The raison d'être of anomalous dimension

Peter Wittwer
Université de Genève

What will be discussed?

- General remarks about Renormalization Group problems. What is anomalous dimension?

- A simple model with anomalous dimension

- Recent Confusion

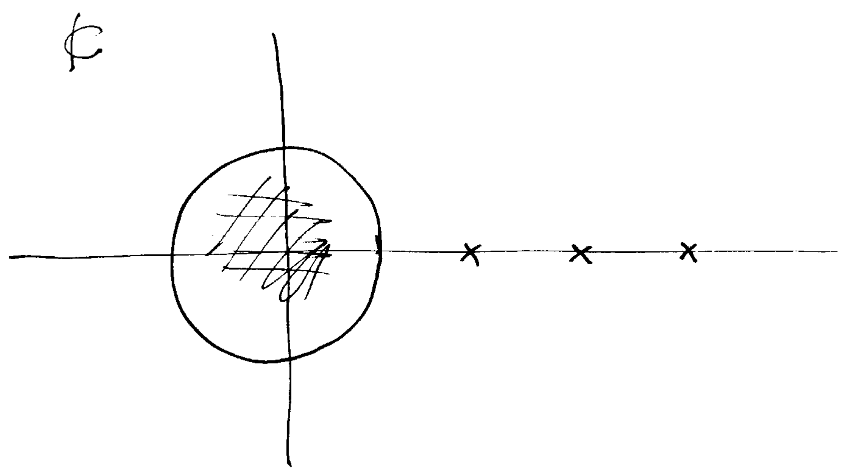
Renormalization Group

- **B** a Banach space

- N a map from $\mathbf{B} \supset U \to \mathbf{B}$

- $NF^* = F^*$ fixed points

spectr($DN(F^*)$)



Symmetry

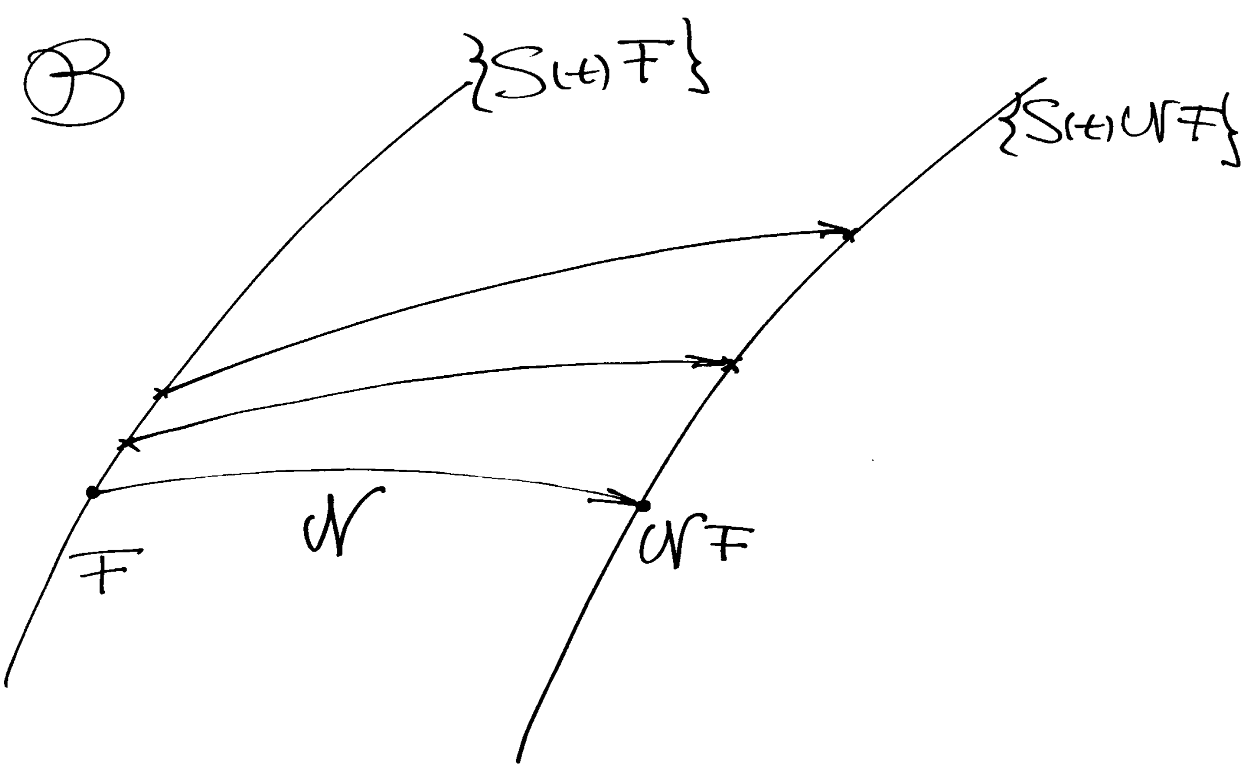
A group

$$t \mapsto S(t), \qquad S(t) : U \to B$$

- $S(0) = 1$
- $S(t_1)S(t_2) = S(t_1 + t_2)$

such that

$$NS(t) = S(\lambda t)N$$



Spectral Consequences

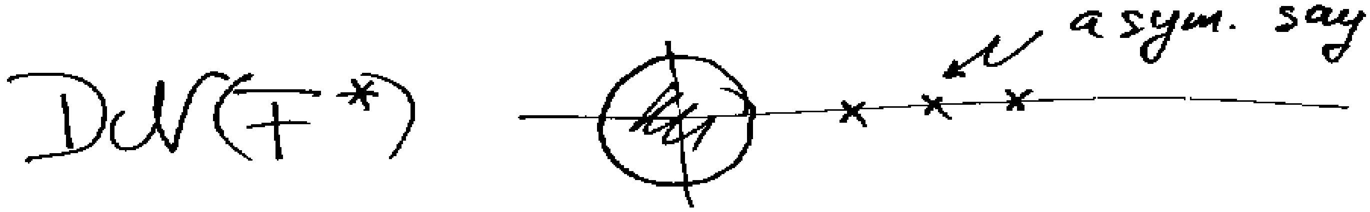
At fixed point F^*,

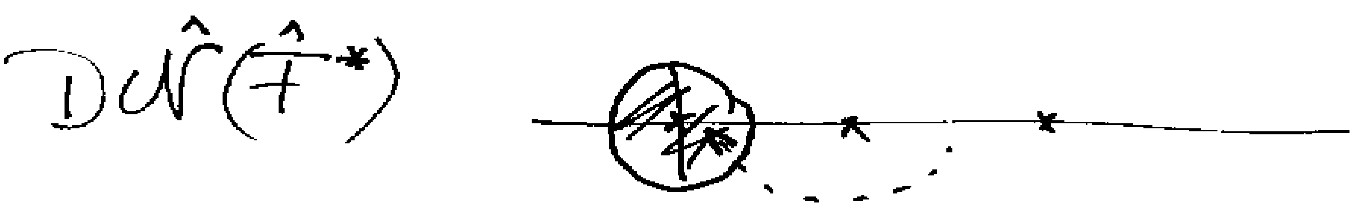
$$NS(t)F^* = S(\lambda t)NF^* = S(\lambda t)F^*$$

$$\frac{d}{dt}\Big|_{t=0}, \qquad H = \left(\frac{d}{dt}S(t)F^*\right)_{t=0}$$

$$DN(F^*)H = \lambda H$$

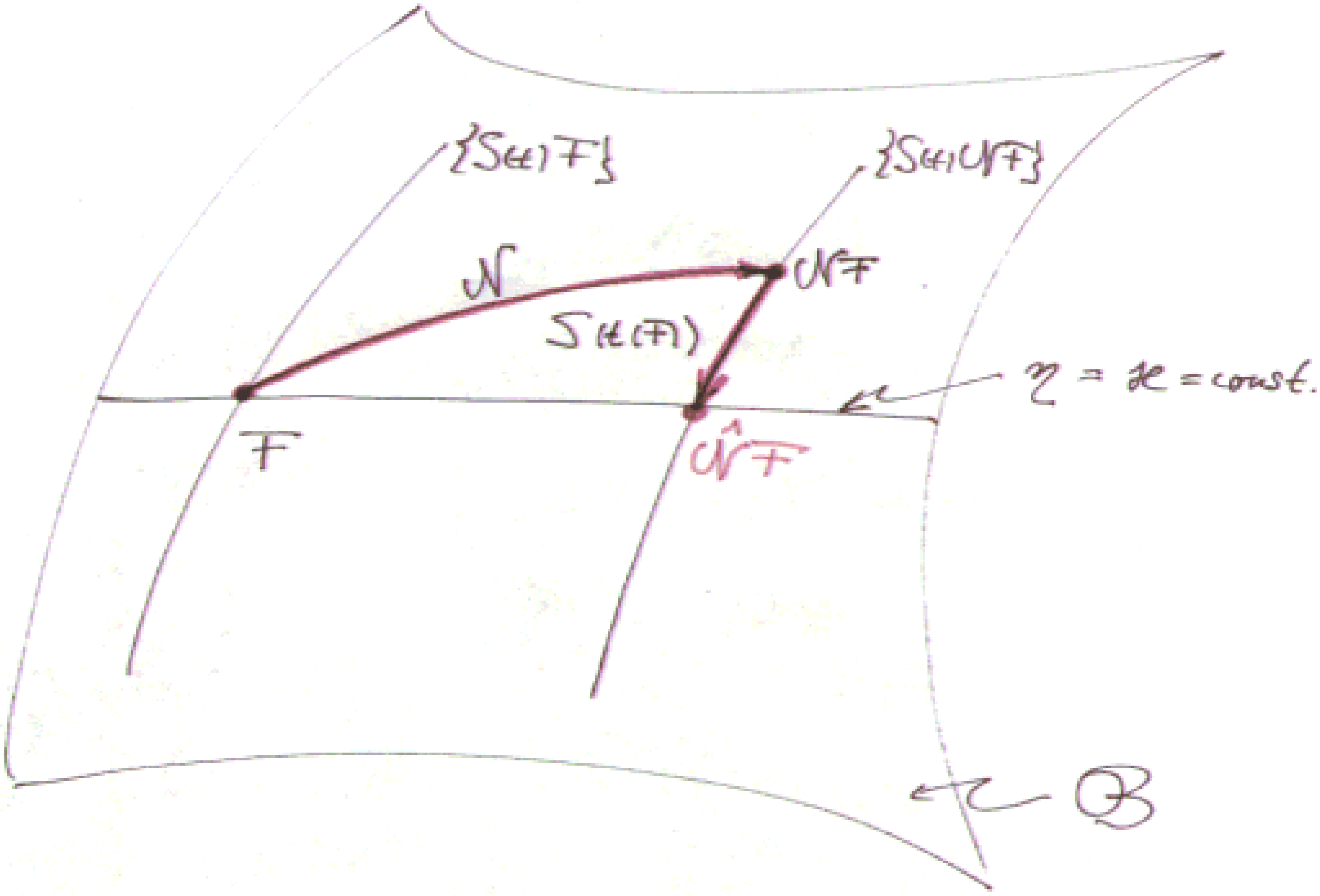
Such an Eigenvalue can be “eliminated”





Divide out Symmetry

normalization $\eta : \mathbf{B} \rightarrow \mathbf{R}$



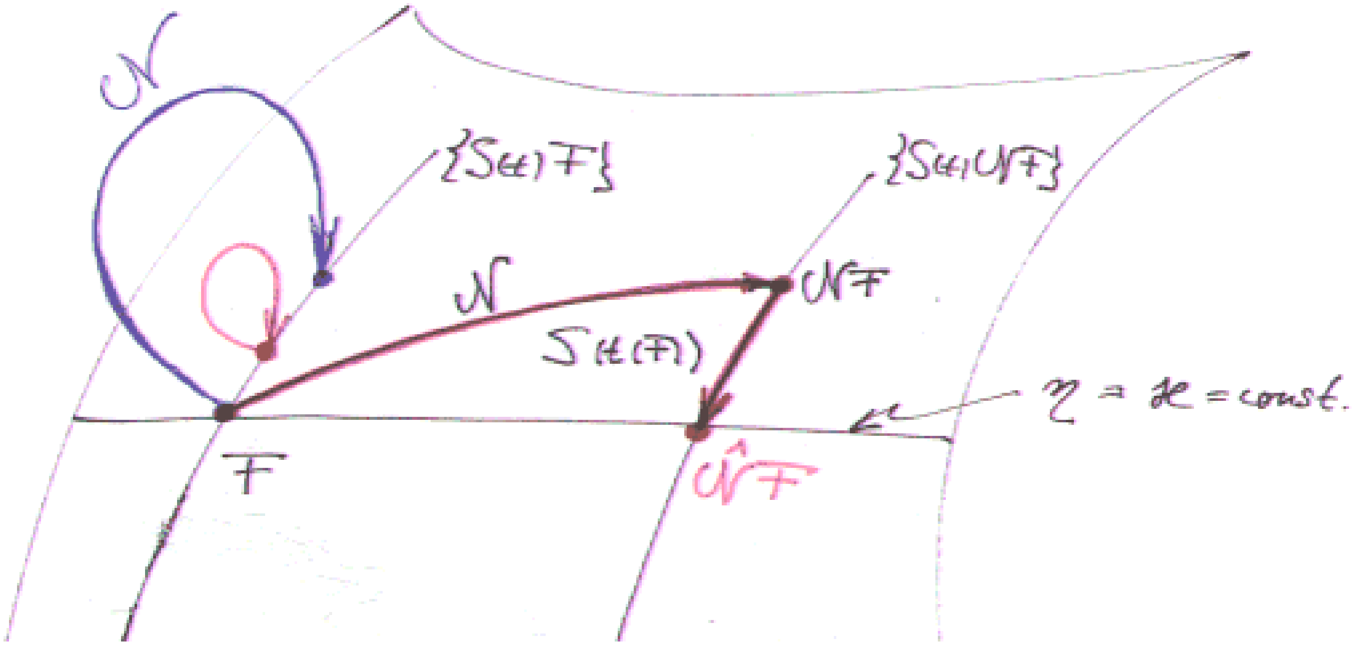
$$\widehat{\mathbf{N}}(F) = S(t(F))\mathbf{N}(F)$$

Solution of the Original Problem

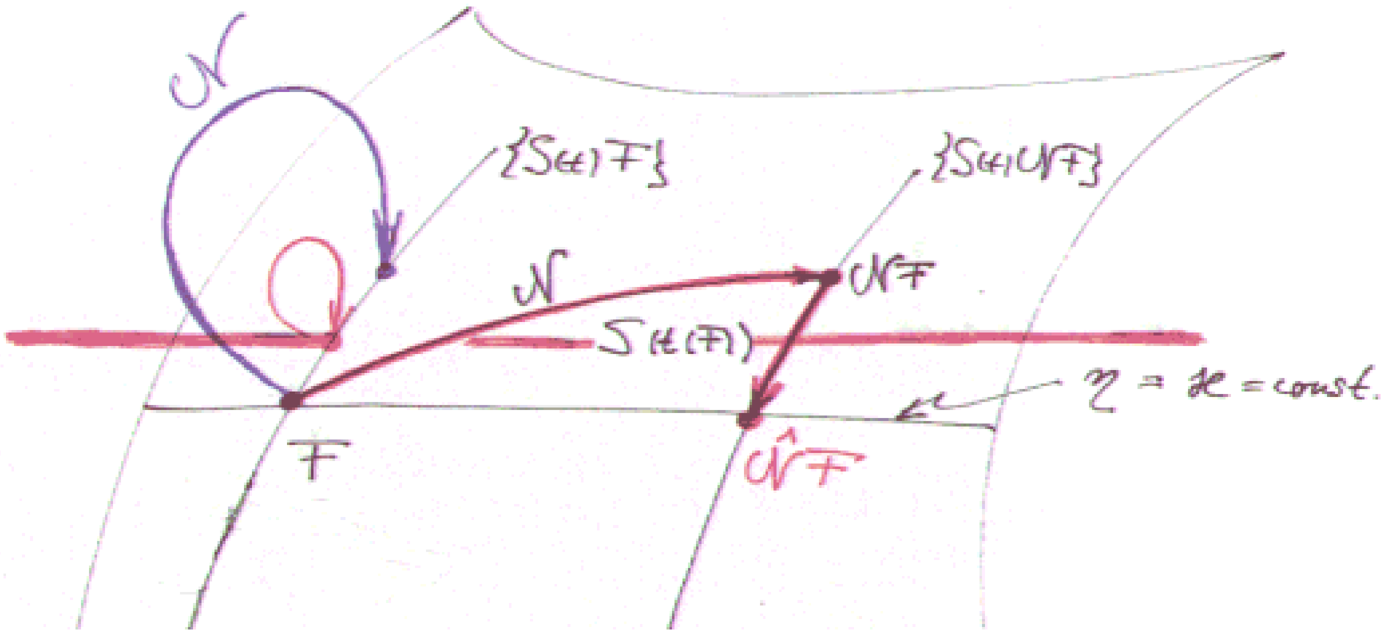
If $\widehat{\mathbf{N}}\widehat{F}^* = \widehat{F}^*$

then $\mathbf{N}F^* = F^*$

for $F^* = S(t(F^*)/(\lambda - 1))\,\widehat{F}^*$



Invariant Subsets



Anomalous Dimension

$$\widehat{=}$$

Eigenvalues equal to one

$$t \to S(t)$$

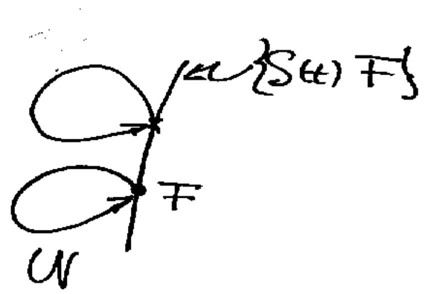
$$\mathbf{N}\,S(t) = S(t)\,\mathbf{N}$$

if $\mathbf{N}F^* = F^*$

then $\mathbf{N}F_t^* = F_t^*$

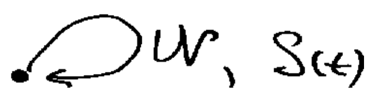
for all $F_t^* = S(t)F^*$

a line of fixed points



or

$$S(t)F^* = F^*,\ \forall t$$



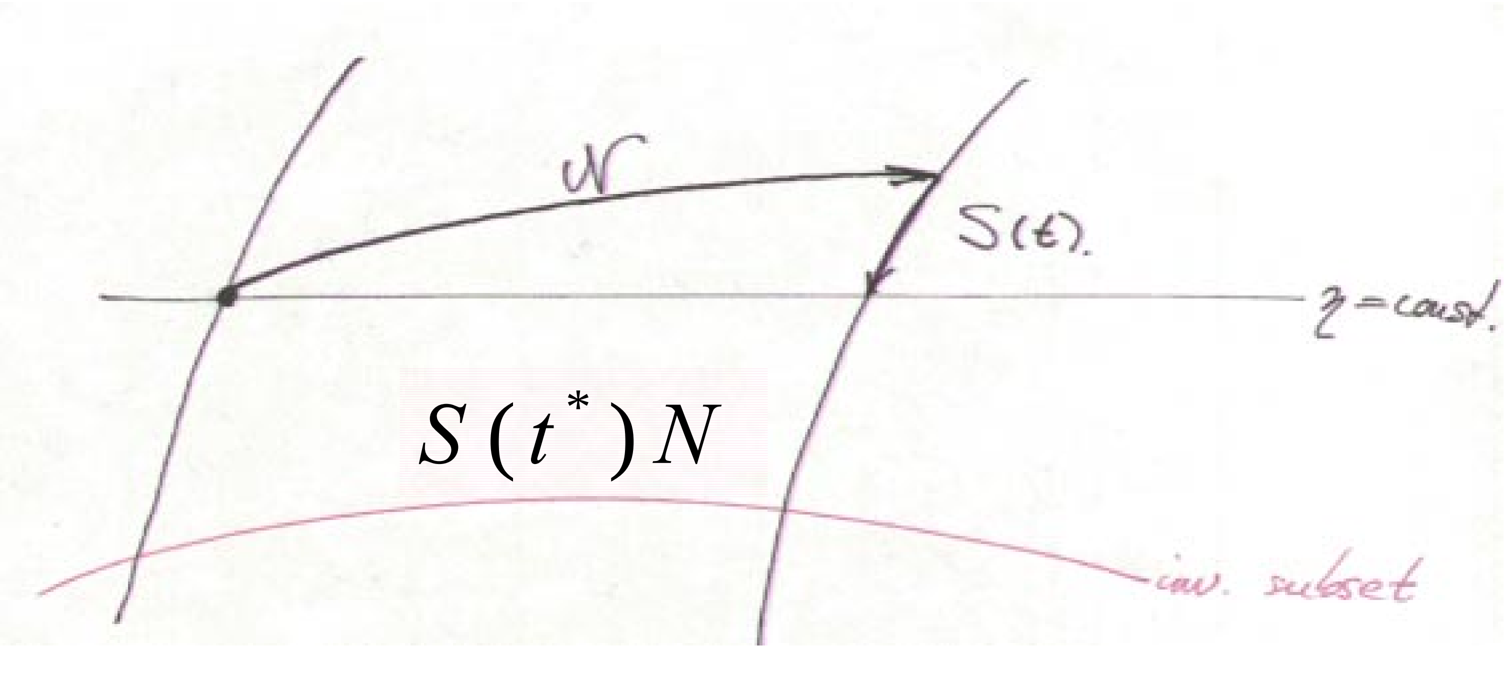
Typical Situation

It is typically naïve to expect a given **N** with a commuting symmetry S(t) to have fixed points!

But $\hat{\mathbf{N}} = S(t)\mathbf{N}$

may have fixed points for particular values of t

(anomalous dimension)



A Trivial Example

$\mathbf{B} = \mathbf{R}^n$

$\mathbf{N} = L$, a pos. sym. $n \times n$ matrix,

$\mathrm{S(t)F} = \mathrm{e}^{\mathrm{t}} F, \qquad F \in R^n$

- The only solution of NF=F is F=0, and S(t)F=F.

- S(t)N has fixed points for

  $t = -\log(\lambda)$, where

  $\lambda \in \{\text{Eigenvalue of L}\}$

Stable Distributions

$$\mathbf{B} = L_1(\mathbf{R}, dx)$$

$$NF = F*F$$

$$\sigma(t)F(x) = e^t F(x)$$

$$N\sigma(t) = \sigma(2t)N$$

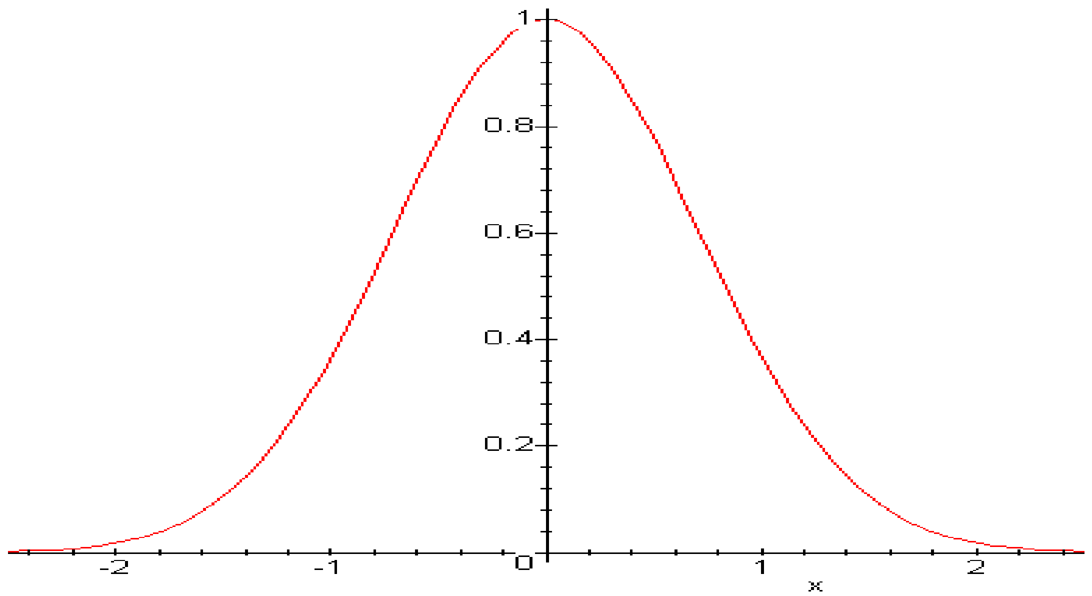
$$S(t)F(x) = e^t F(e^t x)$$

$$NS(t) = S(t)N$$

Gaussien Case

$$R = R_1 + R_2$$

$$f(x) = \frac{1}{\sqrt{\pi}} \exp(-x^2)$$



$$f(x) = \sqrt{2}\; f^{*2}(\sqrt{2}x)$$

$$= (\mathrm{S}(\sqrt{2})\mathrm{f}^{*2})(x)$$

Nonlinear Averages

$$R = \frac{1}{\frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4}}$$

$$\mathbf{B} = \mathrm{L}_1(\mathbf{R}^+, dx) \oplus \mathbf{R}$$

$$NF = T(T(F * F) * T(F * F)) \ ,$$

$$TF(x) = \frac{1}{x^2} F(\frac{1}{x})$$

$$S(t)F(x) = e^t F(e^t x)$$

$$NS(t) = S(t)N$$

Alain Schenkel Jan Wehr

Theorem [SWW]

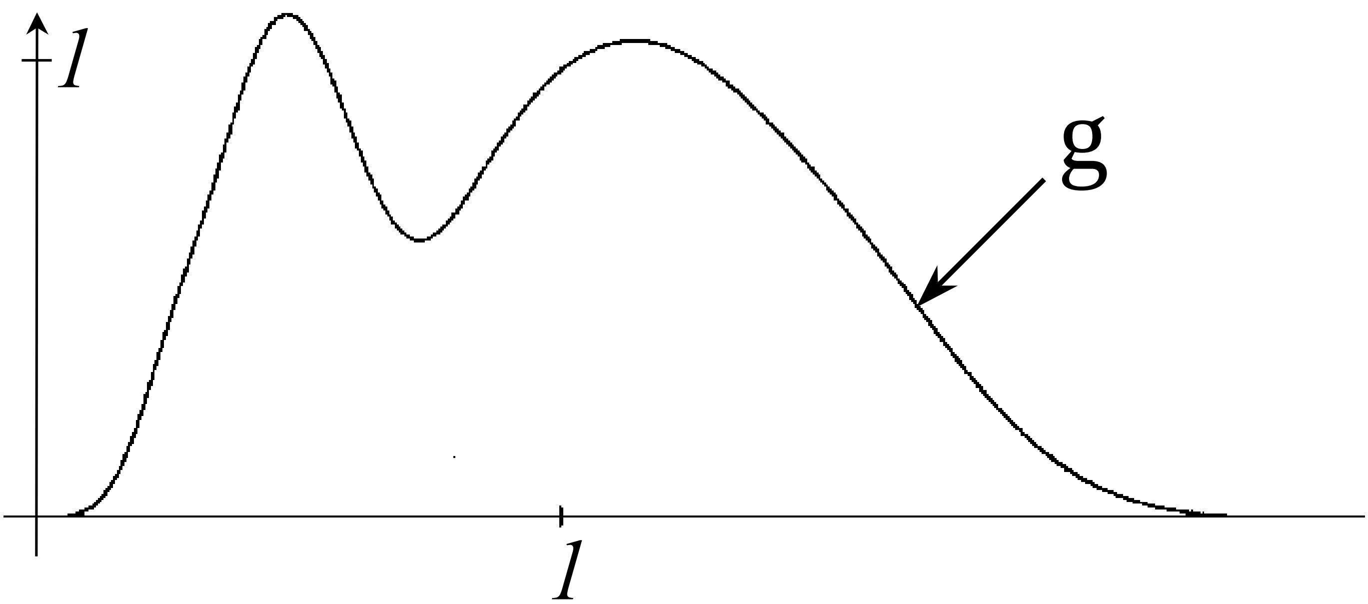
Let $a = 0.618... = (\sqrt{5} - 1)/2$

$\exists \mu \in [1.7562036, 1.7562047]$

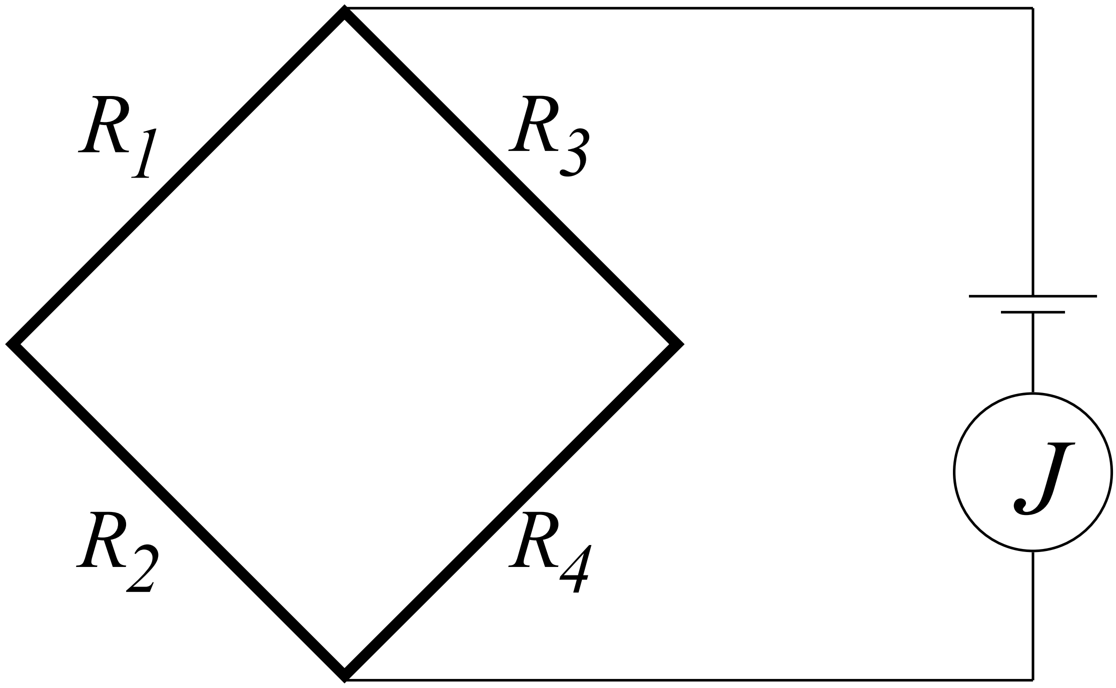
such that

$$F = S(\log(\mu))NF$$

$$F(x) = a \cdot \delta_\infty(x) + (1-a)g(x)$$



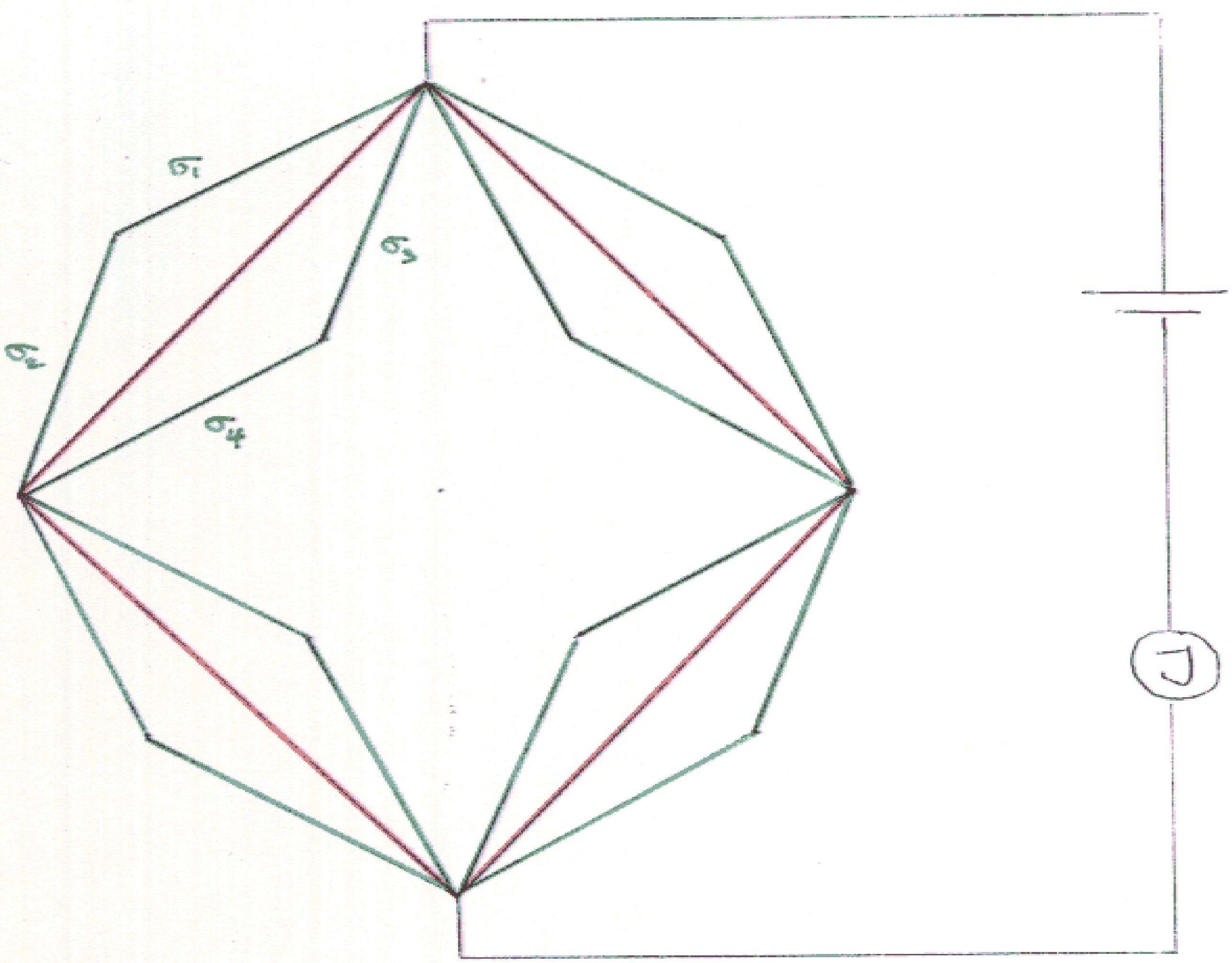
Motivation



Kirchhoff ´s laws:

$$R = \frac{1}{\frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4}}$$

Statistical Mechanics



$$\sigma_i = 0 \quad \text{prob. } 1-\varepsilon$$

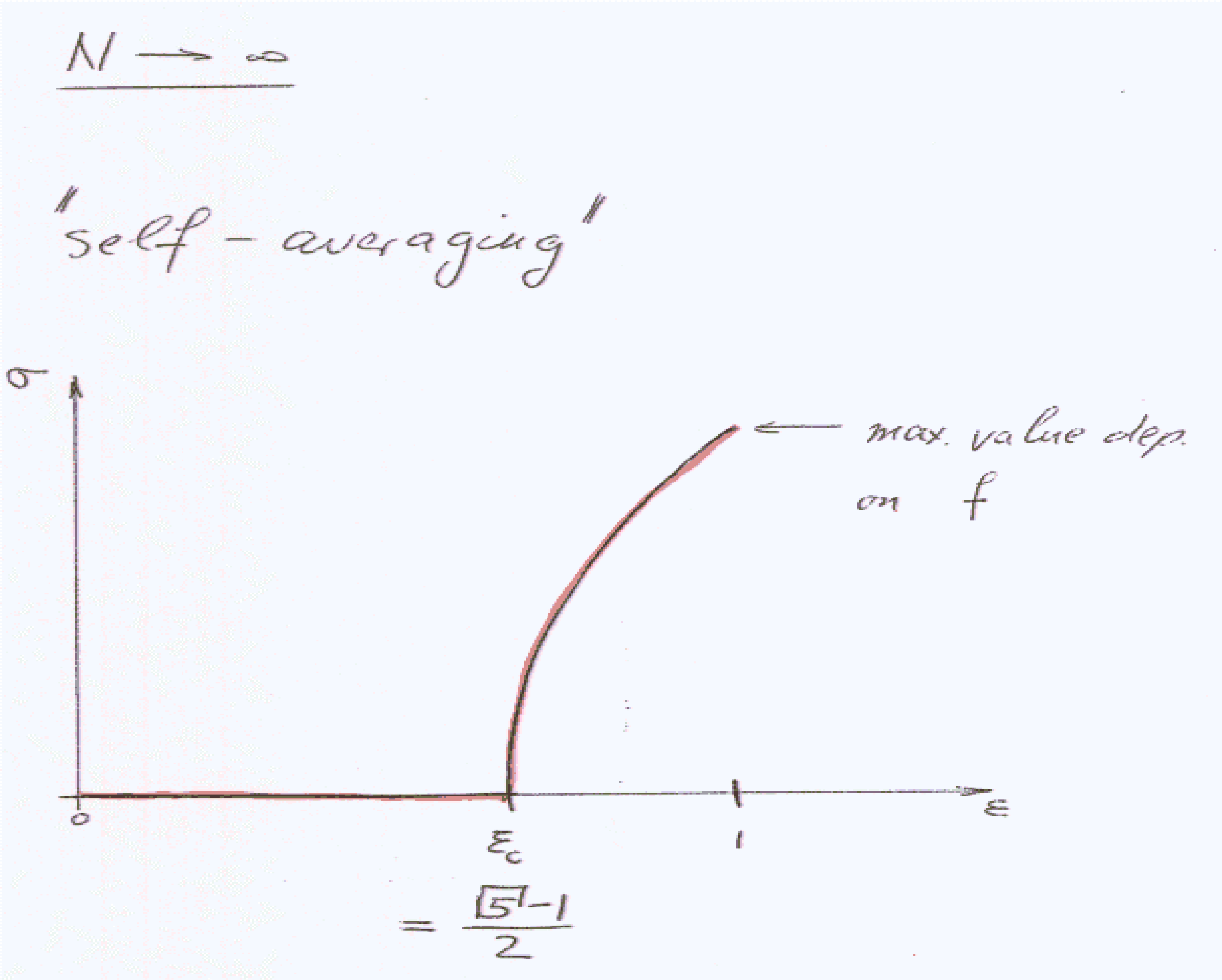
$$\in [a,b] \quad \text{prob. } \varepsilon \cdot \int_a^b f(x)\, dx$$

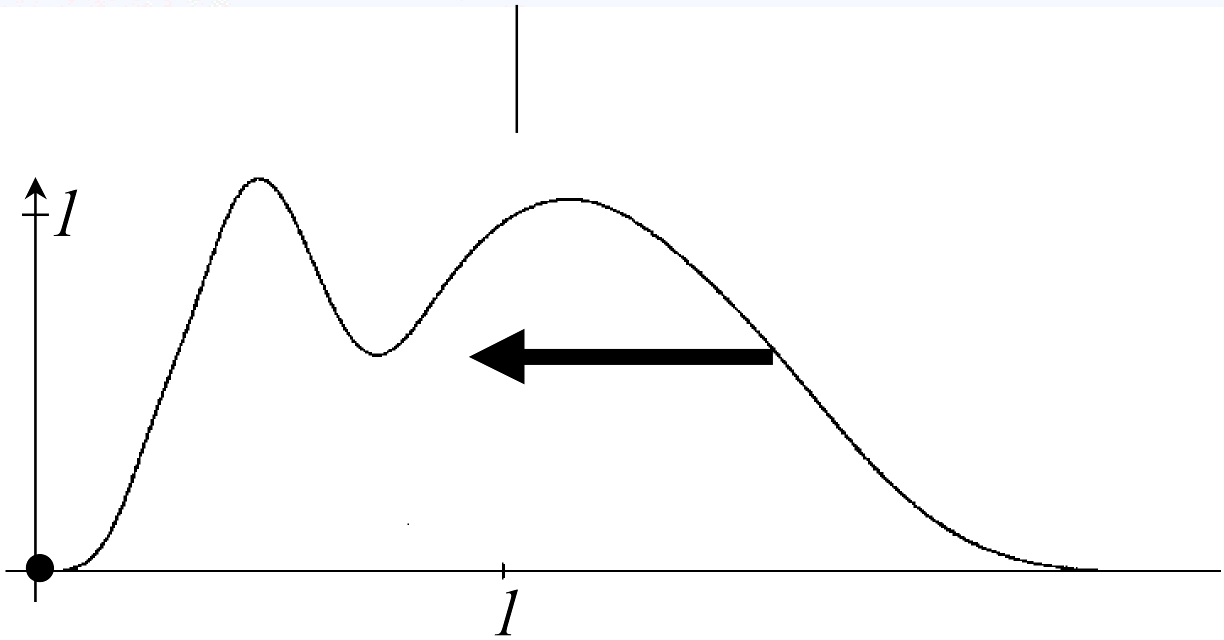
$$\rotatebox{90}{$\in$}$$

$$L_1(\mathbb{R}^+, \quad dx)$$

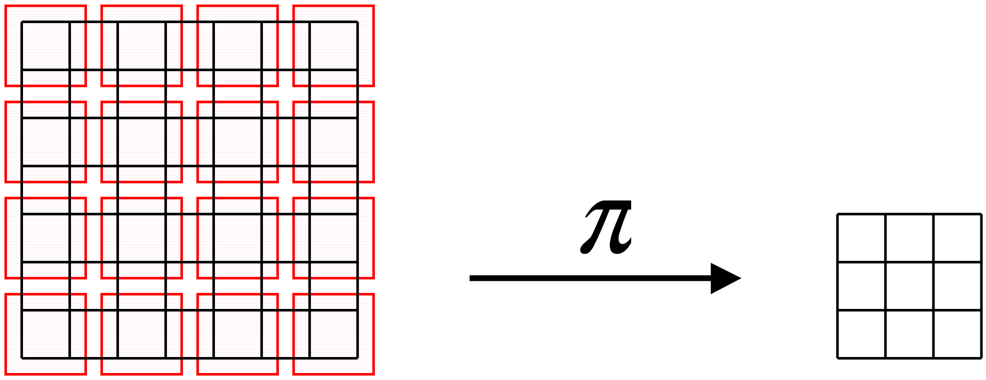
$$(= \infty \quad \text{prob. } \ldots)$$

Phase Transition





Field Theory



Λ_n $\qquad\qquad$ Λ_{n-1}

$$\Psi : \Lambda_n \to \mathbf{R} \qquad\qquad \Phi : \Lambda_{n-1} \to \mathbf{R}$$

$$\Phi(x) = (A\Psi)(x) = \Sigma_{y \in \pi^{-1}(x)} \Psi(y)$$

μ_n a prob. Measure on $S'(\mathbf{R}^{\Lambda_n})$

$$\mu_{n-1}(\Phi) = (N\mu_n)(\Phi) =$$

$$= \left[\int_{S'(\mathbf{R}^{\Lambda_n})} \delta(\Phi - A\Psi)\mu_n(\Psi)\right] D\Phi$$

Symmetry

$$(S_n(t)\mu_n)(\Psi) = e^{|\Lambda_n|t}\mu_n(e^t\Psi)$$

$$NS_n = S_{n-1}N$$

$$(\widehat{N}\mu_n)(\Phi) =$$

$$\left[\int_{S'(\mathbf{R}^{\Lambda_n})} \delta(\Phi - \alpha A\Psi)\mu_n(\Psi)\right] D\Phi$$

Related to anomalous dimension exponent

Recent Confusion

Today's Most popular choice for N

$$(N\mu_n)(\Phi) = \left[\int_{S'(\mathbf{R}^{\Lambda_n})} e^{-\|\Phi - A\Psi\|^2} \mu_n(\Psi)\right] \frac{D\Phi}{\text{norm.}}$$

What happened to the Symmetry ?

It is still there, but has to be calculated !

$$(N_\alpha\mu_n)(\Phi) = \left[\int_{S'(\mathbf{R}^{\Lambda_n})} e^{-\|\Phi - \alpha A\Psi\|^2} \mu_n(\Psi)\right] \frac{D\Phi}{\text{norm.}}$$

Does not correspond to a symmetry!

For each value of α a different class of models!

References

- H.Koch, P.Wittwer, Hierarchical Models

$$(N_\alpha f)(x) = \int_{\mathbf{R}} e^{-y^2} f(\alpha x + y)^2 dy$$

$$(N_\alpha f)(x) = \int_{\mathbf{R}} e^{-y^2} f(\alpha x + y) f(\alpha x - y) dy$$

- D. Brydges, J. Dimock, T.R. Hurd

$$\mu(\Phi) = F(\Phi)\mu_{G_\varepsilon}(\Phi)$$

$$G_\varepsilon(p) = \frac{1}{|p|^{2+\varepsilon}} \qquad \Rightarrow \alpha = \alpha(\varepsilon)$$

None of these models has nontrivial anomalous dimension exponents!