Computer-Assisted Proofs in Analysis and Programming in Logic

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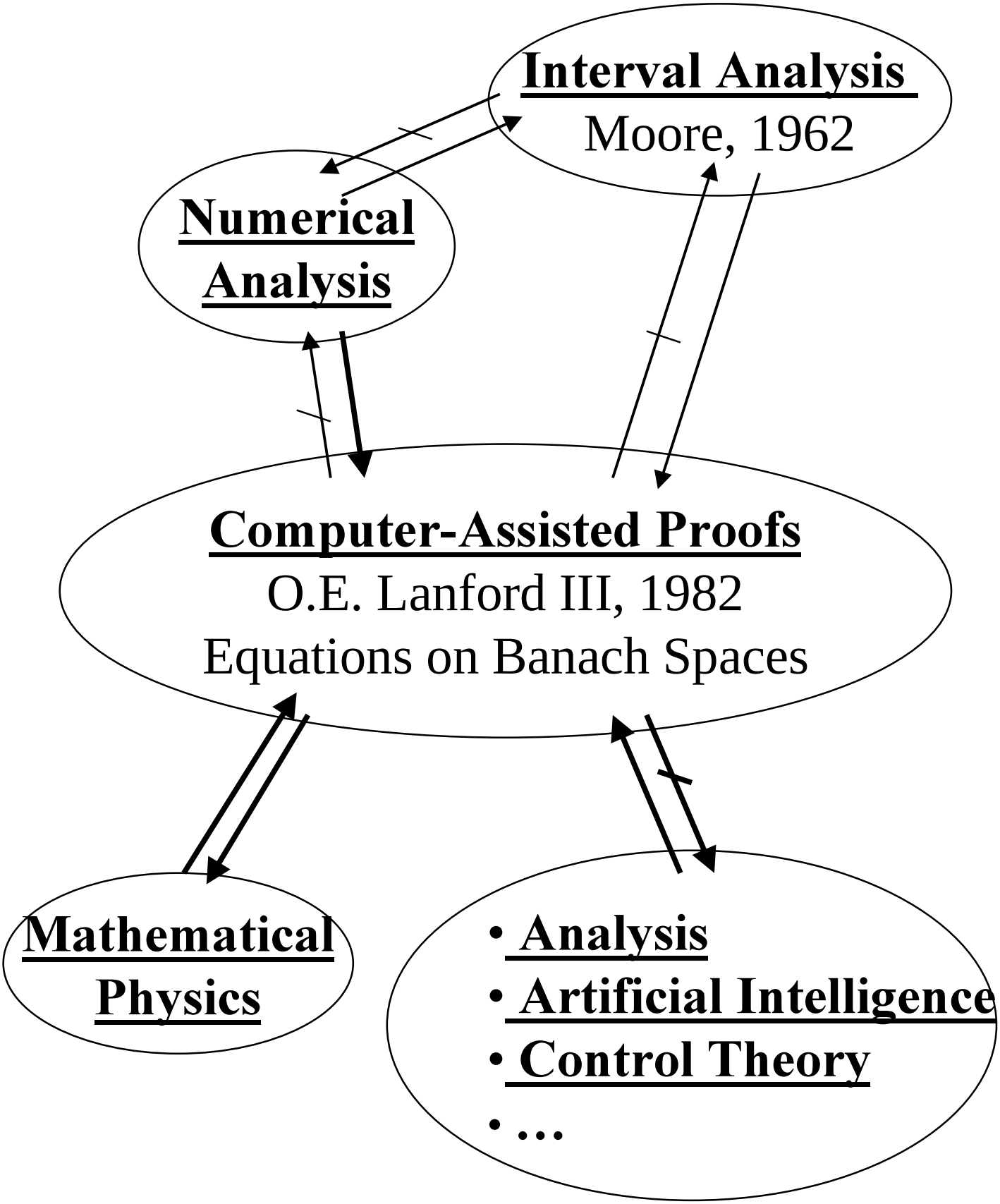
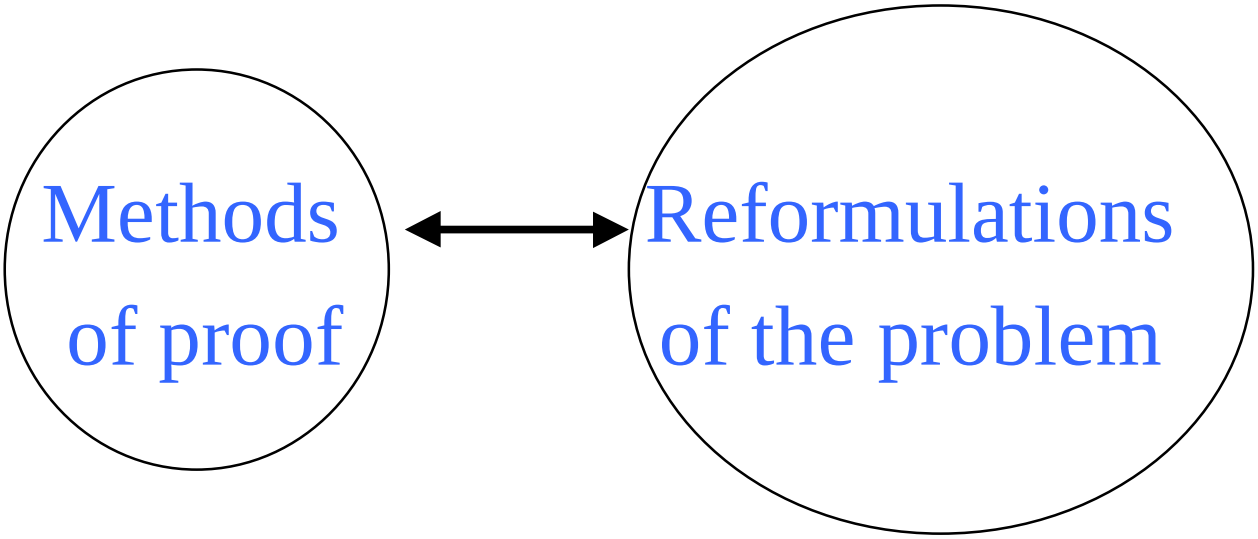
Why this Colloquium?



How do we prove theorems?

Problem

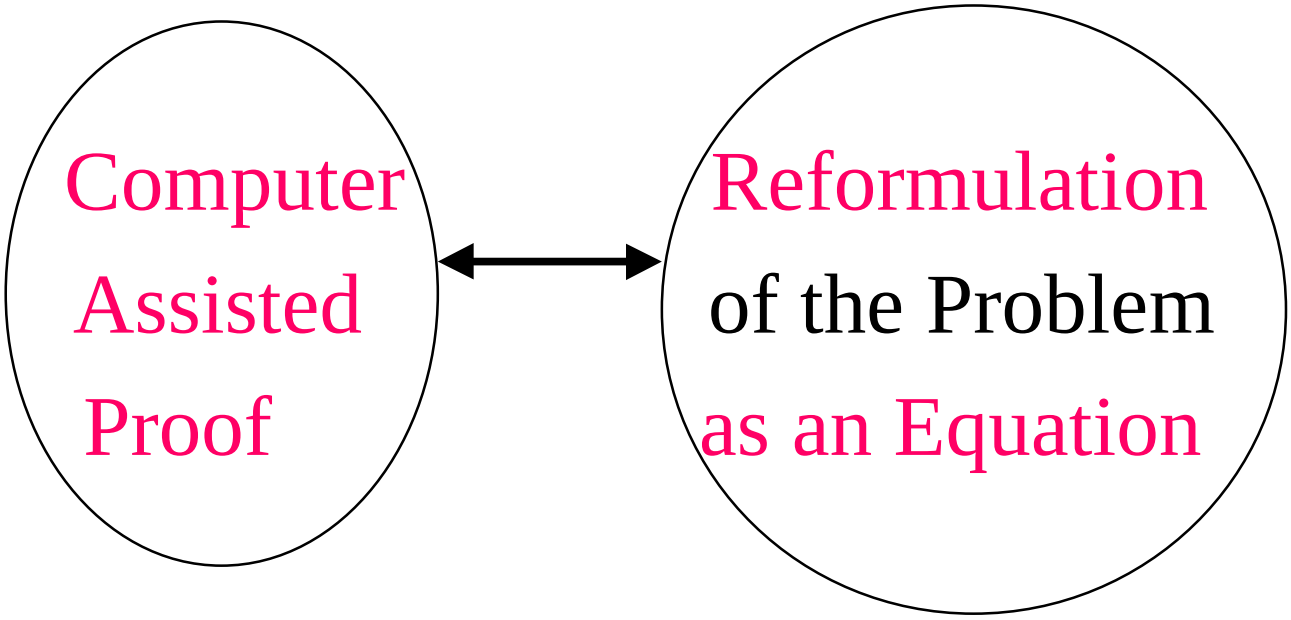


Solution

(Theorem)

What is a Computer-Assisted Proof?

Problem



Solution

(Theorem)

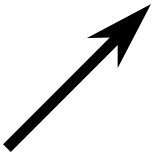
What Kind of Equations?

Prove existence of solutions for:

$$N(f) = f$$

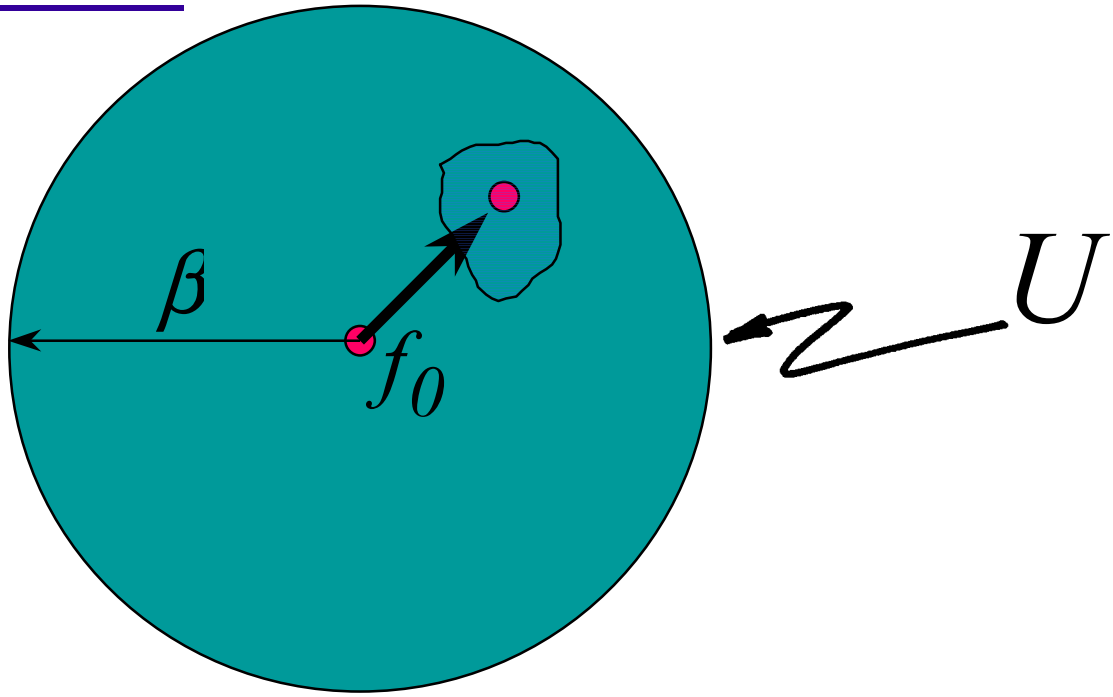
$f \in B$ Banach space

$DN(f)$ reasonable



C.M.P. can be put into practice near the solution

C.M.P.



$$\| DM(f)h \| \leq \rho < 1$$

where $\{f \in U\}$ and $\{h \in B \mid \|h\| \leq 1\}$

$$\|M(f_0) - f_0\| \leq \varepsilon < (1-\rho)\beta$$

Compute with Sets

How to Proceed

- f_0 approximate solution

- $M(f)=f-L(N(f)-f)$

$$L \approx (DN(f_0)-1)^{-1}$$

$$DM(f) \approx 0$$

- hypothesis of the C.M.P. satisfied

“Building Blocks”

- Decompose

$$M = M_1 \circ M_2 \cdots$$

- Bound “Factors”

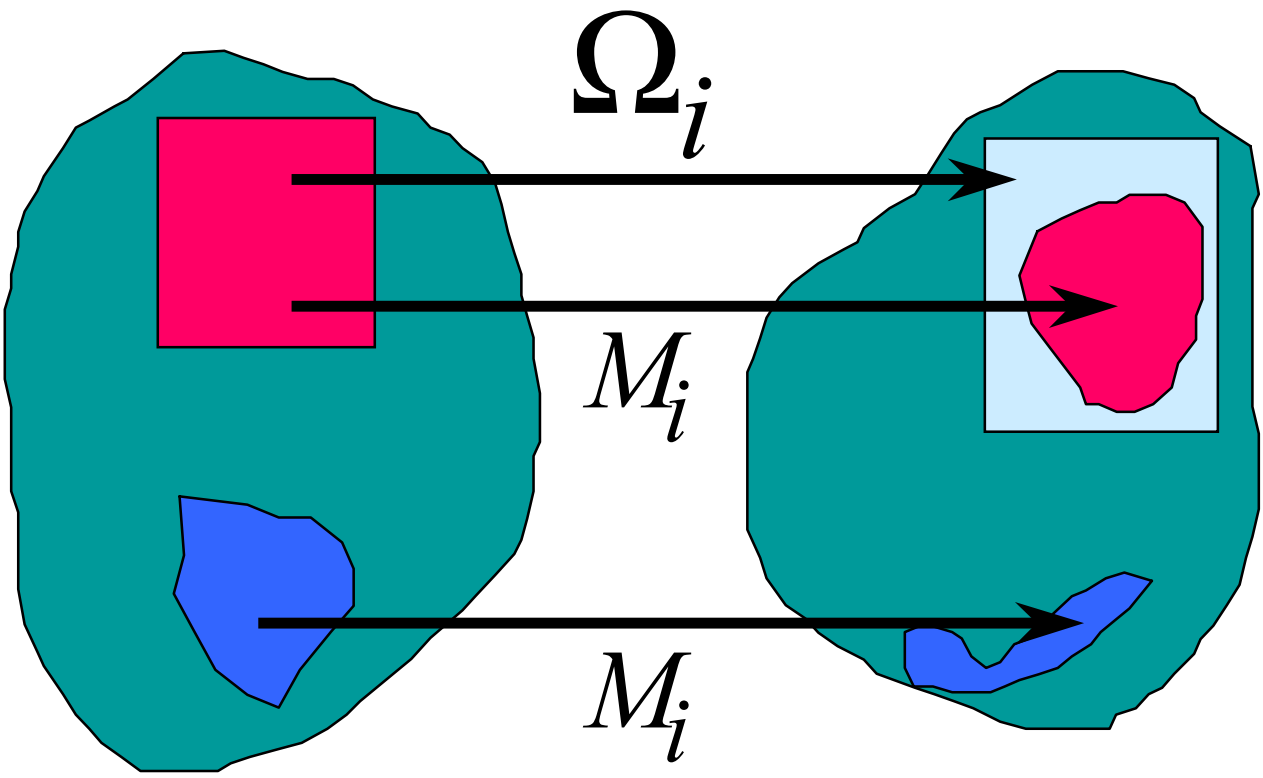
$$M_i, \quad DM_i$$

- Reassemble

$$M = M_1 \circ M_2 \cdots$$

$$DM = \text{apply chain rule}$$

Bound = Set-Map



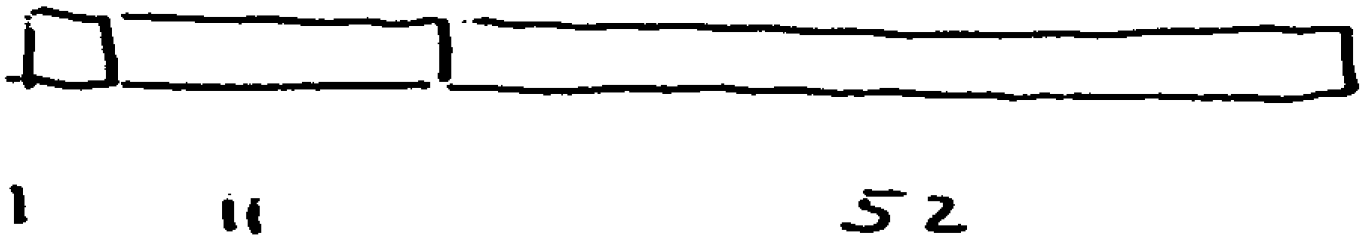
Def.: $\Omega_i \geq M_i$

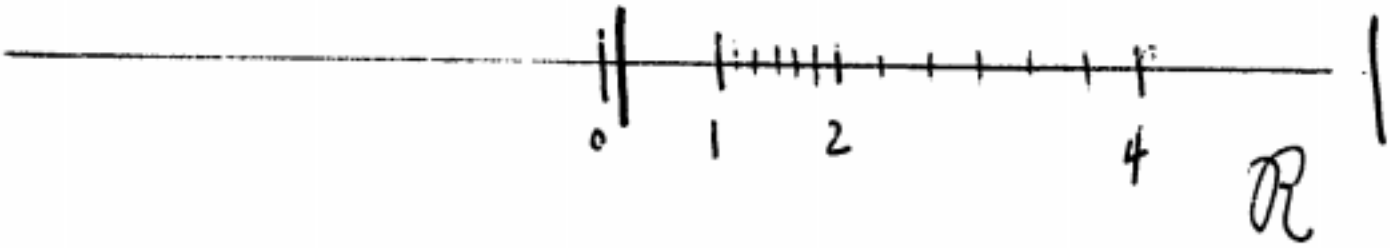
$$D(\Omega_i) \subseteq D(M_i)$$

- $\Omega_i(S) \supseteq M_i(S)$

$$\forall S \in D(\Omega_i)$$

Floating Point Arithmetic (IEEE)





$$r_1 \oplus_f r_2 = \text{rounding}\,(r_1 \oplus r_2)$$

Standard Sets for **R**

std(R) is the collection of all closed intervals i(X,Y) of the form

$$i(X,Y)=\{r \in \mathbf{R} \mid X \leq r \leq Y\}$$

with $X \leq Y$ elements of $\Re$

Example of a Bound

$$-(I) = \{r \in \mathbb{R} \mid r = -x \text{ for some } x \in I\}.$$

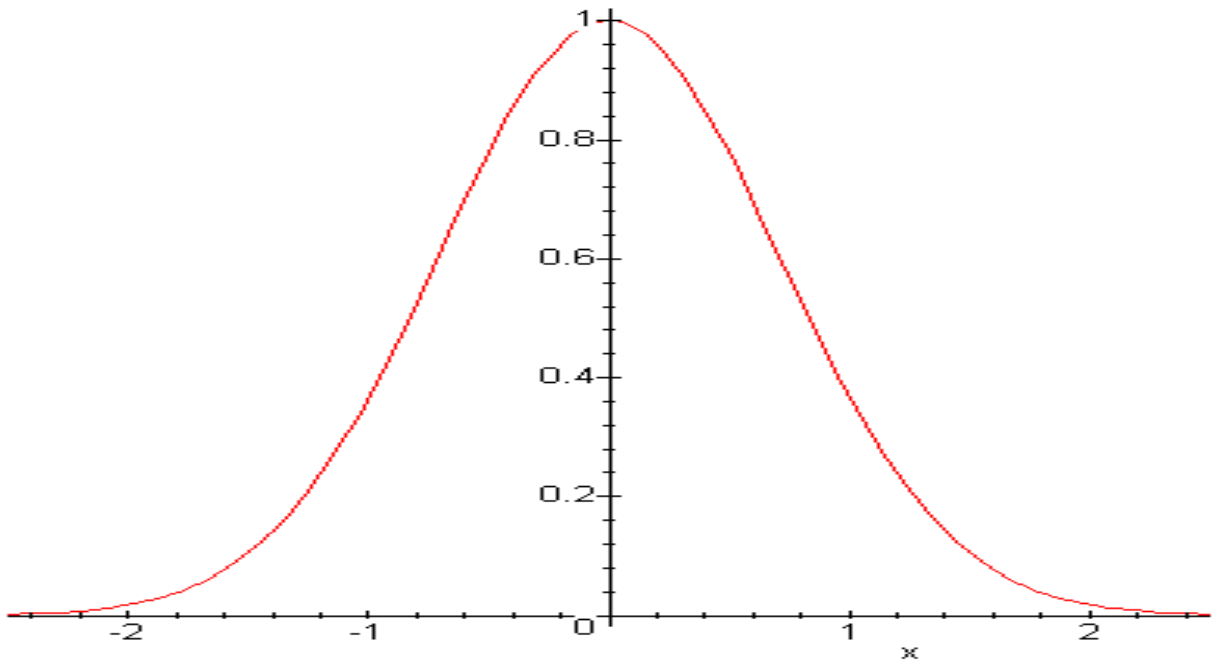
```
i(X1,Y1) contains -i(X,Y):-
 X1 is -Y,
 Y1 is -X,
 !.
```

Prolog

Stable Distributions

$$R = R_1 + R_2$$

$$f(x) = \frac{1}{\sqrt{\pi}} \exp(-x^2)$$



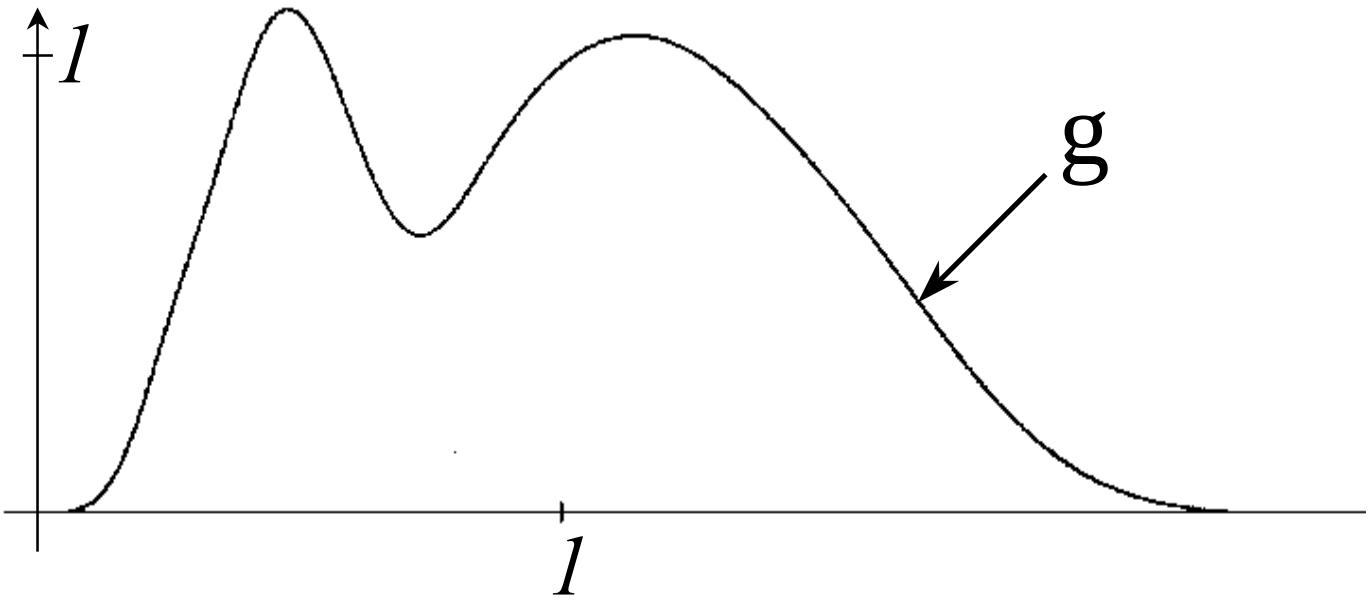
$$f(x) = f^{*2}(\sqrt{2}x)$$

Alain Schenker Jan Wehr

Theorem [SWW]

$$R = \frac{1}{\frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4}}$$

$$f(x) = a \cdot \delta_\infty(x) + (1-a) g(x)$$



$$g(x) = \mu\left(c_1 g^{*2} + c_2 T(T(g^{*2})^{*2})(\mu x)\right)$$

$$Tg(x) = \frac{1}{x^2} g\left(\frac{1}{x}\right)$$

Constants

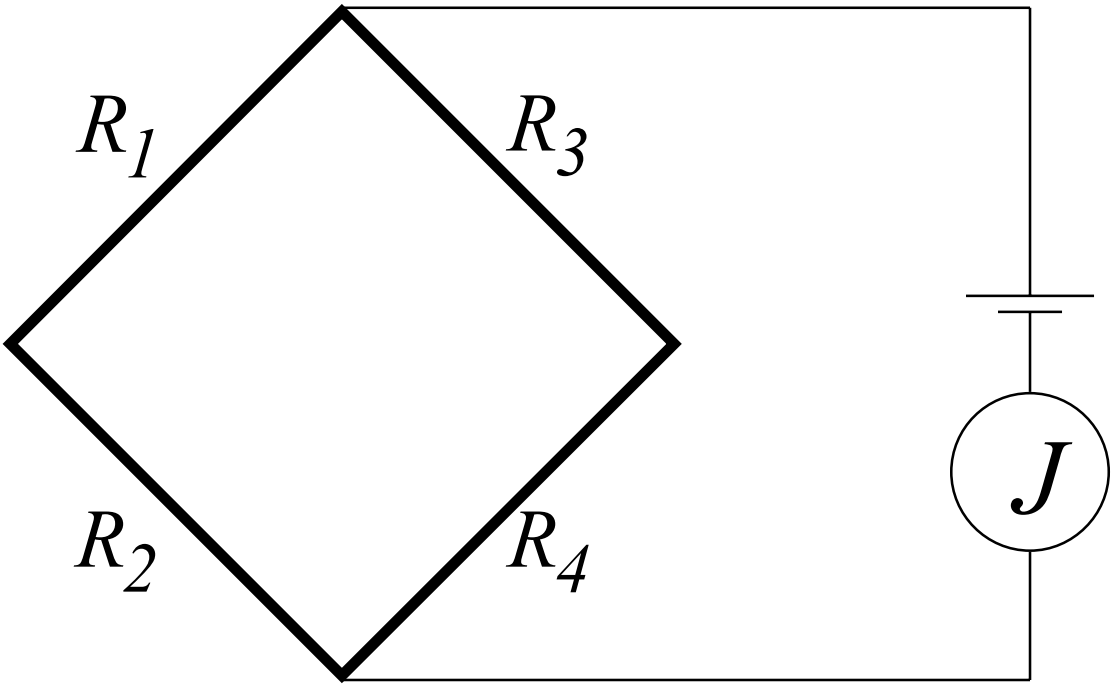
$$a = 0.618... \qquad = (\sqrt{5} - 1)/2$$

$$c_1 = 1 - a^3$$

$$c_2 = a^3$$

$$\exists \mu \in [1.7562036, 1.7562047]$$

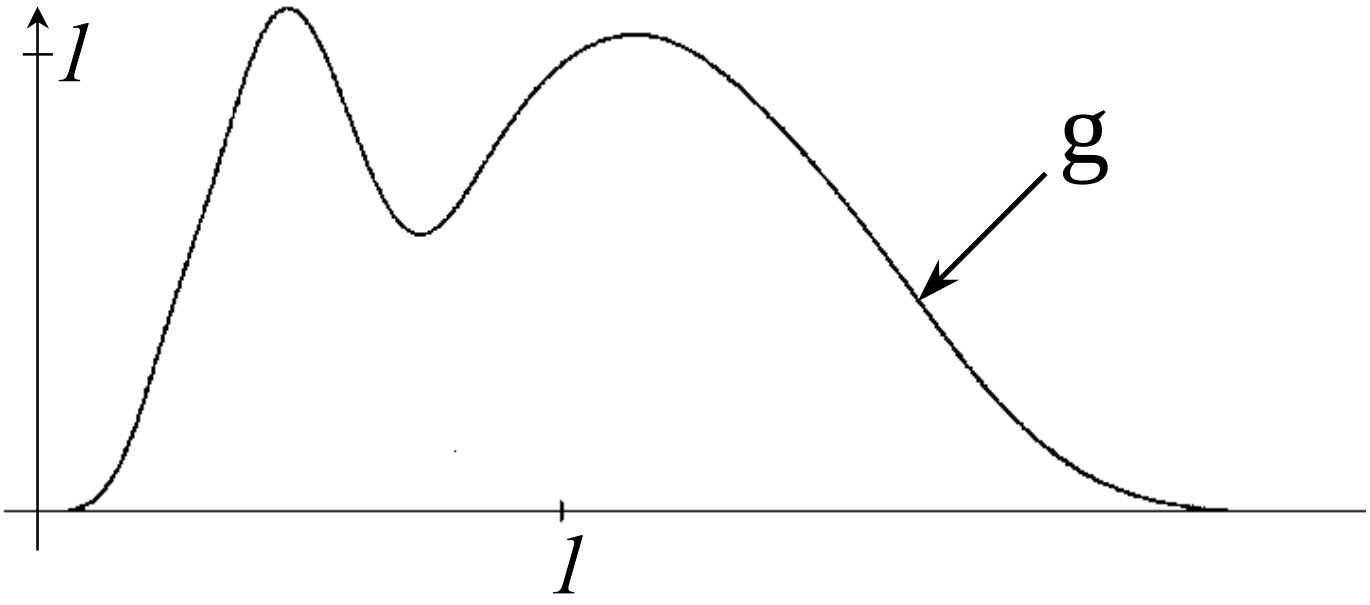
Motivation



Kirchhoff ´s laws:

$$R = \frac{1}{\frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4}}$$

Choice of Function Space



$$Ng(x) = \mu(c_1\, g^{*2} + c_2\, T(T(g^{*2})^{*2}))(\mu x)$$

$$Tg(x) = \frac{1}{x^2}\, g(\frac{1}{x}) \qquad \mu = 1.756\ldots$$

$$B_{\alpha,\beta} = L_1(R^+, w_{\alpha,\beta}(x)\, dx)$$

$$w_{\alpha,\beta}(x) = \exp(\alpha/x + \beta x)$$

Decomposition

$$N(g)(x) = (c_1(g^{*2}) + c_2(T(g^{*2})^{*2})(\mu x)$$
$$= ((S \circ A \circ M \circ \Gamma \circ P_2 \circ U \circ \Gamma \circ P_1)(g))(x)$$

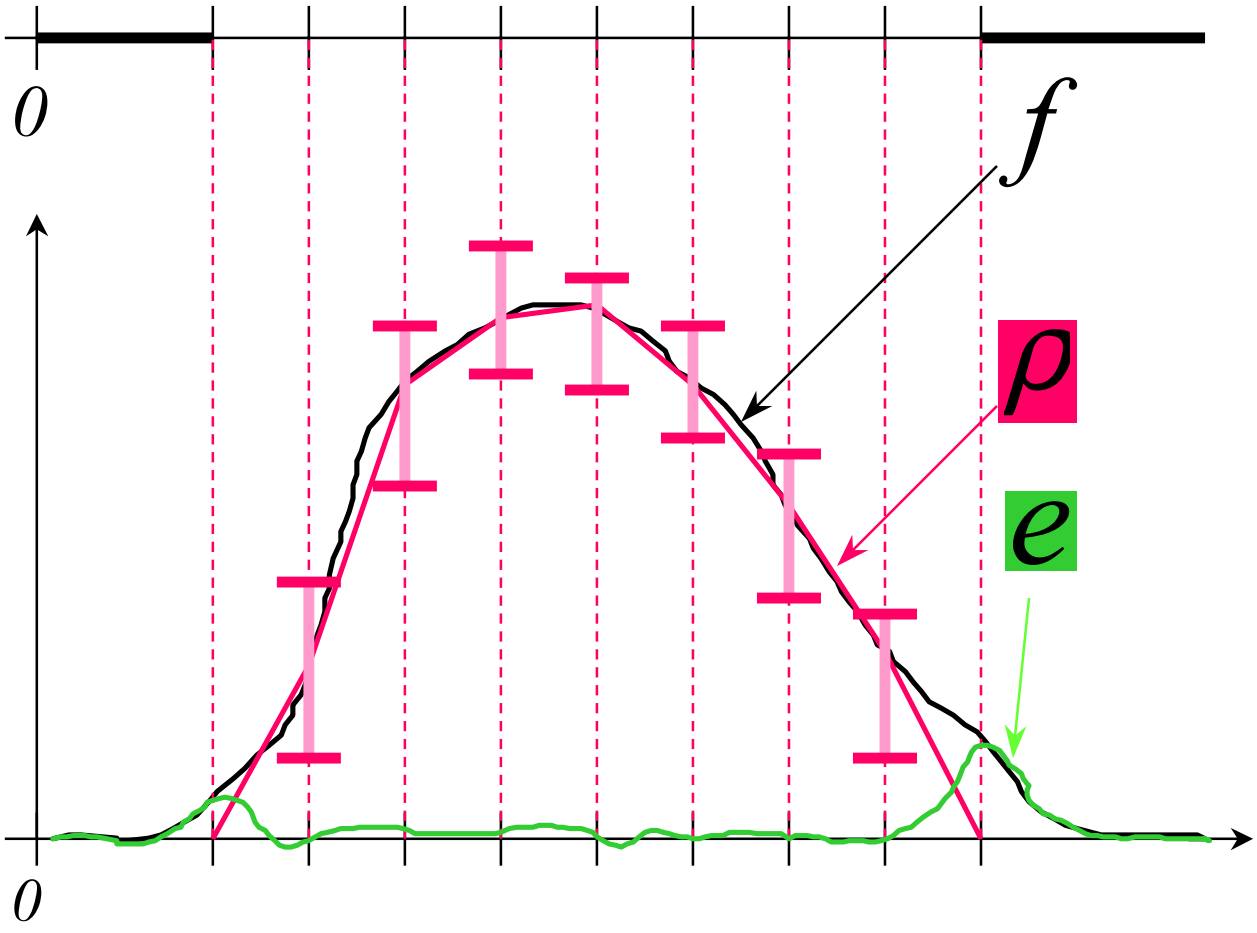
$$P_1 g = (p(g), p(g)) ,$$
$$\Gamma(f,g) = (f, T(g)) ,$$
$$U(f,g) = (f,g) ,$$
$$P_2(f,g) = (f, p(g)) ,$$
$$M(f,g) = (c_1 f, c_2 g) ,$$
$$A(f,g) = f + g ,$$
$$S(f)(x) = \mu f(\mu x) ,$$

$$B_{\alpha,\beta} \to B_{4\alpha,\beta} : \; p(g)(x) = g^{*2}(x) .$$

Standard Sets (Choice)

$$B = L_1(R^+, w(x)\,dx)$$

$$w(x) = \exp(1/x + x)$$



$$f = \rho + e, \quad \rho \in \cdots,$$

$$\|e\| \le \varepsilon$$

Bound on Convolution

$$p(f)(x) = (f * f)(x)$$

$$f = \rho + e$$

$$f * f = \rho * \rho + 2 * \rho * e + e * e$$

$$= \tilde{\rho} + \tilde{e}$$

Lemma: $\rho * \rho$ an explicit cubic spline.

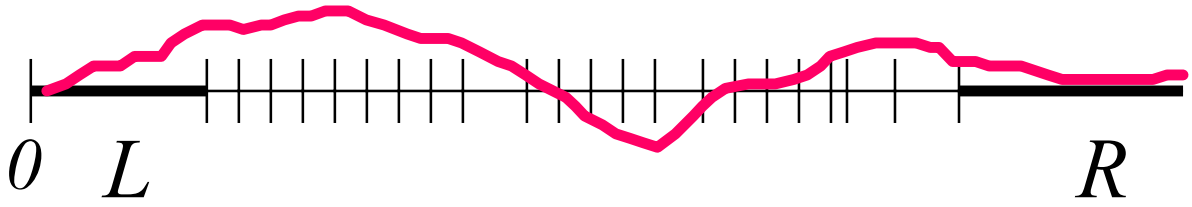
$$\|\tilde{e}\| \leq \ \ldots$$

this is the only work one does!

Bound on Derivative

$$(Dp(f)h)(x) = 2(f * h)(x)$$

$$f = \rho + e, \qquad \|e\| \le \beta$$



$$h = h_L + h_R + h_\perp + \sum_{i=0}^{N} h_i \chi_i$$

Lemma:

$$\|\rho * h_\perp\| < \delta^2 \cdot \|\rho''\| \cdot \|h_\perp\|$$

Choice of Programming Language


```
C..calculate sTGJDZF and sTDZFOGAMMA
      sT3=sDIFF(sINV(sZ),sQUOT(sPOWER(sRHOP,2),sR))
      sT3=sINV(sPOWER(sSQRT(sT3),7))
      sT5=sQUOT(sPROD(sT3,sT4)
     *   ,sPROD(sPOWER(sETA,4),sPOWER(sDIFF(sONE,sT1),3)))
      sT7=sQUOT(sQUOT(siCONST(7),sTWO),sPOWER(sSQRT(sT6),9))
      sT8=sQUOT(sPROD(sSUM(sTWO,sT1),sT7)
     *   ,sPROD(sPOWER(sETA,6),sPOWER(sDIFF(sONE,sT1),5)))
      sFAC=sPROD(sQUOT(siCONST(5),sTWO),sPROD(s3HALF,sPROD(sHALF
     *    ,sSQRT(sPI))))
      sT9=sPROD(sPROD(sPROD(sTWO,sFAC),sCTT),sSUM(sT5,sT8))
      sTGJDZF=sUPPER(sPROD(sPROD(sPOWER(sRHOP,3),sT9),sGGAMMA))
      sTDZFOGAMMA=sPROD(sTWO,sPROD(sPROD(sCTT,sFAC)
     *           ,sPROD(sQUOT(sPOWER(sRHOP,4),sPOWER(sETA,3)),sT3)))
C..calculate coefficients sJ1, sJ2 and sJ3
      sT1=sQUOT(sGGAMMA,sPOWER(sRHOP,3))
      sT1=DCMPLX(-rMAXABS(sT1),rMAXABS(sT1))
      sT2=sQUOT(sT1,sPOWER(sRHOGAMMA,2))
      sJ1H=sPROD(sFPOO,sPROD(sFOUR,sT1))
      sJ1 =sSUM(sJ01,sJ1H)
      sJ2H=sPROD(sFP01,sPROD(siCONST(5),sT1))
      sJ2 =sSUM(sJ02,sJ2H)
      sJ3H=sSUM(sPROD(sFPOO,sSUM(sPROD(sT1,sQUOT(siCONST(12)
     *    ,sKAPPA)),sPROD(sT2,siCONST(6)))),sPROD(sFP01
     *    ,sPROD(siCONST(6),sT1)))
      sJ3 =sSUM(sJ03,sJ3H)
C..calculate bound for general
      sT1=sSUM(sSUM(sSUM(sGDZJ,sPROD(sSUM(sPROD(sABS(sFPOO),sGAMMA)
     *   ,sSUM(sPROD(sABS(sFP01),s2GAMMA),sPROD(sABS(sFP02)
     *   ,sPROD(sGAMMA,s2GAMMA)))),sGDZGAMMA))
     *   ,sSUM(sPROD(sTDZFOGAMMA,sGDZGAMMA)
     *   ,sPROD(sSUM(sABS(sFPOO),sSUM(sPROD(sABS(sFP01),sSUM(sGAMMA
     *   ,sGAMMA)),sPROD(sABS(sFP02),sSUM(sPROD(sTHREE,s2GAMMA)
     *   ,sSUM(sPROD(sTHREE,sPROD(sGAMMA,sGHGAMMA))
     *   ,sPOWER(sGHGAMMA,2))))))),sPROD(sDZGAMMA,sGGAMMA))))
     *   ,sPROD(sTGJDZF,sDZGAMMA))
      sT2=sSUM(sSUM(sPROD(sJ1H,sPOWER(sRHOP,3))
     *   ,sPROD(sJ2H,sPOWER(sRHOP,4))),sPROD(sJ3H
     *   ,sPOWER(sRHOP,5)))
      sGDZJTOT=sCONST(rMAXABS(sSUM(sT1,sT2)))
C..calculate bound for higher order
      sT1=sSUM(sSUM(sHDZJ,sSUM(sPROD(sDZFOGAMMA,sHDZGAMMA)
     *   ,sPROD(sSUM(sABS(sFPOO),sSUM(sPROD(sABS(sFP01)
     *   ,sSUM(sGAMMA,sGAMMA)),sPROD(sABS(sFP02),sSUM(sPROD(sTHREE
     *   ,s2GAMMA),sSUM(sPROD(sTHREE,sPROD(sGAMMA,sGHGAMMA))
     *   ,sPOWER(sGHGAMMA,2))))))),sPROD(sDZGAMMA,sHGAMMA))))
     *   ,sPROD(sQUOT(sCT,sC),sPROD(sHJDZF,sDZGAMMA)))
      sHDZJTOT=sCONST(rMAXABS(sT1))
C..invert Ecalle's equation: lower order coefficients
      sDZF01=sQUOT(sJ1,sPROD(sFOUR,sGA3))
      sDZF02=sQUOT(sJ2,sPROD(siCONST(5),sGA3))
      sDZF03=sQUOT(sDIFF(sJ3,sPROD(sSUM(sPROD(s3HALF
     *      ,sQUOT(sGA5,sGA3)),sPROD(sQUOT(sTHREE,sFOUR),sGA3))
     *      ,sJ1)),sPROD(siCONST(6),sGA3))
C..add additional contribution to general
      CALL FsDILAT(vGAMMA,sRHO1,vT1)
      CALL FsMULT(vT1,sRHOGAMMA,vT1)
      CALL FDZsDILAT(vGAMMA,sRHO1,vT2)
      CALL FMULT(vT1,vT2,vT2)
      CALL FEQUAL(vT2,vT3)
```

Prolog
(PROgramming in LOGic)

even if r_1 and r_2 are in S, as we shall now assume. However, as far as bounds are concerned, it is sufficient to find an interval `i(X1,Y1)` that contains $r_1 \# r_2$, given an interval `i(X,Y)` that contains the computed value for $r_1 \# r_2$. The following predicate `enlarges/2` serves this purpose:

```
i(X1,Y1) enlarges i(X,Y):-
 X1 is_a_safe_lower_bound_on X,
 Y1 is_a_safe_upper_bound_on Y,
 !.
```

To be more precise, if $X \leq Y$ are given representable numbers (not necessarily in the safe range), then `i(X1,Y1) enlarges i(X,Y)` is satisfied if and only if the indicated safe lower bound `X1` and safe upper bound `Y1` can be found, in which case `i(X1,Y1)` is a standard set with the above-mentioned property, as explained in §3. A typical application of `enlarges/2` is given in the next clause.

Consider now the sum function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. The corresponding set map assigns to a pair (I_2, I_3) in $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})$ the set

$$I_2 + I_3 = \{r \in \mathbb{R} \mid r = x + y \text{ for some } x \in I_2 \,,\, y \in I_3\} \tag{5.3}$$

in $\mathcal{P}(\mathbb{R})$. The following clause defines a bound on this map:

```
i(X1,Y1) contains i(X2,Y2)+i(X3,Y3):-
 X is X2+X3,
 Y is Y2+Y3,
 i(X1,Y1) enlarges i(X,Y),
 !.
```

As indicated in §4, the domain of this bound is the set of all pairs `(I2, I3)` in $\mathrm{std}(\mathbb{R}) \times \mathrm{std}(\mathbb{R})$ for which `I1 contains I2+I3` is true. In this case, the domain is determined by the predicate `enlarges/2`, which is used in order to compensate for possible rounding errors introduced by `is/2` and to ensure that the returned result `I1` is again a standard set.

The same description applies to our bound on the product of (sets of) real numbers if $+$ is replaced by $*$. The bound itself is defined as follows:

```
i(X1,Y1) contains i(X2,Y2)*i(X3,Y3):-
 A is X2*X3,
 B is X2*Y3,
 C is Y2*X3,
 D is Y2*Y3,
 sort_numbers([A,B,C,D],[X,_,_,Y]),
 i(X1,Y1) enlarges i(X,Y),
 !.
```

Here, we have used that the product of two standard sets I_2 and I_3 is an interval and that the

Bounds on + and *

```
i(X1,Y1) contains i(X2,Y2)+i(X3,Y3):-
 X is X2+X3,
 Y is Y2+Y3,
 i(X1,Y1) enlarges i(X,Y),
 !.
```

```
i(X1,Y1) contains i(X2,Y2)*i(X3,Y3):-
 A is X2*X3,
 B is X2*Y3,
 C is Y2*X3,
 D is Y2*Y3,
 sort_numbers([A,B,C,D],[X,_,_,Y]),
 i(X1,Y1) enlarges i(X,Y),
 !.
```

Chain Rule

```
Z contains (F o G) of X:-
  Y contains G of X,
  Z contains F of Y,
  !.
```

```
D contains (d(F o G) at X) of H:-
 Y contains G of X,
 D contains ((d(F) at Y) o (d(G) at X)) of H,
 !.
```

Summary

- Bounds are set-maps
- Bounds are defined on "standard sets"
- Can compute with bounds
- Bounds can be composed to form new bounds
- Can use chain rule to compute derivative of complicated maps
- Need only worry about simple maps
- Let computer do the rest (Prolog)
- SIAM Review, 38, 4, 1996