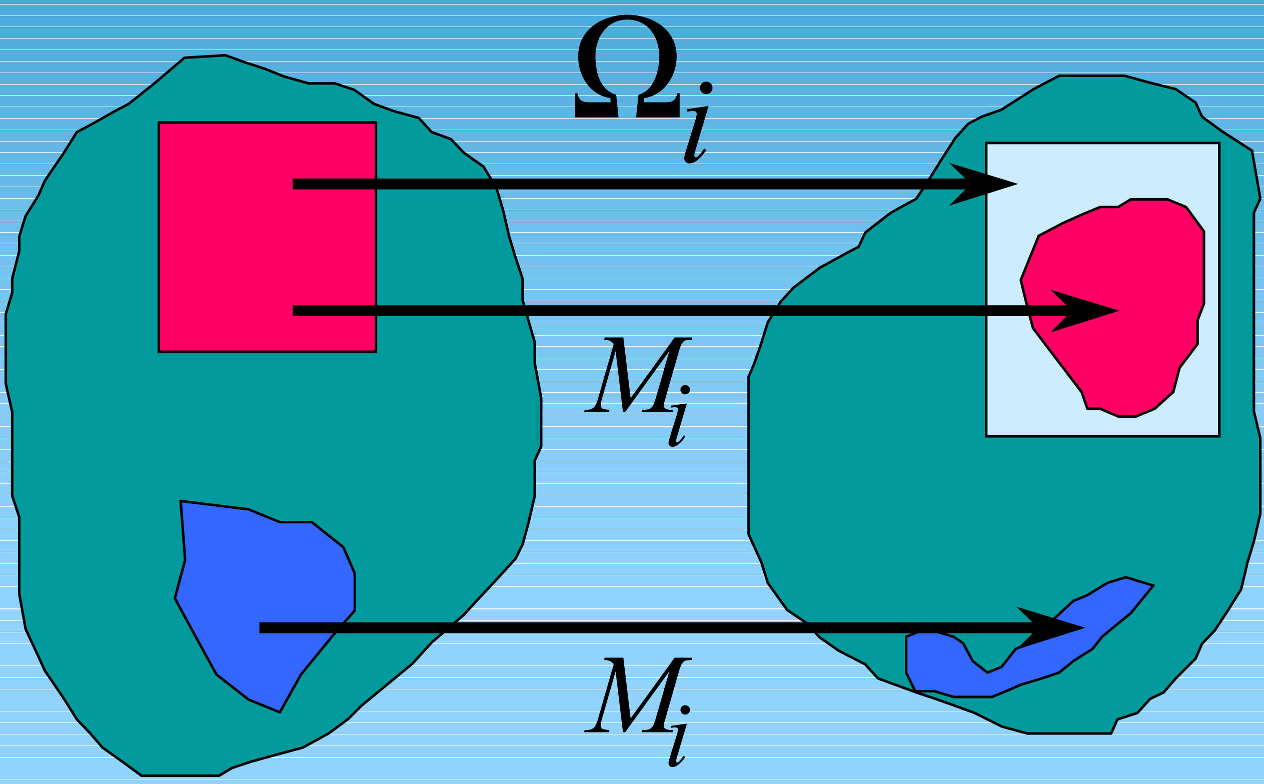
$$M_i, \quad DM_i$$

- Reassemble

$$M = M_1 \circ M_2 \cdots$$

$$DM = \text{apply chain rule}$$

Bound = Set-Map



Def.: $\Omega_i \geq M_i$

$$D(\Omega_i) \subseteq D(M_i)$$

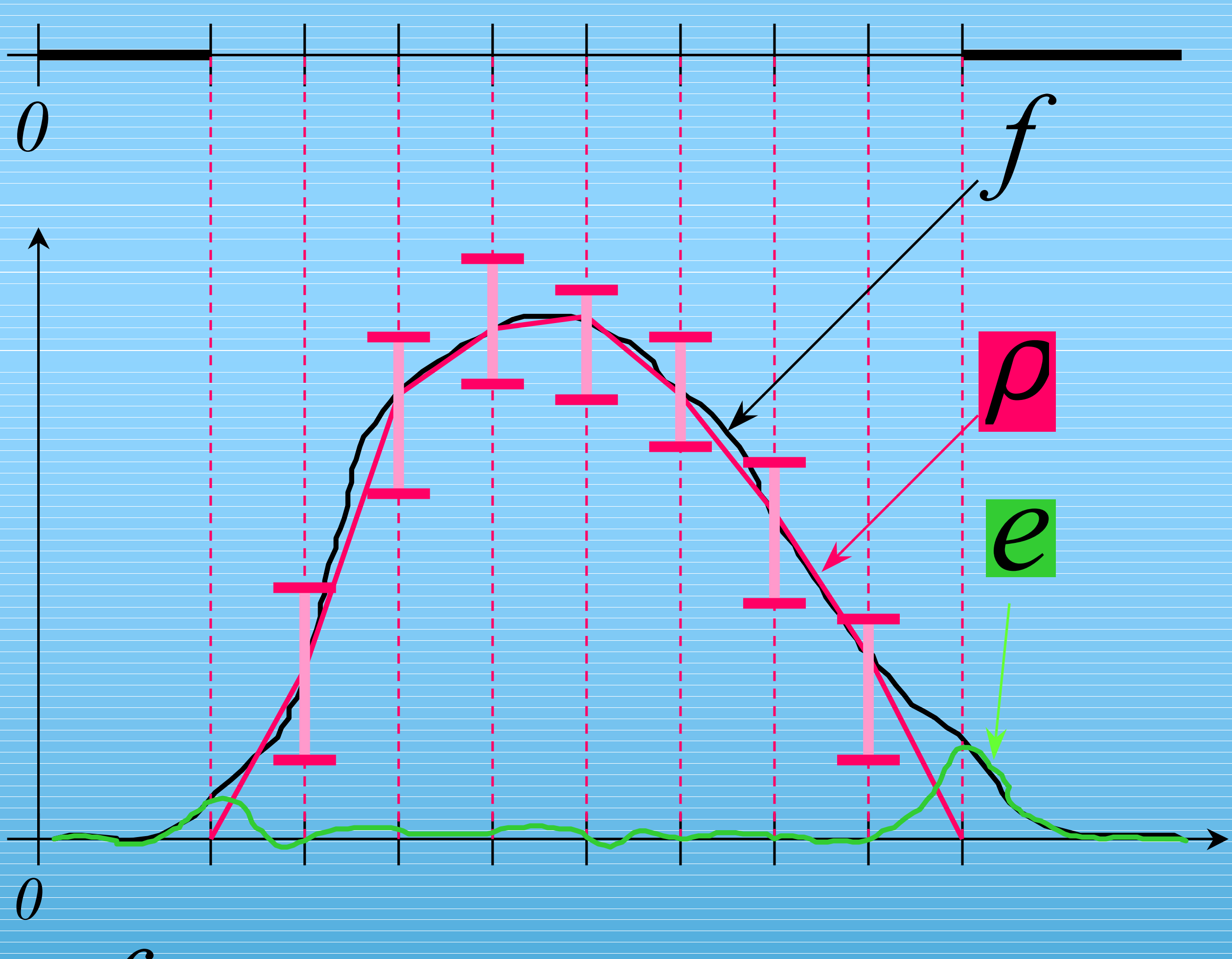
- $\Omega_i(S) \supseteq M_i(S)$

$$\forall S \in D(\Omega_i)$$

Standard Sets (Choice)

$$B = L_1(R^+, w(x)\,dx)$$

$$w(x) = \exp(1/x + x)$$



$$f = \rho + e, \quad \rho \in \cdots,$$

$$\|e\| \le \varepsilon$$

Bound on Convolution

$$C(f)(x) = (f * f)(2x)$$

$$f = \rho + e$$

$$f * f = \rho * \rho + 2 * \rho * e + e * e$$

$$= \tilde{\rho} + \tilde{e}$$

Lemma: $\rho * \rho$ an explicit cubic spline.

$$\|\tilde{e}\| \leq \ \ldots$$

↑

this is the only work one does!

Bound on Derivative

$$(DC(f)h)(x) = 2(f*h)(2x)$$

$$f = \rho + e, \qquad \|e\| \leq \beta$$

$$h = h_L + h_R + h_\perp + \sum_{i=0}^{N} h_i \chi_i$$

Lemma:

$$\|\rho * h_\perp\| < \delta^2 \cdot \|\rho''\| \cdot \|h_\perp\|$$

Summary

- Bounds are set-maps
- Bounds are defined on "standard sets"
- Can compute with bounds
- Bounds can be composed to form new bounds
- Can use chain rule to compute derivative of complicated maps
- Need only worry about simple maps
- Let computer do all this (Prolog)
- SIAM Review, 38, 4, 1996