# Stationary flows in the presence of a wall

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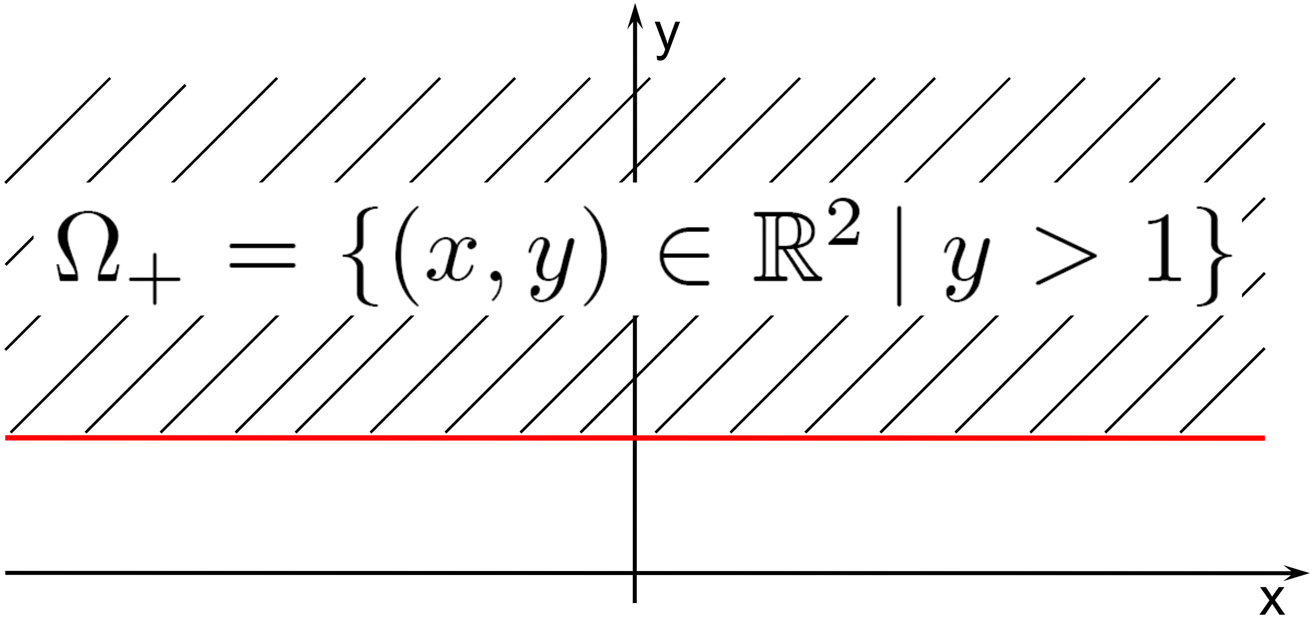
- Motivation and formulation of the problem
- Results
- Proof
- Outlook

# The main mathematical problem




$$\Omega_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 1\}$$

$$-\partial_x \mathbf{u} + \Delta \mathbf{u} = \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p$$

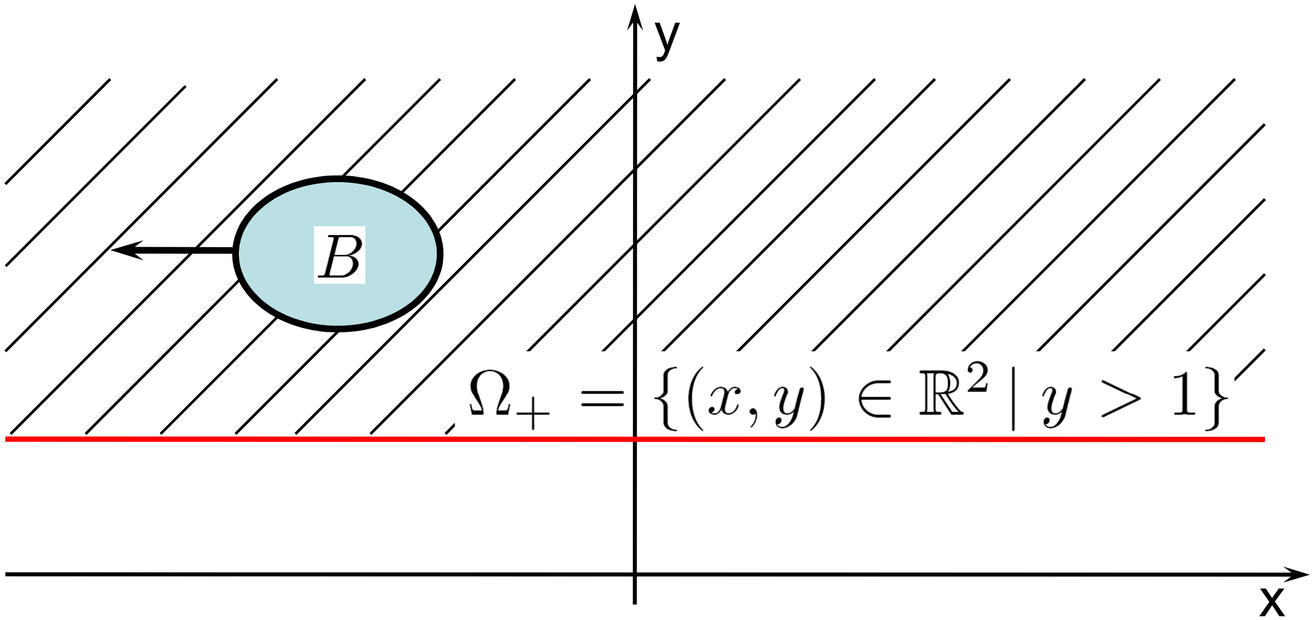
$$\nabla \cdot \mathbf{u} = 0$$

1

$$\mathbf{u}(x, 1) = 0$$

$$\lim_{\mathbf{x} \to \infty} \mathbf{u}(\mathbf{x}) = 0 \qquad \mathbf{F} \text{ of compact support}$$

# Work in progress: the exterior problem



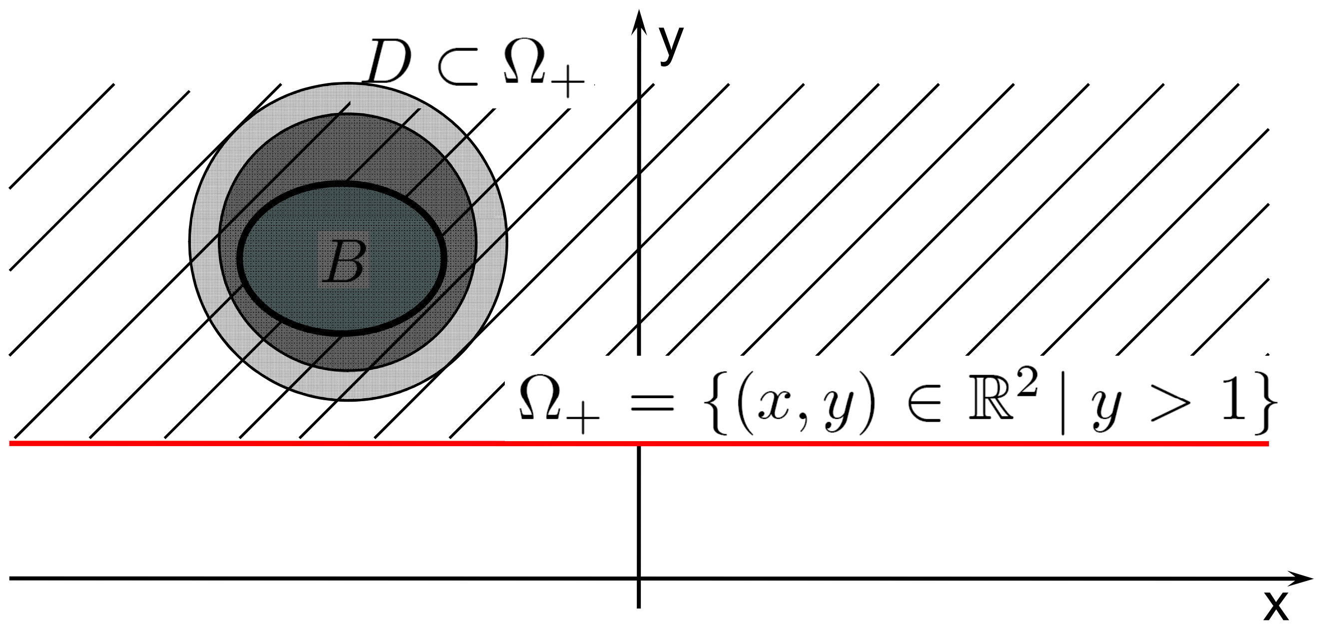

$$\left.\begin{aligned} -\mathbf{u}\cdot\nabla\mathbf{u} - \partial_x\mathbf{u} + \Delta\mathbf{u} - \nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0 \end{aligned}\right\} \quad \Omega = \Omega_+ \setminus B$$

2

$$\mathbf{u}(x,1) = 0$$

$$\lim_{\mathbf{x}\to\infty} \mathbf{u}(\mathbf{x}) = 0 \qquad \mathbf{u}|_{\partial B} = -\mathbf{e}_1 \qquad \mathbf{e}_1 = (1,0)$$

# Connection between 1 and 2




$\tilde{\mathbf{u}}$ be a smooth solution $\quad \tilde{\mathbf{u}} = (-\partial_y \tilde{\psi}, \partial_x \tilde{\psi})$

$\psi = \chi \tilde{\psi} \quad \mathbf{u} = (-\partial_y \psi, \partial_x \psi)$

# Main result

$$
\begin{aligned}
-\partial_x \mathbf{u} + \Delta \mathbf{u} &= \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \\
\nabla \cdot \mathbf{u} &= 0 \\
\mathbf{u}(x,1) &= 0 \\
\lim_{\mathbf{x} \to \infty} \mathbf{u}(\mathbf{x}) &= 0 \qquad \mathbf{F} \text{ of compact support}
\end{aligned}
$$

# Theorem

For all sufficiently small $\mathbf{F} \in C_c^\infty(\Omega_+)$

there exists a solution $\mathbf{u} = (u, v) \in C^2(\Omega_+)$

The solution is unique in $H^1(\Omega_+)$

Proof :

# Reduction to an evolution system I

$$\omega = -\partial_y u + \partial_x v \ ,$$

$$-\partial_x \omega + \Delta \omega = q + \rho \ ,$$

$$\partial_x u + \partial_y v = 0 \ ,$$

$y$ playing the role of time

$$q = \partial_x (u\omega) + \partial_y (v\omega) \ ,$$

$$\rho = -\partial_y F_1 + \partial_x F_2 \ .$$

# Reduction to an evolution system II

$$\partial_y \omega = \bar{\eta} \ ,$$

$$\partial_y \bar{\eta} = -\partial_x^2 \omega + \partial_x \omega + Q \ ,$$

$$\partial_y u = -\omega + \partial_x v \ ,$$

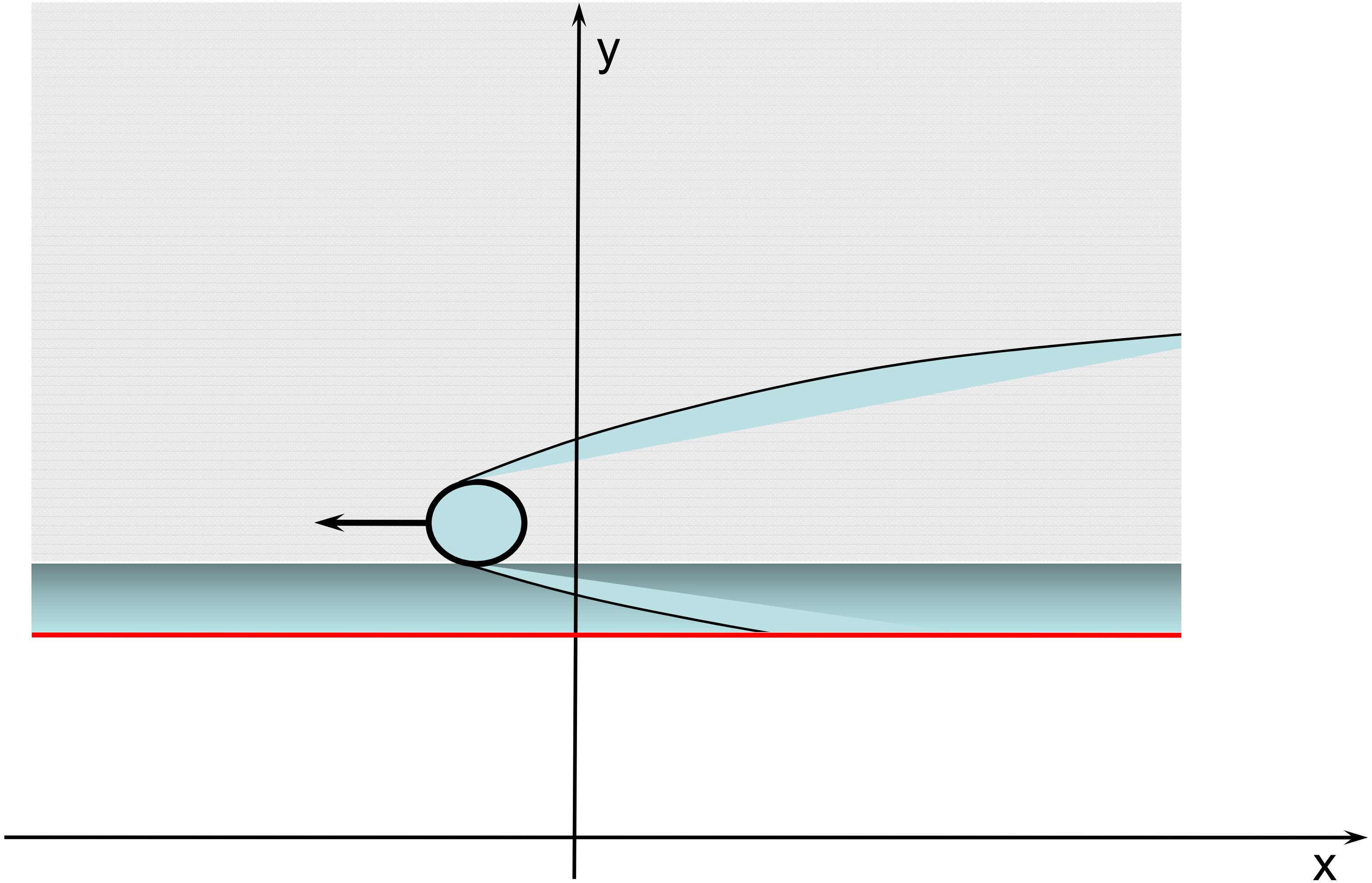
$$\partial_y v = -\partial_x u \ .$$

$$Q = \partial_x Q_0 + \partial_y Q_1$$

$$Q_0 = q_0 + F_2 \qquad q_0 = u\omega$$

$$Q_1 = q_1 - F_1 \qquad q_1 = v\omega$$

# Heuristic aspects



# Decomposition I

$$v = \omega + \psi \ ,$$

$$u = \partial_x^{-1} \left( -\bar{\eta} + \bar{\phi} \right)$$

$$\partial_y \omega = \bar{\eta} \ ,$$

$$\partial_y \bar{\eta} = -\partial_x^2 \omega + \partial_x \omega + \partial_x Q_0 + \partial_y Q_1 \ ,$$

$$\partial_y \psi = -\bar{\phi} \ ,$$

$$\partial_y \bar{\phi} = \partial_x^2 \psi + \partial_x Q_0 + \partial_y Q_1 \ ,$$

# Decomposition II

$$\bar{\eta} = \partial_x \eta + Q_1$$

$$\bar{\phi} = \partial_x \phi + Q_1$$

$$\partial_y \omega = \partial_x \eta + Q_1 \ ,$$

$$\partial_y \eta = -\partial_x \omega + \omega + Q_0 \ ,$$

$$\partial_y \psi = -\partial_x \phi - Q_1 \ ,$$

$$\partial_y \phi = \partial_x \psi + Q_0 \ ,$$

$$Q_0 = q_0 + F_2 \qquad q_0 = u\omega \qquad u = -\eta + \phi$$

$$Q_1 = q_1 - F_1 \qquad q_1 = v\omega \qquad v = \omega + \psi$$

# Fourier transform

*Let* $\hat{f}$ *be a complex valued function in* $L^1(\Omega_+)$

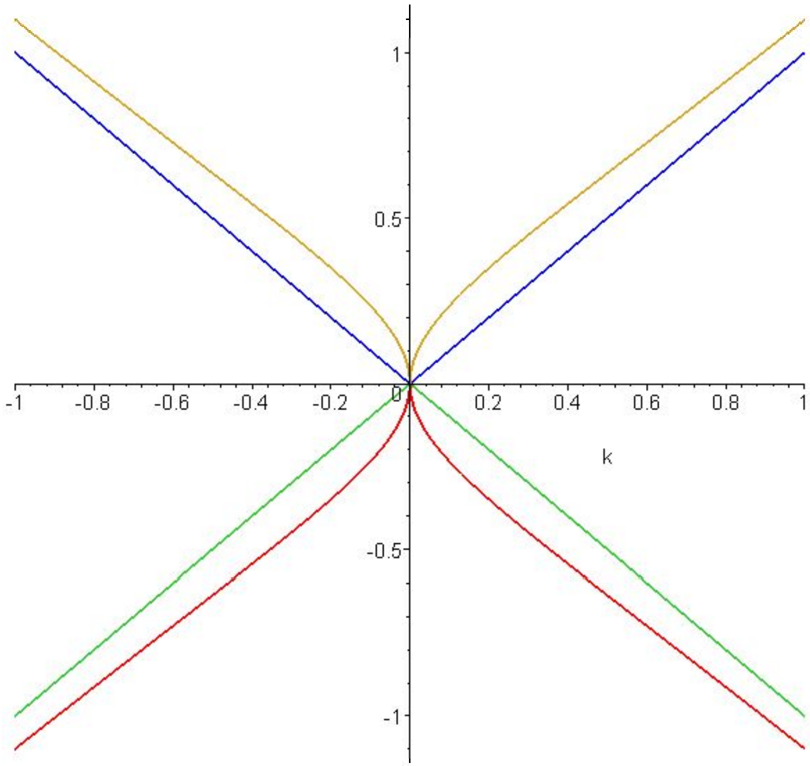
$$\mathcal{F}^{-1}[\hat{f}](x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ikx}\hat{f}(k,y)\,dk$$

$$\partial_y\hat{\omega} = -ik\hat{\eta} + \hat{Q}_1\ ,$$

$$\partial_y\hat{\eta} = (ik+1)\hat{\omega} + \hat{Q}_0\ ,$$

$$\partial_y\hat{\psi} = ik\hat{\phi} - \hat{Q}_1\ ,$$

$$\partial_y\hat{\phi} = -ik\hat{\psi} + \hat{Q}_0\ .$$




$$\kappa = \sqrt{k^2 - ik}$$

# Integral equations

$$\hat{\eta} = \sum_{m=0,1} \sum_{n=1,2,3} \hat{\eta}_{n,m} \ , \qquad \hat{\omega} = \sum_{m=0,1} \sum_{n=1,2,3} \hat{\omega}_{n,m} \ ,$$

$$\hat{\phi} = \sum_{m=0,1} \sum_{n=1,2,3} \hat{\phi}_{n,m} \ , \qquad \hat{\psi} = \sum_{m=0,1} \sum_{n=1,2,3} \hat{\psi}_{n,m} \ ,$$

$$\hat{\eta}_{n,m}(k,t) = K_n(k,t-1) \int_{I_n} g_{n,m}(k,s-1)\ \hat{Q}_m(k,s)\ ds \ ,$$

$$\hat{\omega}_{n,m}(k,t) = K_n(k,t-1) \int_{I_n} f_{n,m}(k,s-1)\ \hat{Q}_m(k,s)\ ds \ ,$$

$$\hat{\phi}_{n,m}(k,t) = G_n(k,t-1) \int_{I_n} k_{n,m}(k,s-1)\ \hat{Q}_m(k,s)\ ds \ ,$$

$$\hat{\psi}_{n,m}(k,t) = G_n(k,t-1) \int_{I_n} h_{n,m}(k,s-1)\ \hat{Q}_m(k,s)\ ds \ ,$$

$$K_n(k,\tau) = \frac{1}{2} e^{-\kappa\tau} \ , \ \text{for } n = 1,2 \ ,$$
$$K_3(k,\tau) = \frac{1}{2}\frac{\kappa}{ik}\left(e^{\kappa\tau} - e^{-\kappa\tau}\right) \ ,$$

$$G_n(k,\tau) = \frac{1}{2} e^{-|k|\tau} \ , \ \text{for } n \in 1,2 \ ,$$
$$G_3(k,\tau) = \frac{1}{2}\frac{|k|}{ik}\left(e^{|k|\tau} - e^{-|k|\tau}\right) \ .$$

$$I_1 = [1,t]$$
$$I_n = [t,\infty)$$

# Functional framework I

$\alpha,\ r \geq 0$ and $k \in \mathbb{R}$ $\qquad \mu_{\alpha,r}(k,t) = \dfrac{1}{1 + (|k|\, t^r)^\alpha}$

$\bar{\mu}_\alpha(k,t) = \mu_{\alpha,1}(k,t)$

$\tilde{\mu}_\alpha(k,t) = \mu_{\alpha,2}(k,t)$

$p,\ q \geq 0 \quad \mathcal{B}_{\alpha,p,q} \qquad f \in C(\mathbb{R}_0 \times [1,\infty), \mathbb{C})$

$$\|f;\ \mathcal{B}_{\alpha,p,q}\| = \sup_{t\geq 1} \sup_{k\in\mathbb{R}_0} \frac{|f(k,t)|}{\frac{1}{t^p}\bar{\mu}_\alpha(k,t) + \frac{1}{t^q}\tilde{\mu}_\alpha(k,t)}$$

$$\mathcal{V}_\alpha = \mathcal{B}_{\alpha,\frac{5}{2},1} \times \mathcal{B}_{\alpha,\frac{1}{2},0} \times \mathcal{B}_{\alpha,\frac{1}{2},1}$$

$$\mathcal{B}_\alpha = \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}} \times \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}$$

# Functional framework II

$$\mathcal{C} : \quad \mathcal{V}_\alpha \times \mathcal{V}_\alpha \longrightarrow \mathcal{B}_\alpha ,$$
$$((\tilde{\omega}_1, \tilde{u}_1, \tilde{v}_1), (\tilde{\omega}_2, \tilde{u}_2, \tilde{v}_2)) \longmapsto \left( \frac{1}{2\pi} \left( \tilde{u}_1 * \tilde{\omega}_2 \right), \frac{1}{2\pi} \left( \tilde{v}_1 * \tilde{\omega}_2 \right) \right)$$

$$\mathcal{L} : \quad \mathcal{B}_\alpha \longrightarrow \mathcal{V}_\alpha$$
$$(\tilde{Q}_0, \tilde{Q}_1) \longmapsto (\tilde{\omega}, \tilde{u}, \tilde{v})$$

*A triple $(\tilde{\omega}, \tilde{u}, \tilde{v})$ is called an $\alpha$-solution if:*

$(i)$ $(\tilde{\omega}, \tilde{u}, \tilde{v}) \in \mathcal{V}_\alpha$ ,

$(ii)$ $(\tilde{\omega}, \tilde{u}, \tilde{v}) = \mathcal{L}[\mathcal{C}[(\tilde{\omega}, \tilde{u}, \tilde{v}), (\tilde{\omega}, \tilde{u}, \tilde{v})] + (\tilde{F}_2, -\tilde{F}_1)]$

Existence by the contraction mapping principle

## Work in progress

- compute detailed asymptotic behavior
- put obstacle back in
- link asymptotic behavior to "computable quantities"
- use result to provide artificial boundary conditions for the numerical solution of the problem

# THANK YOU !