Universality in Dynamical Systems

Numerical Analysis of Functional Equations

How to handle Discretization Errors

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Numerical Analysis of Functional Equations

How to handle Discretization Errors

What Kind of Equations?

Any equation of the form

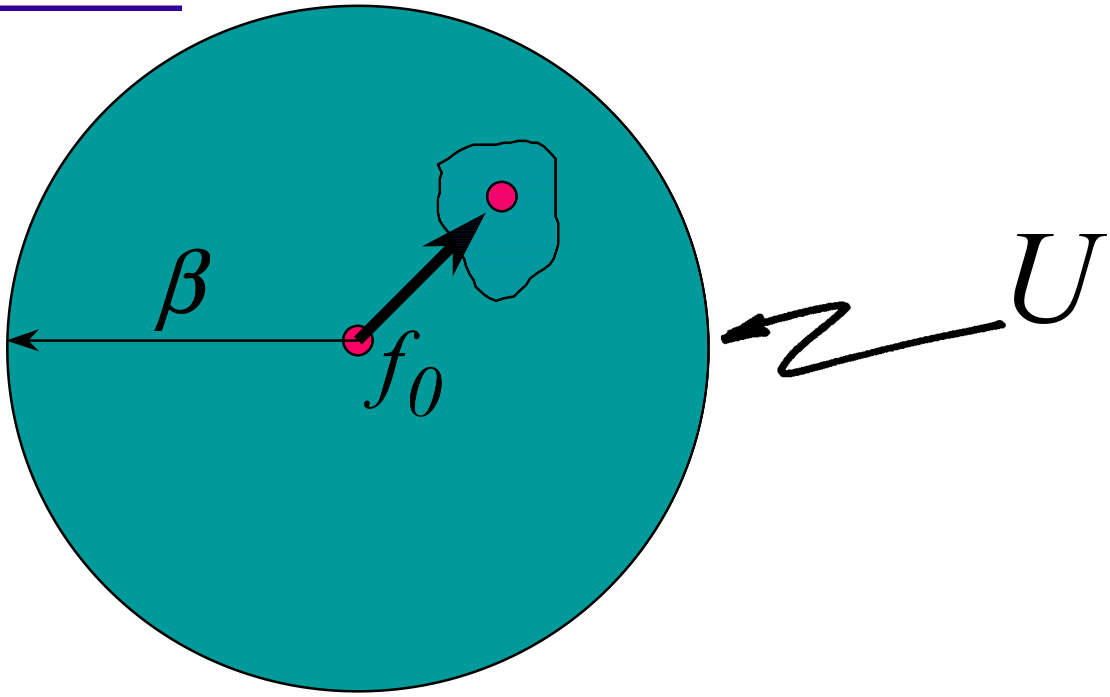
$$N(f) = f$$

$f \in B$ Banach space

$DN(f)$ reasonable

C.M.P. can be put into practice near the solution

C.M.P.



$$\| DM(f)h \| \le \rho < 1$$

where $\{ f \in U \}$ and $\{ h \in B \mid \|h\| \le 1 \}$

$$\|M(f_0) - f_0\| \le \varepsilon < (1-\rho)\beta$$

Compute with Sets

How to Proceed

- f_0 approximate solution

- $$M(f) = f - L(N(f) - f)$$

$$L \approx (DN(f_0) - 1)^{-1}$$

$$DM(f) \approx 0$$

- hypothesis of the C.M.P. satisfied

“Building Blocks”

- Decompose

$$M = M_1 \circ M_2 \cdots$$

- Bound “Factors”

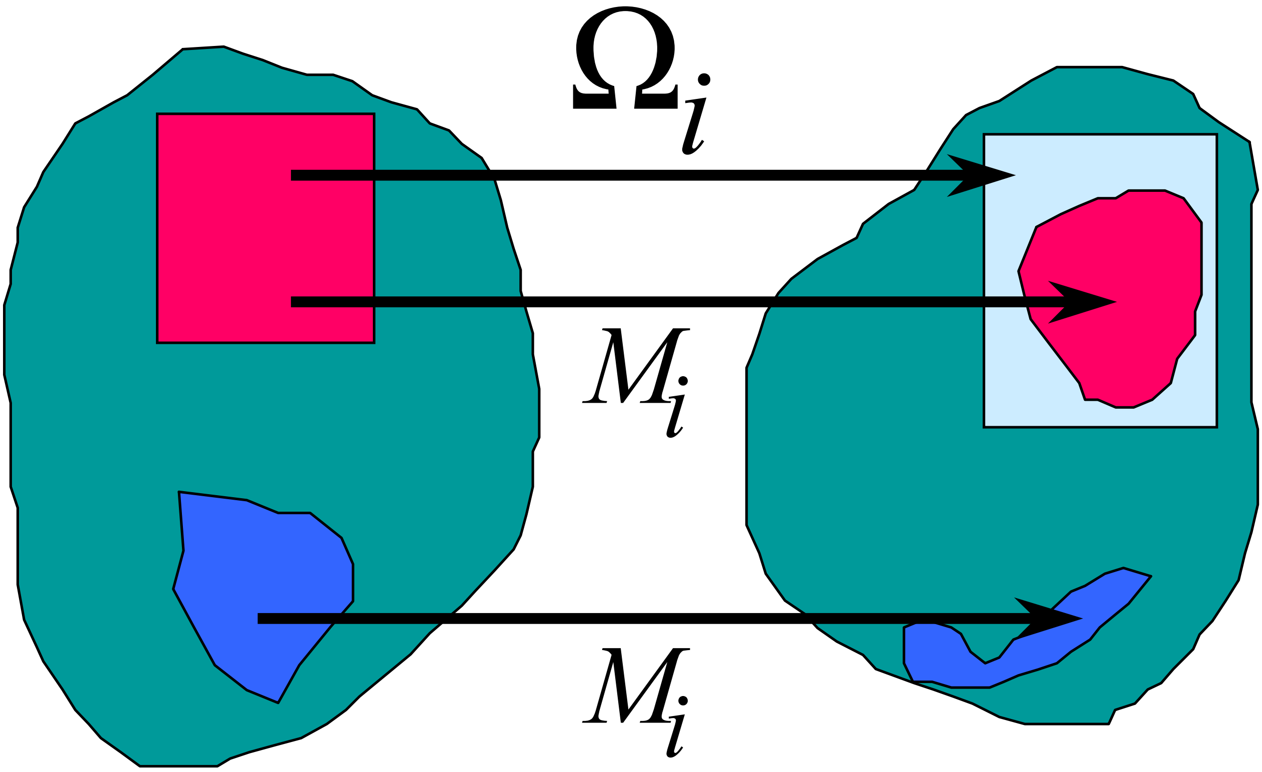
$$M_i, \quad DM_i$$

- Reassemble

$$M = M_1 \circ M_2 \cdots$$

$$DM = \text{apply chain rule}$$

Bound = Set-Map



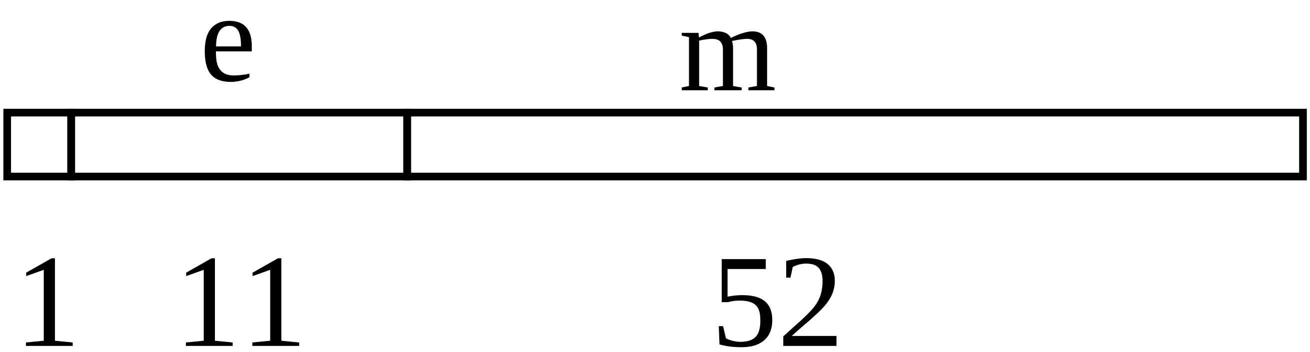
Def.: $\Omega_i \geq M_i$

$$D(\Omega_i) \subseteq D(M_i)$$

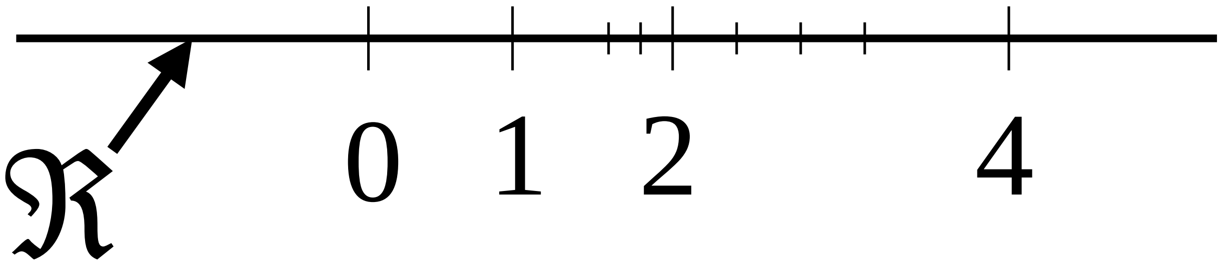
- $\Omega_i(S) \supseteq M_i(S)$

$$\forall S \in D(\Omega_i)$$

Floating Point Arithmetic (IEEE)



$$r = \pm 1.\mathrm{m}\; 2^{\mathrm{e}-1023}$$



$$r_1 \oplus_C r_2 = \mathrm{rounding}(r_1 \oplus r_2)$$

Standard Sets for **R**

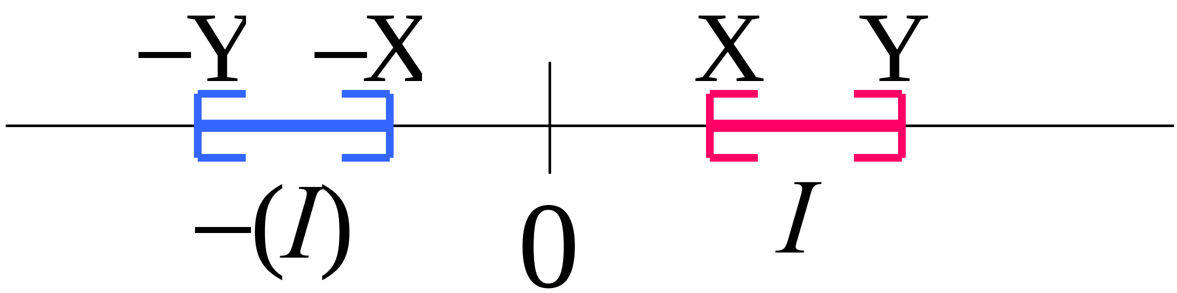
std(R) is the collection of all closed intervals i(X,Y) of the form

i(X,Y)={r∈**R**| X≤ r ≤ Y}

with X ≤ Y elements of $\Re$

Example of a Bound

$$-(I) = \{r \in \mathbb{R} \mid r = -x \text{ for some } x \in I\}.$$




```
i(X1,Y1) contains -i(X,Y):-
 X1 is -Y,
 Y1 is -X,
 !.
```

Prolog

Bounds on + and *

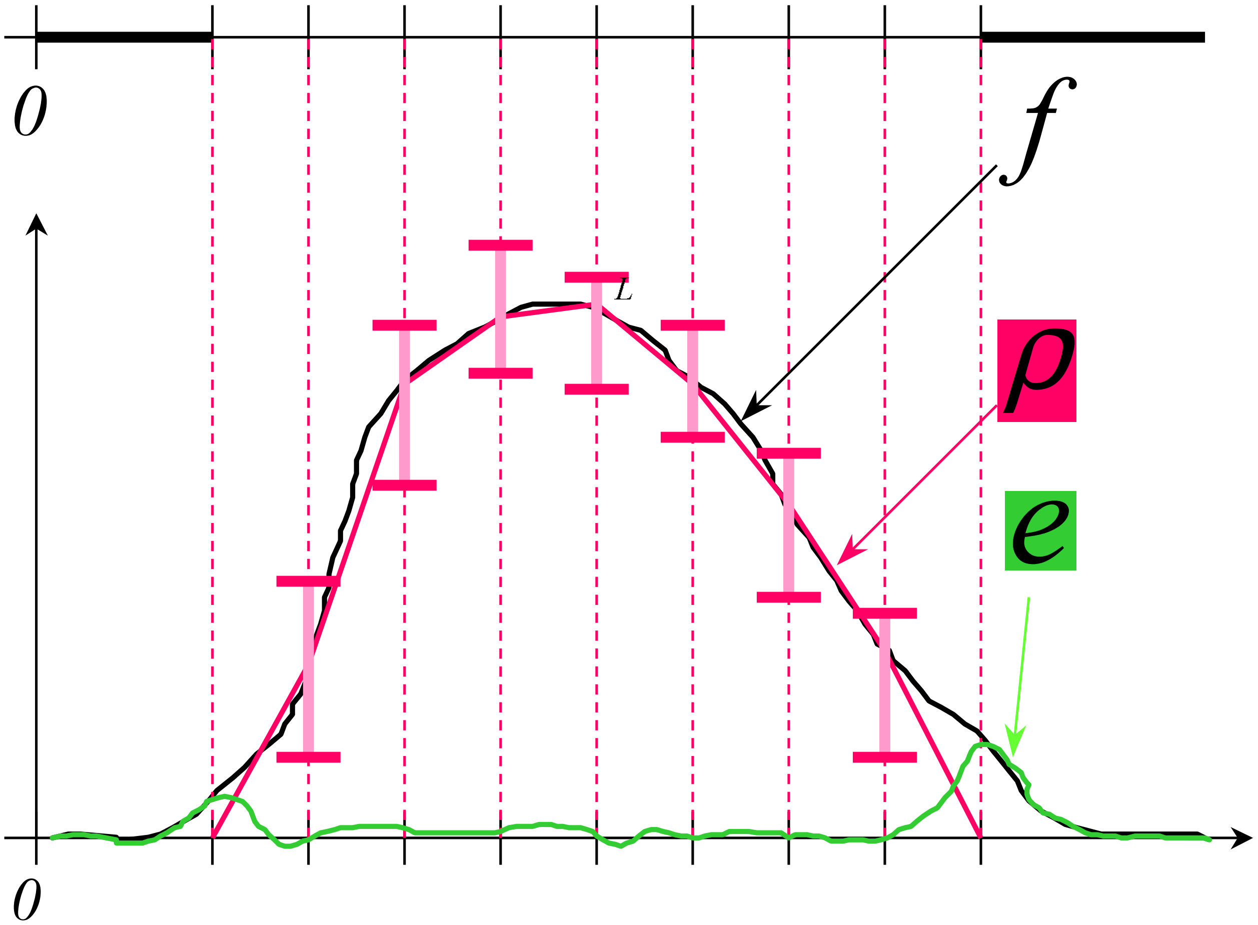
```
i(X1,Y1) contains i(X2,Y2)+i(X3,Y3):-
 X is X2+X3,
 Y is Y2+Y3,
 i(X1,Y1) enlarges i(X,Y),
 !.
```

```
i(X1,Y1) contains i(X2,Y2)*i(X3,Y3):-
 A is X2*X3,
 B is X2*Y3,
 C is Y2*X3,
 D is Y2*Y3,
 sort_numbers([A,B,C,D],[X,_,_,Y]),
 i(X1,Y1) enlarges i(X,Y),
 !.
```

Functions Spaces

$$B = L_1(R^+, w(x)\,dx)$$

$$w(x) = \exp(1/x + x)$$



$$f = \rho + e, \quad \rho \in \cdots,$$

$$\|e\| \le \varepsilon$$

Bound on Convolution

$$p(f)(x) = (f * f)(x)$$

$$f = \rho + e$$

$$f * f = \rho * \rho + 2 * \rho * e + e * e$$

$$= \tilde{\rho} + \tilde{e}$$

Lemma: $\rho * \rho$ an explicit cubic spline.

$$\|\tilde{e}\| \leq \;\ldots$$

↑

Almost no additional work needed!

Choice of Programming Language


```
C..calculate sTGJDZF and sTDZFOGAMMA
      sT3=sDIFF(sINV(sZ),sQUOT(sPOWER(sRHOP,2),sR))
      sT3=sINV(sPOWER(sSQRT(sT3),7))
      sT5=sQUOT(sPROD(sT3,sT4)
     *   ,sPROD(sPOWER(sETA,4),sPOWER(sDIFF(sONE,sT1),3)))
      sT7=sQUOT(sQUOT(siCONST(7),sTWO),sPOWER(sSQRT(sT6),9))
      sT8=sQUOT(sPROD(sSUM(sTWO,sT1),sT7)
     *   ,sPROD(sPOWER(sETA,6),sPOWER(sDIFF(sONE,sT1),5)))
      sFAC=sPROD(sQUOT(siCONST(5),sTWO),sPROD(s3HALF,sPROD(sHALF
     *    ,sSQRT(sPI))))
      sT9=sPROD(sPROD(sPROD(sTWO,sFAC),sCTT),sSUM(sT5,sT8))
      sTGJDZF=sUPPER(sPROD(sPROD(sPOWER(sRHOP,3),sT9),sGGAMMA))
      sTDZFOGAMMA=sPROD(sTWO,sPROD(sPROD(sCTT,sFAC)
     *           ,sPROD(sQUOT(sPOWER(sRHOP,4),sPOWER(sETA,3)),sT3)))
C..calculate coefficients sJ1, sJ2 and sJ3
      sT1=sQUOT(sGGAMMA,sPOWER(sRHOP,3))
      sT1=DCMPLX(-rMAXABS(sT1),rMAXABS(sT1))
      sT2=sQUOT(sT1,sPOWER(sRHOGAMMA,2))
      sJ1H=sPROD(sFP00,sPROD(sFOUR,sT1))
      sJ1 =sSUM(sJ01,sJ1H)
      sJ2H=sPROD(sFP01,sPROD(siCONST(5),sT1))
      sJ2 =sSUM(sJ02,sJ2H)
      sJ3H=sSUM(sPROD(sFP00,sSUM(sPROD(sT1,sQUOT(siCONST(12)
     *    ,sKAPPA)),sPROD(sT2,siCONST(6)))),sPROD(sFP01
     *    ,sPROD(siCONST(6),sT1)))
      sJ3 =sSUM(sJ03,sJ3H)
C..calculate bound for general
      sT1=sSUM(sSUM(sSUM(sGDZJ,sPROD(sSUM(sPROD(sABS(sFP00),sGAMMA)
     *   ,sSUM(sPROD(sABS(sFP01),s2GAMMA),sPROD(sABS(sFP02)
     *   ,sPROD(sGAMMA,s2GAMMA))))),sGDZGAMMA))
     *   ,sSUM(sPROD(sTDZFOGAMMA,sGDZGAMMA)
     *   ,sPROD(sSUM(sABS(sFP00),sSUM(sPROD(sABS(sFP01),sSUM(sGAMMA
     *   ,sGAMMA)),sPROD(sABS(sFP02),sSUM(sPROD(sTHREE,s2GAMMA)
     *   ,sSUM(sPROD(sTHREE,sPROD(sGAMMA,sGHGAMMA))
     *   ,sPOWER(sGHGAMMA,2)))))),sPROD(sDZGAMMA,sGGAMMA))))
     *   ,sPROD(sTGJDZF,sDZGAMMA))
      sT2=sSUM(sSUM(sPROD(sJ1H,sPOWER(sRHOP,3))
     *   ,sPROD(sJ2H,sPOWER(sRHOP,4))),sPROD(sJ3H
     *   ,sPOWER(sRHOP,5)))
      sGDZJTOT=sCONST(rMAXABS(sSUM(sT1,sT2)))
C..calculate bound for higher order
      sT1=sSUM(sSUM(sHDZJ,sSUM(sPROD(sDZFOGAMMA,sHDZGAMMA)
     *   ,sPROD(sSUM(sABS(sFP00),sSUM(sPROD(sABS(sFP01)
     *   ,sSUM(sGAMMA,sGAMMA)),sPROD(sABS(sFP02),sSUM(sPROD(sTHREE
     *   ,s2GAMMA),sSUM(sPROD(sTHREE,sPROD(sGAMMA,sGHGAMMA))
     *   ,sPOWER(sGHGAMMA,2)))))),sPROD(sDZGAMMA,sHGAMMA))))
     *   ,sPROD(sQUOT(sCT,sC),sPROD(sHJDZF,sDZGAMMA)))
      sHDZJTOT=sCONST(rMAXABS(sT1))
C..invert Ecalle's equation: lower order coefficients
      sDZF01=sQUOT(sJ1,sPROD(sFOUR,sGA3))
      sDZF02=sQUOT(sJ2,sPROD(siCONST(5),sGA3))
      sDZF03=sQUOT(sDIFF(sJ3,sPROD(sSUM(sPROD(s3HALF
     *      ,sQUOT(sGA5,sGA3)),sPROD(sQUOT(sTHREE,sFOUR),sGA3))
     *      ,sJ1)),sPROD(siCONST(6),sGA3))
C..add additional contribution to general
      CALL FsDILAT(vGAMMA,sRHO1,vT1)
      CALL FsMULT(vT1,sRHOGAMMA,vT1)
      CALL FDZsDILAT(vGAMMA,sRHO1,vT2)
      CALL FMULT(vT1,vT2,vT2)
      CALL FEQUAL(vT2,vT3)
```

Prolog
(PROgramming in LOGic)

even if r_1 and r_2 are in S, as we shall now assume. However, as far as bounds are concerned, it is sufficient to find an interval `i(X1,Y1)` that contains $r_1 \# r_2$, given an interval `i(X,Y)` that contains the computed value for $r_1 \# r_2$. The following predicate `enlarges/2` serves this purpose:

```
i(X1,Y1) enlarges i(X,Y):-
 X1 is_a_safe_lower_bound_on X,
 Y1 is_a_safe_upper_bound_on Y,
 !.
```

To be more precise, if `X`$\leq$`Y` are given representable numbers (not necessarily in the safe range), then `i(X1,Y1) enlarges i(X,Y)` is satisfied if and only if the indicated safe lower bound `X1` and safe upper bound `Y1` can be found, in which case `i(X1,Y1)` is a standard set with the above-mentioned property, as explained in §3. A typical application of `enlarges/2` is given in the next clause.

Consider now the sum function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. The corresponding set map assigns to a pair (I_2, I_3) in $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})$ the set

(5.3) $$I_2 + I_3 = \{r \in \mathbb{R} \mid r = x + y \text{ for some } x \in I_2 , y \in I_3\}$$

in $\mathcal{P}(\mathbb{R})$. The following clause defines a bound on this map:

```
i(X1,Y1) contains i(X2,Y2)+i(X3,Y3):-
 X is X2+X3,
 Y is Y2+Y3,
 i(X1,Y1) enlarges i(X,Y),
 !.
```

As indicated in §4, the domain of this bound is the set of all pairs (`I2`, `I3`) in std($\mathbb{R}$)$\times$std($\mathbb{R}$) for which `I1 contains I2+I3` is true. In this case, the domain is determined by the predicate `enlarges/2`, which is used in order to compensate for possible rounding errors introduced by `is/2` and to ensure that the returned result `I1` is again a standard set.

The same description applies to our bound on the product of (sets of) real numbers if $+$ is replaced by $*$. The bound itself is defined as follows:

```
i(X1,Y1) contains i(X2,Y2)*i(X3,Y3):-
 A is X2*X3,
 B is X2*Y3,
 C is Y2*X3,
 D is Y2*Y3,
 sort_numbers([A,B,C,D],[X,_,_,Y]),
 i(X1,Y1) enlarges i(X,Y),
 !.
```

Here, we have used that the product of two standard sets I_2 and I_3 is an interval and that the

Discretization errors

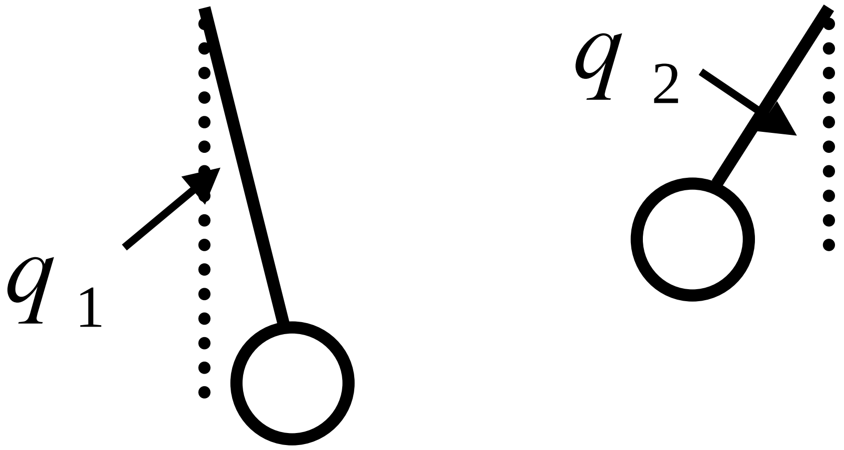
- Bounds are set-maps
- Bounds are defined on "standard sets"
- Can compute with bounds
- Bounds can be composed to form new bounds
- Can use chain rule to compute derivative of complicated maps
- Need only worry about simple maps
- Let computer do the rest
- SIAM Review, 38, 4, 1996
- Alain Schenkel 1998

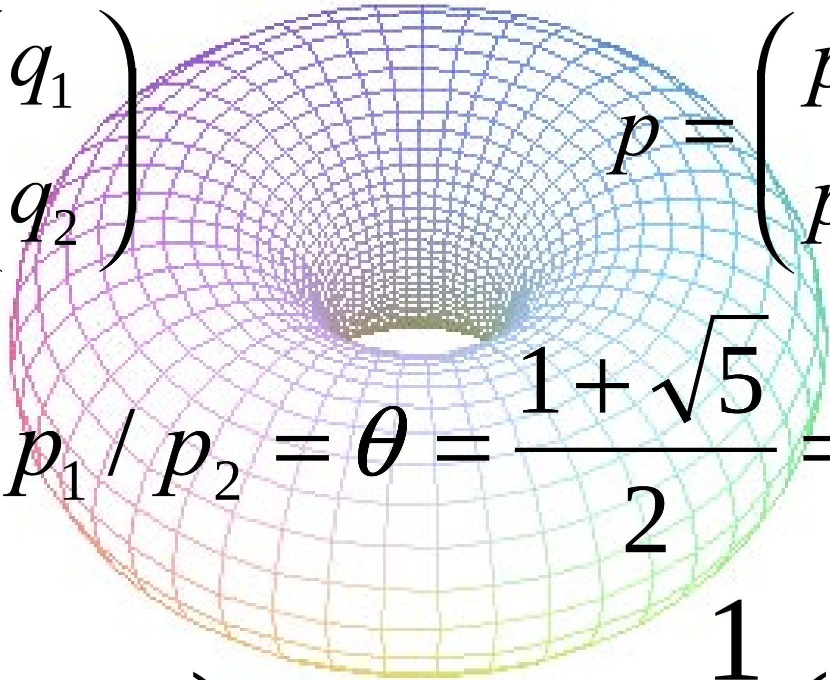
Universality in Dynamical Systems

collaboration with:

Hans Koch
Juan Abad

Hamiltonian Dynamics





$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \qquad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$\theta = p_1 / p_2 = \theta = \frac{1+\sqrt{5}}{2} = 1.61\ldots$$

$$H_0(q,p) = \omega \cdot p + \frac{1}{2}(\Omega \cdot p)^2$$

$$\partial_t p = \nabla_1 H_0 = 0$$

$$\partial_t q = -\nabla_2 H_0 = \omega + (\Omega \cdot p)\Omega$$

Fibonacci Numbers

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$T\omega = \theta\omega\,, \qquad T\Omega = -\frac{1}{\theta}\Omega$$

$$F_1 = F_2 = 1\,, \quad F_{k+2} = F_{k+1} + F_k$$

$$\theta_k = \frac{F_{k+1}}{F_k}\,, \qquad \theta = \lim_{k\to\infty} \theta_k$$

θ_k - tori accumulate on θ - torus

Universality

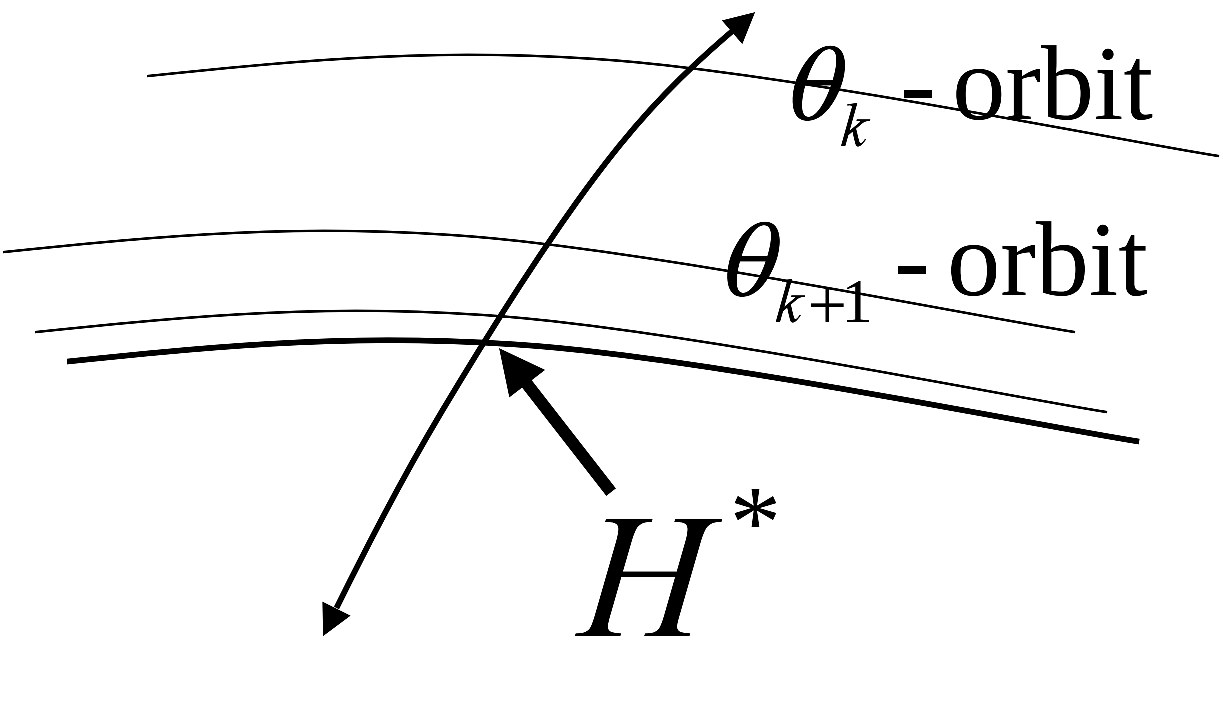
introduce coupling:

$$H = H_0 + \sum_{\substack{\nu \in Z^2 \\ n \geq 0}} h_{\nu,n} e^{i\nu \cdot q} (\Omega \cdot p)^n$$

$$N(H)(q,p) = H(Tq, T^{-1}p)$$

H has θ_k - orbit,

$N(H)$ has θ_{k-1} - orbit



Back to Computer

$$H(q,p)=\sum_{\substack{\nu\in Z^2\\ n\geq 0}} h_{\nu,n}e^{i\nu\cdot q}(\Omega\cdot p)^n$$

$$N(H)(q,p)=H(Tq,T^{-1}p)$$

ill defined, domains!

$$R(H)(q,p)=$$

$$=\frac{\tau}{\mu}H\circ U_H(Tq,\mu T^{-1}p)-\varepsilon$$

- numerical work published
- computer proof in preparation